Langlands Program and TGD

M. Pitkänen[1], February 15, 2007

[1] Department of Physical Sciences, High Energy Physics Division,
PL 64, FIN-00014, University of Helsinki, Finland.
matpitka@rock.helsinki.fi, http://www.physics.helsinki.fi/∼matpitka/.
Recent address: Puutarhurinkatu 10,10960, Hanko, Finland.

Contents

Abstract

Number theoretic Langlands program can be seen as an attempt to unify number theory on one hand and theory of representations of reductive Lie groups on the other hand. So called automorphic functions to which various zeta functions are closely related define the common denominator. Geometric Langlands program tries to achieve a similar conceptual unification in the case of function fields. This program has caught the interest of physicists during last years.

TGD can be seen as an attempt to reduce physics to infinite-dimensional Kähler geometry and spinor structure of the "world of classical worlds" (WCW). Since TGD ce be regarded also as a generalized number theory, it is difficult to escape the idea that the interaction of Langlands program with TGD could be fruitful.

More concretely, TGD leads to a generalization of number concept based on the fusion of reals and various p-adic number fields and their extensions implying also generalization of manifold concept, which inspires the notion of number theoretic braid crucial for the formulation of quantum TGD. TGD leads also naturally to the notion of infinite primes and rationals. The identification of Clifford algebra of WCW as a hyper-finite factors of type II_1 in turn inspires further generalization of the notion of imbedding space and the idea that quantum TGD as a whole emerges from number theory. The ensuing generalization of the notion of imbedding space predicts a hierarchy of macroscopic quantum phases characterized by finite subgroups of $SU(2)$ and by quantized Planck constant. All these new elements serve as potential sources of fresh insights.

1. The Galois group for the algebraic closure of rationals as infinite symmetric group?

The naive identification of the Galois groups for the algebraic closure of rationals would be as infinite symmetric group S_∞ consisting of finite permutations of the roots of a polynomial of infinite degree having infinite number of roots. What puts bells ringing is that the corresponding group algebra is nothing but the hyper-finite factor of type II_1 (HFF). One of the many avatars of this algebra is infinite-dimensional Clifford algebra playing key role in Quantum TGD. The projective representations of this algebra can be interpreted as representations of braid algebra B_∞ meaning a connection with the notion of number theoretical braid.

2. Representations of finite subgroups of S_∞ as outer automorphisms of HFFs

Finite-dimensional representations of $Gal(\overline{Q}/Q)$ are crucial for Langlands program. Apart from one-dimensional representations complex

finite-dimensional representations are not possible if S_∞ identification is accepted (there might exist finite-dimensional l-adic representations). This suggests that the finite-dimensional representations correspond to those for finite Galois groups and result through some kind of spontaneous breaking of S_∞ symmetry.

a) Sub-factors determined by finite groups G can be interpreted as representations of Galois groups or, rather infinite diagonal imbeddings of Galois groups to an infinite Cartesian power of S_n acting as outer automorphisms in HFF. These transformations are counterparts of global gauge transformations and determine the measured quantum numbers of gauge multiplets and thus measurement resolution. All the finite approximations of the representations are inner automorphisms but the limit does not belong to S_∞ and is therefore outer. An analogous picture applies in the case of infinite-dimensional Clifford algebra.

b) The physical interpretation is as a spontaneous breaking of S_∞ to a finite Galois group. One decomposes infinite braid to a series of n-braids such that finite Galois group acts in each n-braid in identical manner. Finite value of n corresponds to IR cutoff in physics in the sense that longer wave length quantum fluctuations are cut off. Finite measurement resolution is crucial. Now it applies to braid and corresponds in the language of new quantum measurement theory to a sub-factor $\mathcal{N} \subset \mathcal{M}$ determined by the finite Galois group G implying non-commutative physics with complex rays replaced by $\mathcal{N}$ rays. Braids give a connection to topological quantum field theories, conformal field theories (TGD is almost topological quantum field theory at parton level), knots, etc..

c) TGD based space-time correlate for the action of finite Galois groups on braids and for the cutoff is in terms of the number theoretic braids obtained as the intersection of real partonic 2-surface and its p-adic counterpart. The value of the p-adic prime p associated with the parton is fixed by the scaling of the eigenvalue spectrum of the modified Dirac operator (note that renormalization group evolution of coupling constants is characterized at the level free theory since p-adic prime characterizes the p-adic length scale). The roots of the polynomial would determine the positions of braid strands so that Galois group emerges naturally. As a matter fact, partonic 2-surface decomposes into regions, one for each braid transforming independently under its own Galois group. Entire quantum state is modular invariant, which brings in additional constraints.

Braiding brings in homotopy group aspect crucial for geometric Langlands program. Another global aspect is related to the modular degrees of freedom of the partonic 2-surface, or more precisely to the regions of partonic 2-surface associated with braids. $Sp(2g, R)$ (g is handle number) can act as transformations in modular degrees of freedom whereas its Langlands dual would act in spinorial degrees of freedom. The outcome would be a coupling between purely local and and global aspects which is necessary since otherwise all information about partonic 2-surfaces as basic objects would be lost. Interesting ramifications of the basic picture about why only three lowest genera correspond to the observed fermion families emerge.

3. Correspondence between finite groups and Lie groups

The correspondence between finite and Lie group is a basic aspect of Langlands.

a) Any amenable group gives rise to a unique sub-factor (in particular, compact Lie groups are amenable). These groups act as genuine outer automorphisms of the group algebra of S_∞ rather than being induced from S_∞ outer automorphism. If one gives up uniqueness, it seems that practically any group G can define a sub-factor: G would define measurement resolution by fixing the quantum numbers which are measured. Finite Galois group G and Lie group containing it and related to it by Langlands correspondence would act in the same representation space: the group algebra of S_∞, or equivalently configuration space spinors. The concrete realization for the correspondence might transform a large number of speculations to theorems.

b) There is a natural connection with McKay correspondence which also relates finite and Lie groups. The simplest variant of McKay correspondence relates discrete groups $G \subset SU(2)$ to ADE type groups. Similar correspondence is found for Jones inclusions with index $\mathcal{M} : \mathcal{N} \leq 4$. The challenge is to understand this correspondence.

i) The basic observation is that ADE type compact Lie algebras with n-dimensional Cartan algebra can be seen as deformations for a direct sum of n SU(2) Lie algebras since SU(2) Lie algebras appear as a minimal set of generators for general ADE type Lie algebra. The algebra results by a modification of Cartan matrix. It is also natural to extend the representations of finite groups $G \subset SU(2)$ to those of $SU(2)$.

ii) The idea would that is that n-fold Connes tensor power transforms the direct sum of n SU(2) Lie algebras by a kind of deformation to a ADE type Lie algebra with n-dimensional Cartan Lie algebra. The deformation would be induced by non-commutativity. Same would occur also for the Kac-Moody variants of these algebras for which the set of generators contains only scaling operator L_0 as an additional generator. Quantum deformation would result from the replacement of complex rays with $\mathcal{N}$ rays, where $\mathcal{N}$ is the sub-factor.

iii) The concrete interpretation for the Connes tensor power would be in terms of the fiber bundle structure $H = M^4_\pm \times CP_2 \rightarrow H/G_a \times G_b$, $G_a \times G_b \subset SU(2) \times SU(2) \subset SL(2,C) \times SU(3)$, which provides the proper formulation for the hierarchy of macroscopic quantum phases with a quantized value of Planck constant. Each sheet of the singular covering would represent single factor in Connes tensor power and single direct $SU(2)$ summand. This picture has an analogy with brane constructions of M-theory.

4. Could there exist a universal rational function giving rise to the algebraic closure of rationals?

One could wonder whether there exists a universal generalized rational function having all units of the algebraic closure of rationals as roots so that S_∞ would permute these roots. Most naturally it would be a ratio of infinite-degree polynomials.

With motivations coming from physics I have proposed that zeros of zeta and also the factors of zeta in product expansion of zeta are algebraic numbers. Complete story might be that non-trivial zeros of Zeta define the closure of rationals. A good candidate for this function is given by $(\xi(s)/\xi(1-s)) \times (s-1)/s)$, where $\xi(s) = \xi(1-s)$ is the symmetrized variant of ζ function having same zeros. It has zeros of zeta as its zeros and poles and product expansion in terms of ratios $(s-s_n)/(1-s+s_n)$ converges everywhere. Of course, this might be too simplistic and might give only the algebraic extension involving the roots of unity given by $exp(i\pi/n)$. Also products of these functions with shifts in real argument might be considered and one could consider some limiting procedure containing very many factors in the product of shifted ζ functions yielding the universal rational function giving the closure.

5. What does one mean with S_∞?

There is also the question about the meaning of S_∞. The hierarchy of infinite primes suggests that there is entire infinity of infinities in number theoretical sense. Any group can be formally regarded as a permutation group. A possible interpretation would be in terms of algebraic closure of rationals and algebraic closures for an infinite hierarchy of polynomials to which infinite primes can be mapped. The question concerns the interpretation of these higher Galois groups and HFFs. Could one regard these as local variants of S_∞ and does this hierarchy give all algebraic groups, in particular algebraic subgroups of Lie groups, as Galois groups so that almost all of group theory would reduce to number theory even at this level?

Be it as it may, the expressive power of HFF:s seem to be absolutely marvellous. Together with the notion of infinite rational and generalization of number concept they might unify both mathematics and physics!

1 Introduction

Langlands program [5, 6, 7, 8] is an attempt to unify number theory and representation theory of groups and as it seems all mathematics. About related topics I know frustratingly little at technical level. Zeta functions and theta functions [10, 11, 12, 13], and more generally modular forms [14] are the connecting notion appearing both in number theory and in the theory of automorphic representations of reductive Lie groups. The fact that zeta functions have a key role in TGD has been one of the reasons for my personal interest.

The vision about TGD as a generalized number theory [E1, E2, E3, C1, C2] gives good motivations to learn the basic ideas of Langlands program. I hasten to admit that I am just a novice with no hope becoming a master of the horrible technicalities involved. I just try to find whether the TGD framework could allow new physics inspired insights to Langlands program and whether the more abstract number theory relying heavily on the representations of Galois groups could have a direct physical counterpart in TGD Universe and help to develop TGD as a generalized number theory vision. After these apologies I however dare to raise my head a little bit and say aloud that mathematicians might get inspiration from physics inspired new insights.

The basic vision is that Langlands program could relate very closely to the unification of physics as proposed in TGD framework [1, 2, 3]. TGD can indeed be seen both as infinite-dimensional geometry, as a generalized number theory involving several generalizations of the number concept, and as an algebraic approach to physics relying on the unique properties of hyper finite factors of type II_1 so that unification of mathematics would obviously fit nicely into this framework. The fusion of real and various p-adic physics based on the generalization of the number concept, the notion of number theoretic braid, hyper-finite-factors of type II_1 andsub-factors, and the notion of infinite prime, inspired a new view about how to represent finite Galois groups and how to unify the number theoretic and geometric Langlands programs.

1.1 Langlands program very briefly

Langlands program [6] states that there exists a connection between number theory and automorphic representations of a very general class of Lie groups known as reductive groups (groups whose all representations are fully reducible). At the number theoretic side there are Galois groups characterizing extensions of number fields, say rationals or finite fields. Number theory involves also so called automorphic functions to which zeta functions carrying arithmetic information via their coefficients relate via so called Mellin transform $\sum_n a_n n^s \rightarrow \sum_n a_n z^n$ [13].

Automorphic functions, invariant under modular group $SL(2,Z)$ or subgroup $\Gamma_0(N) \subset SL(2,Z)$ consisting of matrices

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \ , \ c \ mod \ N = 0 \ ,$$

emerge also via the representations of groups $GL(2,R)$. This generalizes also to higher dimensional groups $GL(n,R)$. The dream is that all number theoretic zeta functions could be understood in terms of representation theory of reductive groups. The highly non-trivial outcome would be possibility to deduce very intricate number theoretical information from the Taylor coefficients of these functions.

Langlands program relates also to Riemann hypothesis and its generalizations. For instance, the zeta functions associated with 1-dimensional algebraic curve on finite field F_q, $q = p^n$, code the numbers of solutions to the equations defining algebraic curve in extensions of F_q which form a hierarchy of finite fields F_{q^m} with $m = kn$ [12]: in this case Riemann hypothesis has been proven.

It must be emphasized that algebraic 1-dimensionality is responsible for the deep results related to the number theoretic Langlands program as far as 1-dimensional function fields on finite fields are considered [12, 7]. In fact, Langlands program is formulated only for algebraic extensions of 1-dimensional function fields.

One might also conjecture that Langlands duality for Lie groups reflects some deep duality on physical side. For instance, Edward Witten is working with the idea that geometric variant of Langlands duality could correspond to the dualities discovered in the framework of YM theories and string models. In particular, Witten proposes that electric-magnetic duality which indeed relates gauge group and its dual, provides a physical correlate for the Langlands duality for Lie groups and could be understood in terms of topological version of four-dimensional $N = 4$ super-symmetric YM theory [21]. Interestingly, Witten assigns surface operators to the 2-D surfaces of 4-D space-time. This brings unavoidably in mind partonic 2-surfaces and TGD as $N = 4$ super-conformal almost topological QFT. In this chapter it will be proposed that super-symmetry might correspond to the Langlands duality in TGD framework.

1.2 Questions

Before representing in more detail the TGD based ideas related to Langlands correspondence it is good to summarize the basic questions which Langlands program stimulates.

1.2.1 Could one give more concrete content to the notion of Galois group of algebraic closure of rationals?

The notion of Galois group for algebraic closure of rationals $Gal(\overline{Q}/Q)$ is immensely abstract and one can wonder how to make it more explicit? Langlands program adopts the philosophy that this group could be defined only via its representations. The so called automorphic representations constructed in terms of adeles. The motivation comes from the observation that the subset of adeles consisting of Cartesian product of invertible p-adic integers is a structure isomorphic with the maximal abelian subgroup of $Gal(\overline{Q}/Q)$ obtained by dividing $Gal(\overline{Q}/Q)$ with its commutator subgroup. Representations of finite abelian Galois groups are obtained as homomorphisms mapping infinite abelian Galois group to its finite factor group. In this approach the group $Gal(\overline{Q}/Q)$ remains rather abstract and adeles seem to define a mere auxiliary technical tool although it is clear that so called l-adic representations for Galois groups are are natural also in TGD framework.

This raises some questions.

a) Could one make $Gal(\overline{Q}/Q)$ more concrete? For instance, could one identify it as an infinite symmetric group S_∞ consisting of finite permutations of infinite number of objects? Could one imagine some universal polynomial of infinite degree or a universal rational function resulting as ratio of polynomials of infinite degree giving as its roots the closure of rationals?

b) S_∞ has only single normal subgroup consisting of even permutations and corresponding factor group is maximal abelian group. Therefore finite non-abelian Galois groups cannot be represented via homomorphisms to factor groups. Furthermore, S_{infty} has only infinite-dimensional non-abelian irreducible unitary representations as a simple argument to be discussed later shows.

What is highly non-trivial is that the group algebras of S_∞ and closely related braid group B_∞ define hyper-finite factors of type II_1 (HFF). Could sub-factors characterized by finite groups G allow to realize the representations of finite Galois groups as automorphisms p HFF? The interpretation would be in terms of "spontaneous symmetry breaking" $Gal(\overline{Q}/Q) \rightarrow G$.

Could it be possible to get rid of adeles in this manner?

c) Could one find a concrete physical realization for the action of S_∞? Could the permuted objects be identified as strands of braid so that a braiding of Galois group to infinite braid group B_∞ would result? Could the outer automorphism action of Galois group on number theoretic braids defining the basic structure of quantum TGD allow to realize Galois groups physically as Galois groups of number theoretic braids associated with subset of algebraic points defined by the intersection of real and p-adic partonic 2-surface? The requirement that mathematics is able to represent itself physically would provide the reason for the fact that reality and various p-adicities intersect along subsets of rational and algebraic points only.

1.2.2 Could one understand the correspondences between the representations of finite Galois groups and reductive Lie groups?

Langlands correspondence involves a connection between the representations of finite-dimensional Galois groups and reductive Lie groups.

a) Could this correspondence result via an extension of the representations of finite groups in infinite dimensional Clifford algebra to those of reductive Lie groups identified for instance as groups defining sub-factors (any compact group can define a unique sub-factor)? If Galois groups and reductive groups indeed have a common representation space, it might be easier to understand Langlands correspondence.

b) Is there some deep difference between between general Langlands correspondence and that for $GL(2, F)$ and could this relate to the fact that subgroups of $SU(2)$ define sub-factors with quantized index $\mathcal{M} : \mathcal{N} \leq 4$.

c) McKay correspondence [36] relates finite subgroups of compact Lie groups to compact Lie group (say finite sub-groups of SU(2) to ADE type Lie-algebras or Kac-Moody algebras). TGD approach leads to a general heuristic explanation of this correspondence in terms of Jones inclusions and Connes tensor product. Could sub-factors allow to understand Langlands correspondence for general reductive Lie groups as both the fact that any compact Lie group can define a unique sub-factor and an argument inspired by McKay correspondence suggest.

1.2.3 Could one unify geometric and number theoretic Langlands programs?

There are two Langlands programs: algebraic [5, 7] and geometric [7, 8] one corresponding to ordinary number fields and function fields. The natural question is whether and how these approaches could be unified.

a) Could the discretization based on the notion of number theoretic braids induce the number theoretic Langlands from geometric Langlands so that the two programs could be unified by the generalization of the notion of number field obtained by gluing together reals with union of reals and various p-adic numbers fields and their extensions along common rationals and algebraics. Certainly the fusion of p-adics and reals to a generalized notion of number should be essential for the unification of mathematics.

b) Could the distinction between number fields and function fields correspond to two kinds of sub-factors corresponding to finite subgroups $G \subset SU(2)$ and SU(2) itself leaving invariant the elements of imbedded algebra? This would obviously generalize to imbeddings of Galois groups to arbitrary compact Lie group. Could gauge group algebras contra Kac Moody algebras be a possible physical interpretation for this. Could the two Langlands programs correspond to two kinds of ADE type hierarchies defined by Jones inclusions? Could minimal conformal field theories with finite number of primary fields correspond to algebraic Langlands and full string theory like conformal field theories with infinite number of primary fields to geometric Langlands? Could this difference correspond to sub-factors defined by disrete groups and Lie groups?

c) Could the notion of infinite rational [4] be involved with this unification? Infinite rationals are indeed mapped to elements of rational function fields (also algebraic extensions of them) so that their interpretation as quantum states of a repeatedly second quantized arithmetic super-symmetric quantum field theory might provide totally new mathematical insights.

1.2.4 Is it really necessary to replace groups $GL(n, F)$ with their adelic counterparts?

If the group of invertible adeles is not needed or allowed then a definite deviation from Langlands program is implied. It would seem that multiplicative adeles (ideles) are not favored by TGD view about the role of p-adic number fields. The l-adic representations of p-adic Galois groups corresponding to single p-adic prime l emerge however naturally in TGD framework.

a) The 2×2 Clifford algebra could be easily replaced with its adelic version. A generalization of Clifford algebra would be in question and very much analogous to $GL(2, A)$ in fact. The interpretation would be that real numbers are replaced with adeles also at the level of imbedding space and space-time. This interpretation does not conform with the TGD based view about the relationship between real and p-adic degrees of freedom. The physical picture is that H is 8-D but has different kind of local topologies and that spinors are in some sense universal and independent of number field.

b) Configuration space spinors define a hyper-finite factor of type II_1. It is not clear if this interpretation continues to make sense if configuration space spinors (fermionic Fock space) are replaced with adelic spinors. Note that this generalization would require the replacement of the group algebra of S_{infty} with its adelic counterpart.

2 Basic concepts and ideas related to the number theoretic Langlands program

The basic ideas of Langlands program are following.

a) $Gal(\overline{Q}/Q)$ is a poorly understood concept. The idea is to define this group via its representations and construct representations in terms of group $GL(2, A)$ and more generally $GL(n, A)$, where A refers to adeles. Also representations in any reductive group can be considered. The so called automorphic representations of these groups have a close relationship to the modular forms [14], which inspires the conjecture that n-dimensional representations of $Gal(\overline{Q}/Q)$ are in 1-1 correspondence with automorphic representations of $GL(n, A)$.

b) This correspondence predicts that the invariants characterizing the n-dimensional representations of $Gal(\overline{Q}/Q)$ *resp.* $GL(n, A)$ should correspond to each other. The invariants at Galois sides are the eigenvalues of Frobenius conjugacy classes Fr_p in $Gal(\overline{Q}/Q)$. The non-trivial implication is that in the case of l-adic representations the latter must be algebraic numbers. The ground states of the representations of $Gl(n, R)$ are in turn eigen states of so called Hecke operators $H_{p,k}$, $k = 1, .., n$ acting in group algebra of $Gl(n, R)$. The eigenvalues of Hecke operators for the ground states of representations must correspond to the eigenvalues of Frobenius elements if Langlands correspondence holds true.

c) The characterization of the K-valued representations of reductive groups in terms of Weil group W_F associated with the algebraic extension K/F allows to characterize the representations in terms of homomorphisms of Weil group to the Langlands dual $G_L(F)$ of $G(F)$.

2.1 Correspondence between n-dimensional representations of $Gal(\overline{F}/F)$ and representations of $GL(n, A_F)$ in the space of functions in $GL(n,F)\backslash GL(n,A_F)$

The starting point is that the maximal abelian subgroup $Gal(Q^{ab}/Q)$ of the Galois group of algebraic closure of rationals is isomorphic to the infinite product $\hat{Z} = \prod_p Z_p^{\times}$, where $Z_p^{\times}$ consists of invertible p-adic integers [7].

By introducing the ring of adeles one can transform this result to a slightly different form. Adeles are defined as collections $((f_p)_{p\in P}, f_\infty)$, P denotes primes, $f_p \in Q_p$, and $f_\infty \in R$, such that $f_p \in Z_p$ for all p for all but finitely many primes p. It is easy to convince oneself that one has $A_Q = (\hat{Z} \otimes_Z Q) \times R$ and $Q^{\times}\backslash A_Q = \hat{Z} \times (R/Z)$. The basic statement of abelian class field theory is that abelian Galois group is isomorphic to the group of connected components of $F^{\times}\backslash A_F^{\times}$.

This statement can be transformed to the following suggestive statement:

1) *1-dimensional representations of $Gal(\overline{F}/F)$ correspond to representations of $GL(1, A_F)$ in the space of function defined in $GL(1,F)\backslash GL(1,A_F)$.*

The basic conjecture of Langlands was that this generalizes to n-dimensional representations of $Gal(\overline{F}/F)$.

2) *The n-dimensional representations of $Gal(\overline{F}/F)$ correspond to representations of $GL(n, A_F)$ in the space of functions defined in $GL(n,F)\backslash GL(n,A_F)$.*

This relation has become known as Langlands correspondence.

It is interesting to relate this approach to that discussed in this chapter.

a) In TGD framework adeles do not seem natural although p-adic number fields and l-adic representations have a natural place also here. The new view about numbers is of course an essentially new element allowing geometric interpretation.

b) The irreducible representations of $Gal(\overline{F}, F)$ are assumed to reduce to those for its finite subgroup G. If $Gal(\overline{F}, F)$ is identifiable as S_∞, finite dimensional representations cannot correspond to ordinary unitary representations since, by argument to be represented later, their dimension is of order order $n \rightarrow \infty$ at least. Finite Galois groups can be however interpreted as a sub-group of outer automorphisms defining a sub-factor of $Gal(\overline{Q}, Q)$ interpreted as HFF. Outer automorphisms result at the limit $n \rightarrow \infty$ from a diagonal imbedding of finite Galois group to its n^{th} Cartesian power acting as automorphisms in S_∞. At the limit $n \rightarrow \infty$ the imbedding does not define inner automorphisms anymore. Physicist would interpret the situation as a spontaneous symmetry breaking.

c) These representations have a natural extension to representations of $Gl(n, F)$ and of general reductive groups if also realized as point-wise symmetries of sub-factors of HFF. Continuous groups correspond to outer automorphisms of group algebra of S_∞ not inducible from outer automorphisms of S_{infty}. That finite Galois groups and Lie groups act in the same representation space should provide completely new insights to the understanding of Langlands correspondence.

d) The l-adic representations of $Gal(\overline{Q}/Q)$ could however change the situation. The representations of finite permutation groups in R and in p-adic number fields $p < n$ are more complex and actually not well-understood [29]. In the case of elliptic curves [7] (say $y^2 = x^3 + ax + b$, a, b rational numbers with $4a^3 + 27b^2 \neq 0$) so called first etale cohomology group is Q_l^2 and thus 2-dimensional and it is possible to have 2-dimensional representations $Gal(\overline{Q}/Q) \rightarrow GL(2, Q_l)$. More generally, l-adic representations σ of of $Gal(\overline{F}/F) \rightarrow GL(n, \overline{Q_l})$ is assumed to satisfy the condition that there exists a finite extension $E \subset \overline{Q}_l$ such that σ factors through a homomorphism to $GL(n, E)$.

Assuming $Gal(\overline{Q}/Q) = S_\infty$, one can ask whether l-adic or adelic representations and the representations defined by outer automorphisms of subfactors might be two alternative manners to state the same thing.

2.1.1 Frobenius automorphism

Frobenius automorphism is one of the basic notions in Langlands correspondence. Consider a field extension K/F and a prime ideal v of F (or prime p in case of ordinary integers). v decomposes into a product of prime ideals of K: $v = \prod w_k$ if v is unramified and power of this if not. Consider unramified case and pick one w_k and call it simply w. Frobenius automorphisms Fr_v is by definition the generator of the the Galois group $Gal(K/w, F/v)$, which reduces to Z/nZ for some n.

Since the decomposition group $D_w \subset Gal(K/F)$ by definition maps the ideal w to itself and preserves F point-wise, the elements of D_w act like the elements of $Gal(O_K/w, O_F/v)$ (O_X denotes integers of X). Therefore there exists a natural homomorphism $D_w : Gal(K/F) \rightarrow Gal(O_K/w, O_F/v)$ ($= Z/nZ$ for some n). If the inertia group I_w identified as the kernel of the homomorphism is trivial then the Frobenius automorphism Fr_v, which by definition generates $Gal(O_K/w, O_F/v)$, can be regarded as an element of D_w and $Gal(K/F)$. Only the conjugacy class of this element is fixed since any w_k can be chosen. The significance of the result is that the eigenvalues of

Fr_p define invariants characterizing the representations of $Gal(K/F)$. The notion of Frobenius element can be generalized also to the case of $Gal(\overline{Q}/Q)$ [7]. The representations can be also l-adic being defined in $GL(n, E_l)$ where E_l is extension of Q_l. In this case the eigenvalues must be algebraic numbers so that they make sense as complex numbers.

Two examples discussed in [7] help to make the notion more concrete.

a) For the extensions of finite fields $F = G(p, 1)$ Frobenius automorphism corresponds to $x \rightarrow x^p$ leaving elements of F invariant.

b) All extensions of Q having abelian Galois group correspond to so called cyclotomic extensions defined by polynomials $P_N(x) = x^N + 1$. They have Galois group $(Z/NZ)^{\times}$ consisting of integers $k < n$ which do not divide n and the degree of extension is $\phi(N) = |Z/NZ^{\times}|$, where $\phi(n)$ is Euler function counting the integers $n < N$ which do not divide N. Prime p is unramified only if it does not divide n so that the number of "bad primes" is finite. The Frobenius equivalence class Fr_p in $Gal(K/F)$ acts as raising to p^{th} power so that the Fr_p corresponds to integer $p \ mod \ n$.

2.1.2 Automorphic representations and automorphic functions

In the following I want to demonstrate that I have at least tried to do my home lessons by trying to reproduce the description of [7] for the route from automorphic adelic representations of $GL(2, R)$ to automorphic functions defined in upper half-plane.

1. Characterization of the representation

The representations of $GL(2, Q)$ are constructed in the space of smooth bounded functions $GL(2, Q)\backslash GL(2, A) \rightarrow C$ or equivalently in the space of $GL(2, Q)$ left-invariant functions in $GL(2, A)$. A denotes adeles and $GL(2, A)$ acts as right translations in this space. The argument generalizes to arbitrary number field F and its algebraic closure $\overline{F}$.

a) Automorphic representations are characterized by a choice of compact subgroup K of $GL(2, A)$. The motivating idea is the central role of double coset decompositions $G = K_1 A K_2$, where K_i are compact subgroups and A denotes the space of double cosets $K_1 g K_2$ in general representation theory. In the recent case the compact group $K_2 \equiv K$ is expressible as a product $K = \prod_p K_p \times O_2$. For each unramified prime p one has $K_p = GL(2, Z_p)$. For ramified primes K_p consists of $SL(2, Z_p)$ matrices with $c \in p^{n_p} Z_p$. Here p^{n_p} is the divisor of conductor N corresponding to p. K-finiteness condition states that the right action of K on f generates a finite-dimensional vector space.

b) The representation functions are eigen functions of the Casimir operator C of $gl(2,R)$ with eigenvalue ρ so that irreducible representations of $gl(2,R)$ are obtained. An explicit representation of Casimir operator is given by

$$C = \frac{X_0^2}{4} + X_+X - +X_-X_+ \ ,$$

where one has

$$X_0 \left(\begin{array}{cc} 0 & i \\ -i & 0 \end{array} \right) \ , \left(\begin{array}{cc} 1 & \mp i \\ \mp i & -1 \end{array} \right) \ .$$

d) The center $A^\times$ of $GL(2,A)$ consists of $A^\times$ multiples of identity matrix and it is assumed $f(gz) = \chi(z)f(g)$, where $\chi : A^\times \rightarrow C$ is a character providing a multiplicative representation of $A^\times$.

e) Also the so called cuspidality condition

$$\int_{Q\backslash NA} f(\left(\begin{array}{cc} 1 & u \\ 0 & 1 \end{array} \right) g)du = 0$$

is satisfied [7]. Note that the integration measure is adelic. Note that the transformations appearing in integrand are an adelic generalization of the 1-parameter subgroup of Lorentz transformations leaving invariant light-like vector. The condition implies that the modular functions defined by the representation vanish at cusps at the boundaries of fundamental domains representing copies $H_u/\Gamma_0(N)$ where N is the conductor. The "basic" cusp corresponds to $\tau = i\infty$ for the "basic" copy of the fundamental domain.

The groups $gl(2,R)$, $O(2)$ and $GL(2,Q_p)$ act non-trivially in these representations and it can be shown that a direct sum of irreps of $GL(2,A_F) \times gl(2,R)$ results with each irrep occurring only once. These representations are known as cuspidal automorphic representations.

2. From adeles to $\Gamma_0(N)\backslash SL(2,R)$

The path from adeles to the modular forms in upper half plane involves many twists.

a) By so called central approximation theorem the group $GL(2,Q)\backslash GL(2,A)/K$ is isomorphic to the group $\Gamma_0(N)\backslash GL_+(2,R)$, where N is conductor [7]. The group $\Gamma_0(N) \subset SL(2,Z)$ consists of matrices

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \ , \ c \ mod \ N = 0.$$

$_+$ refers to positive determinant. Note that $\Gamma_0(N)$ contains as a subgroup congruence subgroup $\Gamma_0(N)$ consisting of matrices, which are unit matrices modulo N. Congruence subgroup is a normal subgroup of $SL(2,Z)$ so that also $SL(2,Z)/\Gamma(N)$ is group. Physically $\Gamma(N)$ would be rather interesting alternative for $\Gamma_0(N)$ as a compact subgroup and the replacement $K_p = \Gamma_0(p^{k_p}) \rightarrow \Gamma(p^{k_p})$ of p-adic groups adelic decomposition is expected to guarantee this.

b) Central character condition together with assumptions about the action of K implies that the smooth functions in the original space are completely determined by their restrictions to $\Gamma_0(N)\backslash SL(2,R)$ so that one gets rid of the adeles.

3. From $\Gamma_0(N)\backslash SL(2,R)$ to upper half-plane $H_u = SL(2,R)/SO(2)$

The representations of $(gl(2,C), O(2))$ come in four categories corresponding to principal series, discrete series, the limits of discrete series, and finite-dimensional representations [7]. For the discrete series representation π giving square integrable representation in $SL(2,R)$ one has $\rho = k(k-1)/4$, where $k > 1$ is integer. As sl_2 module, π_∞ is direct sum of irreducible Verma modules with highest weight $-k$ and lowest weight k. The former module is generated by a unique, up to a scalar, highest weight vector v_∞ such that

$$X_0 v_\infty = -k v_\infty \ \ , \ \ X_+ v_\infty = 0 \ \ .$$

The latter module is in turn generated by the lowest weight vector

$$\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) v_\infty \ \ .$$

This means that entire module is generated from the ground state v_∞, and one can focus to the function ϕ_π on $\Gamma_0(N)\backslash SL(2,R)$ corresponding to this vector. The goal is to assign to this function $SO(2)$ invariant function defined in the upper half-plane $H_u = SL(2,R)/SO(2)$, whose points can be parameterized by the numbers $\tau = (a+bi)/(c+di)$ determined by $SL(2,R)$ elements. The function $f_\pi(g) = \phi_\pi(g)(ci+d)^k$ indeed is SO(2) invariant since the phase $exp(ik\phi)$ resulting in $SO(2)$ rotation by ϕ is compensated by the phase resulting from $(ci+d)$ factor. This function is not anymore $\Gamma_0(N)$ invariant but transforms as

$$f_\pi((a\tau+b)/(c\tau+d)) = (c\tau+d)^k f_\pi(\tau)$$

under the action of $\Gamma_0(N)$ The highest weight condition $X_+ v_\infty$ implies that f is holomorphic function of τ. Such functions are known as modular forms of weight k and level N. It would seem that the replacement of $\Gamma_0(N)$ suggested by physical arguments would only replace $H_u/\Gamma_0(N)$ with $H_u/\Gamma(N)$.

f_π can be expanded as power series in the variable $q = exp(2\pi\tau)$ to give

$$f_\pi(q) \ = \ \sum_{n=0}^{\infty} a_n q^n \ . \tag{1}$$

Cuspidality condition means that f_π vanishes at the cusps of the fundamental domain of the action of $\Gamma_0(N)$ on H_u. In particular, it vanishes at $q = 0$ which which corresponds to $\tau = -\infty$. This implies $a_0 = 0$. This function contains all information about automorphic representation.

2.1.3 Hecke operators

Spherical Hecke algebra (which must be distinguished from non-commutative Hecke algebra associated with braids) can be defined as algebra of $GL(2, Z_p)$ bi-invariant functions on $GL(2, Q_p)$ with respect to convolution product. This algebra is isomorphic to the polynomial algebra in two generators $H_{1,p}$ and $H_{2,p}$ and the ground states v_p of automorphic representations are eigenstates of these operators. The normalizations can be chosen so that the second eigenvalue equals to unity. Second eigenvalue must be an algebraic number. The eigenvalues of Hecke operators $H_{p,1}$ correspond to the coefficients a_p of the q-expansion of automorphic function f_π so that f_π is completely determined once these coefficients carrying number theoretic information are known [7].

The action of Hecke operators induces an action on the modular function in the upper half-plane so that Hecke operators have also representation as what is known as classical Hecke operators. The existence of this representation suggests that adelic representations might not be absolutely necessary for the realization of Langlands program.

2.2 Some remarks about the representations of $Gl(n)$ and of more general reductive groups

The simplest representations of $Gl(n, R)$ have the property that the Borel group B of upper diagonal matrices is mapped to diagonal matrices consisting of character ξ which decomposes to a product of characters χ_k associated with diagonal elements b_k of B defining homomorphism

$$b_k \rightarrow sgn(b)^{m(k)}|b_k|^{ia_k}$$

to unit circle if a_k is real. Also more general, non-unitary, characters can be allowed. The representation itself satisfies the condition $f(bg) = \chi(b)f(g)$. Thus n complex parameters a_k defining a reducible representation of $C^\times$ characterize the irreducible representation.

In the case of $GL(2, R)$ one can consider also genuinely two-dimensional discrete series representations characterized by only single continuous parameter and the previous example represented just this case. These representations are square integrable in the subgroup $SL(2, R)$. Their origin is related to the fact that the algebraic closure of R is 2-dimensional. The so called Weil group W_R which is semi-direct product of complex conjugation operation with $C^\times$ codes for this number theoretically. The 2-dimensional representations correspond to irreducible 2-dimensional representations of W_R in terms of diagonal matrices of $Gl(2, C)$.

In the case of $GL(n, R)$ the representation is characterized by integers n_k: $\sum n_k = n$ characterizing the dimensions $n_k = 1, 2$ of the representations of W_R. For $Gl(n, C)$ one has $n_k = 1$ since Weil group W_C is obviously trivial in this case.

In the case of a general reductive Lie group G the homomorphisms of W_R to the Langlands dual G_L of G defined by replacing the roots of the root lattice with their duals characterize the automorphic representations of G.

The notion of Weil group allows also to understand the general structure of the representations of $GL(n, F)$ in $GL(n, K)$, where F is p-adic number field and K its extension. In this case Weil group is a semi-direct product of Galois group of $Gal(K/F)$ and multiplicative group $K^\times$. A very rich structure results since an infinite number of extensions exists and the dimensions of discrete series representations.

The deep property of the characterization of representations in terms of Weyl group is functoriality. If one knows the homomorphisms $W_F \to G$ and $G \to H$ then the composite homomorphism defines an automorphic representation of H. This means that irreps of G can be passed to those of H by homomorphism [5].

3 TGD inspired view about Langlands program

In this section a general TGD inspired vision about Langlands program is described. The fusion of real and various p-adic physics based on the generalization of the number concept, the notion of number theoretic braid, hyper-finite-factors of type II_1 and their sub-factors, and the notion of infinite prime, lead to a new view about how to represent finite Galois groups and how to unify the number theoretic and geometric Langlands programs.

3.1 What is the Galois group of algebraic closure of rationals?

Galois group is essentially the permutation group for the roots of an irreducible polynomial. It is a a subgroup of symmetric group S_n, where n is the degree of polynomial. One can also imagine the notion of Galois group $Gal(\overline{Q}/Q)$ for the algebraic closure of rationals but the concretization of this notion is not easy.

3.1.1 $Gal(\overline{Q}/Q)$ as infinite permutation group?

The maximal abelian subroup of $Gal(\overline{Q}/Q)$, which is obtained by dividing with the normal subgroup of even permutations, is identifiable as a product of multiplicative groups $Z_p^\times$ of invertible p-adic integers $n = n_0 + pZ$, $n_0 \in \{1, ..p-1\}$ for all p-adic primes and can be understood reasonably via its isomorphism to the product $\hat{Z} = \prod_p Z_p$ of multiplicative groups Z_p of invertible p-adic integers, one factor for each prime p [6, 7, 5].

Adeles [15] are identified as the subring of $(\hat{Z} \otimes_Z Q) \times R$ containing only elements for which the elements of Q_p belong to Z_p except for a finite number of primes so that the number obtained can be always represented as a product of element of $\hat{Z}$ and point of circle R/Z: $A = \hat{Z} \times R/Z$. Adeles define a multiplicative group $A^\times$ of ideles and $GL(1, A)$ allow to construct representations $Gal(Q^{ab}/Q)$.

It is much more difficult to get grasp on $Gal(\overline{Q}/Q)$. The basic idea of Langlands program is that one should try to understand $Gal(\overline{Q}/Q)$ through its representations rather than directly. The natural hope is that n-dimensional representations of $Gal(\overline{Q}/Q)$ could be realized in $GL(n, A)$.

1. $Gal(\overline{Q}/Q)$ as infinite symmetric group?

One could however be stubborn and try a different approach based on the direct identification $Gal(\overline{Q}/Q)$. The naive idea is that $Gal(\overline{Q}/Q)$ could in some sense be the Galois group of a polynomial of infinite degree. Of course, for mathematical reasons also a rational function defined as a ratio of this kind of polynomials could be considered so that the Galois group could be assigned to both zeros and poles of this function. In the generic case this group would be an infinite symmetric group S_∞ for an infinite number of objects containing only permutations for subsets containing a finite number of objects. This group could be seen as the first guess for $Gal(\overline{Q}/Q)$.

S_∞ can be defined by generators e_m representing permutation of $m^{th}$and $(m+1)^{th}$ object satisfying the conditions

$$\begin{aligned} e_m e_m &= e_n e_m \text{ for } |m-n|>1 , \\ e_n e_{n+1} e_n &= e_n e_{n+1} e_n e_{n+1} \text{ for } n=1,...,n-2 \ , \\ e_n^2 &= 1 \ . \end{aligned} \quad (2)$$

By the definition S_∞ can be expected to possess the basic properties of finite-dimensional permutation groups. Conjugacy classes, and thus also irreducible unitary representations, should be in one-one correspondence with partitions of n objects at the limit $n \rightarrow \infty$. Group algebra defined by complex functions in S_∞ gives rise to the unitary complex number based representations and the smallest dimensions of the irreducible representations are of order n and are thus infinite for S_∞. For representations based on real and p-adic number based variants of group algebra situation is not so simple but it is not clear whether finite dimensional representations are possible.

S_n and obviously also S_∞ allows an endless number of realizations since it can act as permutations of all kinds of objects. Factors of a Cartesian and tensor power are the most obvious possibilities for the objects in question. For instance, S_n allows a representation as elements of rotation group $SO(n)$ permuting orthonormalized unit vectors e_i with components $(e_i)^k = \delta_i^k$. This induces also a realization as spinor rotations in spinor space of dimension $D = 2^{d/2}$.

2. Group algebra of S_∞ as HFF

The highly non-trivial fact that the group algebra of S_∞ is hyper-finite factor of type II_1 (HFF) [33] suggests a representation of permutations as permutations of tensor factors of HFF interpreted as an infinite power of finite-dimensional Clifford algebra. The minimal choice for the finite-dimensional Clifford algebra is $M^2(C)$. In fermionic Fock space representation of infinite-dimensional Clifford algebra e_i would induce the transformation $(b^\dagger_{m,i}, b^\dagger_{m,i+1}) \rightarrow (b^\dagger_{m,i+1}, b^\dagger_{m,i})$. If the index m is lacking, the representation would reduce to the exchange of fermions and representation would be abelian.

3. Projective representations of S_∞ as representations of braid group B_∞

S_n can be extended to braid group B_n by giving up the condition $e_i^2 = 1$ for the generating permutations of the symmetric group. Generating permutations are represented now as homotopies exchanging the neighboring strands of braid so that repeated exchange of neighboring strands induces a sequence of twists by π. Projective representations of S_∞ could be interpreted as representations of B_∞. Note that odd and even generators commute mutually and for unitary representations either of them can be diagonalized and are represented as phases $exp(i\phi)$ for braid group. If $exp(i\phi)$ is not a root of unity this gives effectively a polynomial algebra and the polynomials subalgebras of these phases might provide representations for the Hecke operators also forming commutative polynomial algebras.

The additional flexibility brought in by braiding would transform Galois group to a group analogous to homotopy group and could provide a connection with knot and link theory [21, 22] and topological quantum field theories in general [20]. Finite quantum Galois groups would generate braidings and a connection with the geometric Langlands program where Galois groups are replaced with homotopy groups becomes suggestive [7, 8].

4. What does one mean with S_∞?

There is also the question about the meaning of S_∞. The hierarchy of infinite primes suggests that there is an entire infinity of infinities in number theoretical sense. After all, any group can be formally regarded as a permutation group. A possible interpretation would be in terms of algebraic closure of rationals and algebraic closures for an infinite hierarchy of polynomials to which infinite primes can be mapped. The question concerns the interpretation of these higher Galois groups and HFFs. Could one regard these as local variants of S_∞ and does this hierarchy give all algebraic groups, in particular algebraic subgroups of Lie groups, as Galois groups so that almost all of group theory would reduce to number theory even at this level?

3.1.2 The group algebra of Galois group of algebraic closure of rationals as hyper-finite factor of type II$_1$

The most natural framework for constructing unitary irreducible representations of Galois group is its group algebra. In the recent case this group algebra would be that for S_∞ or B_∞ if braids are allowed. What puts bells ringing is that the group algebra of S_∞ is a hyper-finite factor of type II$_1$ isomorphic as a von Neumann algebra to the infinite-dimensional Clifford algebra [33], which in turn is the basic structures of quantum TGD whose localized version might imply entire quantum TGD. The very close relationship with the braid group makes it obvious that same holds true for corresponding braid group B_∞. Indeed, the group algebra of an infinite discrete group defines under very general conditions HFF. One of these conditions is so called amenability [31]. This correspondence gives hopes of understanding the Langlands correspondence between representations of discrete Galois groups and the representations of $GL(n, F)$ (more generally representations of reductive groups).

Thus it seems that configuration space spinors (fermionic Fock space) could naturally define a finite-dimensional spinor representation of finite-dimensional Galois groups associated with the number theoretical braids. Inclusions $\mathcal{N} \subset \mathcal{M}$ of hyper-finite factors realize the notion of finite measurement resolution and give rise to finite dimensional representations of finite groups G leaving elements of $\mathcal{N}$ invariant. An attractive idea is that these groups are identifiable as Galois groups.

The identification of the action of G on $\mathcal{M}$ as homomorphism $G \rightarrow Aut(\mathcal{M})$ poses strong conditions on it. This is discussed in the thesis of Jones [38] which introduces three algebraic invariants for the actions of finite group in hyperfinite-factors of type II_1, denoted by $\mathcal{M}$ in the sequel. In general the action reduces to inner automorphism of $\mathcal{M}$ for some normal subgroup $H \subset G$: this group is one of the three invariants of G action. In general one has projective representation for H so that one has $u_{h_1}u_{h_2} = \mu(h_1, h_2)u_{h_1h_2}$, where $\mu(h_1)$ is a phase factor which satisfies cocyle conditions coming from associativity.

a) The simplest action is just a unitary group representation for which $g \in G$ is mapped to a unitary operator u_g in $\mathcal{M}$ acting in $\mathcal{M}$ via adjoint action $m \rightarrow u_g m u_g^\dagger = Ad(u_g)m$. In this case one has $H = G$. In this case the fixed point algebra does not however define a factor and there is no natural reduction of the representations of $Gal(\overline{Q}/Q)$ to a finite subgroup.

b) The exact opposite of this situation outer action of G mean $H = \{e\}$. All these actions are conjugate to each other. This gives gives rise to two kinds of sub-factors and two kinds of representations of G. Both actions of Galois group could be realized either in the group or braid algebra of $Gal(\overline{Q}/Q)$ or in infinite dimensional Clifford algebra. In neither case the action be inner automorphic action $u \rightarrow gug^\dagger$ as one might have naively expected. This is crucial for circumventing the difficulty caused by the fact that $Gal(\overline{Q}/Q)$ identified as S_∞ allows no finite-dimensional complex representation.

c) The first sub-factor is $\mathcal{M}^G \subset \mathcal{M}$ corresponding, where the action of G on $\mathcal{M}$ is outer. Outer action defines a fixed point algebra for all finite groups G. For $D = \mathcal{M} : \mathcal{N} < 4$ only finite subgroups $G \subset SU(2)$ would be represented in this manner. The index identifiable as the fractal dimension of quantum Clifford algebra having $\mathcal{N}$ as non-abelian coefficients is $D = 4cos^2(\pi/n)$. One can speak about quantal representation of Galois group. The image of Galois group would be a finite subgroup of $SU(2)$ acting as spinor rotations of quantum Clifford algebra (and quantum spinors) regarded as a module with respect to the included algebra invariant under inner automorphisms. These representations would naturally correspond to 2-dimensional representations having very special role for the simple reason that the algebraic closure of reals is 2-dimensional.

d) Second sub-factor is isomorphic to $\mathcal{M}^G \subset (\mathcal{M} \otimes L(H))^G$. Here $L(H)$ is the space of linear operators acting in a finite-dimensional representation space H of a unitary irreducible representation of G. The action of G is a tensor product of outer action and adjoint action. The index of the inclusion is $dim(H)^2 \geq 1$ [39] so that the representation of Galois group can be said to be classical (non-fractal).

e) The obvious question is whether and in what sense the outer automorphisms represent Galois subgroups. According to [38] the automorphisms belong to the completion of the group of inner automorphisms of HFF. Identifying HFF as group algebra of S_∞, the interpretation would be that outer automorphisms are obtained as diagonal embeddings of Galois group to $S_n \times S_n \times ....$ If one includes only a finite number of these factors the outcome is an inner automorphisms so that for all finite approximations inner automorphisms are in question. At the limit one obtains an automorphisms which does not belong to S_∞ since it contains only finite permutations. This identification is consistent with the identification of the outer automorphisms as diagonal embedding of G to an infinite tensor power of sub-Clifford algebra of Cl_∞.

This picture is physically very appealing since it means that the ordering of the strands of braid does not matter in this picture. Also the reduction of the braid to a finite number theoretical braid at space-time level could be interpreted in terms of the periodicity at quantum level. From the point of view of physicist this symmetry breaking would be analogous to a spontaneous symmetry breaking above some length scale L. The cutoff length scale L would correspond to the number N of braids to which finite Galois group G acts and corresponds also to some p-adic length scale.

One might hope that the emergence of finite groups in the inclusions of hyper-finite factors could throw light into the mysterious looking finding that the representations of finite Galois groups and unitary infinite-dimensional automorphic representations of $GL(n, R)$ are correlated by the connection between the eigenvalues of Frobenius element Fr_p on Galois side and eigenvalues of commuting Hecke operators on automorphic side. The challenge would be to show that the action of Fr_p as outer automorphism of group algebra of S_∞ or B_∞ corresponds to Hecke algebra action on configuration space spinor fields or in modular degrees of freedom associated with partonic 2-surface.

3.1.3 Could there exist a universal rational function having $Gal(\overline{Q}/Q)$ as the Galois group of its zeros/poles?

The reader who is not fascinated by the rather speculative idea about a universal rational function having $Gal(\overline{Q}/Q)$ as a permutation group of its zeros and poles can safely skip this subsection since it will not be needed anywhere else in this chapter.

a) Taking the idea about permutation group of roots of a polynomial of infinite order seriously, one could require that the analytic function defining the Galois group should behave like a polynomial or a rational function with rational coefficients in the sense that the function should have an everywhere converging expansion in terms of products over an infinite number of factors $z - z_i$ corresponding to the zeros of the numerator and possible denominator of a rational function. The roots z_i would define an extension of rationals giving rise to the entire algebraic closure of rationals. This is a tall order and the function in question should be number theoretically very special.

b) One can speculate even further. TGD has inspired the conjecture that the non-trivial zeros $s_n = 1/2 + iy_n$ of Riemann zeta [10] (assuming Riemann hypothesis) are algebraic numbers and that also the numbers p^{s_n}, where p is any prime, and thus local zeta functions serving as multiplicative building blocks of ζ have the same property [E8]. The story would be perfect if these algebraic numbers would span the algebraic closure of rationals.

The symmetrized version of Riemann zeta defined as $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ satisfying the functional equation $\xi(s) = \xi(1-s)$ and having only the trivial zeros could appear as a building block of the rational function in question. The function

$$f(s) = \frac{\xi(s)}{\xi(s+1)} \times \frac{s-1}{s}$$

has non-trivial zeros s_n of ζ as zeros and their negatives as $-s_n$ as poles. There are no other zeros since trivial zeros as well as the zeros at $s = 0$ and $s = 1$ are eliminated. Using Stirling formula one finds that $\xi(s)$ grows as s^s for real values of $s \to \infty$. The growths of the numerator and denominator compensate each other at this limit so that the function approaches constant equal to one for $Re(s) \rightarrow \infty$.

If $f(s)$ indeed behaves as a rational function whose product expansion converges everywhere it can be expressed in terms of its zeros and poles as

$$\begin{aligned} f(s) &= \prod_{n>0} A_n(s) \ , \\ A_n &= \frac{(s-s_n)(s-\overline{s}_n)}{(1+s-s_n)(1+s-\overline{s}_n)} \ . \end{aligned} \tag{3}$$

The product expansion seems to converge for any finite value of s since the terms A_n approach unity for large values of $|s_n| = |1/2 + iy_n|$. $f(s)$ has $s_n = 1/2 + iy_n$ indeed has zeros and $s_n = -1/2 + iy_n$ as poles.

c) This proposal might of course be quite too simplistic. For instance, one might argue that the phase factors p^{iy} associated with the non-trivial zeros give only roots of unity multiplied by Gaussian integers. One can however imagine more complex functions obtained by forming products of $f(s)$ with its shifted variants $f(s+\Delta)$ with algebraic shift Δ in, say, the interval $[-1/2, 1/2]$. Some kind of limiting procedure using a product of this kind of functions might give the desired universal function.

3.2 Physical representations of Galois groups

It would be highly desirable to have concrete physical realizations for the action of finite Galois groups. TGD indeed provides two kinds of realizations of this kind. For both options there are good hopes about the unification of number theoretical and geometric Galois programs obtained by replacing permutations with braiding homotopies and by discretization of continuous situation to a finite number theoretic braids having finite Galois groups as automorphisms.

3.2.1 Number theoretical braids and the representations of finite Galois groups as outer automorphisms of braid group algebra

Number theoretical braids [E1, C1, C2] are in a central role in the formulation of quantum TGD based on general philosophical ideas which might apply to both physics and mathematical cognition and, one might hope, also to a good mathematics.

An attractive idea inspired by the notion of the number theoretical braid is that the symmetric group S_n might act on roots of a polynomial represented by the strands of braid and could thus be replaced by braid group.

The basic philosophy underlying quantum TGD is the notion of finite resolution, both the finite resolution of quantum measurement and finite cognitive resolution [C1, C2]. The basic implication is discretization at space-time level and finite-dimensionality of all mathematical structures which can be represented in the physical world. At space-time level the discretization means that the data involved with the definition of S-matrix comes from a subset of a discrete set of points in the intersection of real and p-adic variants of partonic 2-surface obeying same algebraic equations. Note that a finite number of braids could be enough to code for the information needed to reconstruct the entire partonic 2-surface if it is given by polynomial or rational function having coefficients as algebraic numbers. Entire configuration space of 3-surfaces would be discretized in this picture. Also the reduction of the infinite braid to a finite one would conform with the spontaneous symmetry breaking S_∞ to diagonally imbedded finite Galois group imbedded diagonally.

1. Two objections

Langlands correspondence assumes the existence of finite-dimensional representations of $Gal(\overline{Q}/Q)$. In the recent situation this encourages the idea that the restrictions of mathematical cognition allow to realize only the representations of $Gal(\overline{Q}/Q)$ reducing in some sense to representations for finite Galois groups. There are two counter arguments against the idea.

a) It is good to start from a simple abelian situation. The abelianization of $G(\overline{A}/Q)$ must give rise to multiplicative group of adeles defined as $\hat{Z} = \prod_p Z_p^\times$ where $Z_p^\times$ corresponds to the multiplicative group of invertible p-adic integers consisting of p-adic integers having p-adic norm equal to one. This group results as the inverse limit containing the information about subgroup inclusion hierarchies resulting as sequences $Z^\times/(1+pZ)^\times \subset Z^\times/(1+p^2Z)^\times \subset ..$ and expressed in terms factor groups of multiplicative group of invertible p-adic integers. Z_∞/A_∞ must give the group $\prod_p Z_p^\times$ as maximal abelian subgroup of Galois group. All smaller abelian subgroups of S_∞ would correspond to the products of subgroups of $\hat{Z}^\times$ coming as $Z_p^\times/(1+p^nZ)^\times$. Representations of finite cyclic Galois groups would be obtained by representing trivially the product of a commutator group with a subgroup of $\hat{Z}$. Thus one would obtain finite subgroups of the maximal abelian Galois group at the level of representations as effective Galois groups. The representations would be of course one-dimensional.

One might hope that the representations of finite Galois groups could result by a reduction of the representations of S_∞ to $G = S_\infty/H$ where H is normal subgroup of S_∞. Schreier-Ulam theorem [27] however implies that the only normal subgroup of S_∞ is the alternating subgroup A_∞. Since the braid group B_∞ as a special case reduces to S_∞ there is no hope of obtaining finite-dimensional representations except abelian ones.

b) The identification of $Gal(\overline{Q}/Q) = S_\infty$ is not consistent with the finite-dimensionality in the case of complex representations. The irreducible unitary representations of S_n are in one-one correspondence with partitions of n objects. The direct numerical inspection based on the formula for the dimension of the irreducible representation of S_n in terms of Yang tableau [28] suggests that the partitions for which the number r of summands differs from $r = 1$ or $r = n$ (1-dimensional representations) quite generally have dimensions which are at least of order n. If d-dimensional representations corresponds to representations in $GL(d, C)$, this means that important representations correspond to dimensions $d \to \infty$ for S_∞.

Both these arguments would suggest that Langlands program is consistent with the identification $Gal(\overline{F}, F) = S_\infty$ only if the representations of $Gal(\overline{Q}, Q)$ reduce to those for finite Galois subgroups via some kind of symmetry breaking.

2. Diagonal imbedding of finite Galois group to S_∞ as a solution of problems

The idea is to imbed the Galois group acting as inner automorphisms diagonally to the m-fold Cartesian power of S_n imbedded to S_∞. The limit $m \to \infty$ gives rise to outer automorphic action since the resulting group would not be contained in S_∞. Physicist might prefer to speak about number theoretic symmetry breaking $Gal(\overline{Q}/Q) \to G$ implying that the representations are irreducible only in finite Galois subgroups of $Gal(\overline{Q}/Q)$. The action of finite Galois group G is indeed analogous to that of global gauge transformation group which belongs to the completion of the group of local gauge transformations. Note that G is necessarily finite.

3.2.2 Representation of finite Galois groups as outer automorphism groups of HFFs

Any finite group G has a representation as outer automorphisms of a hyperfinite factor of type II_1 (briefly HFF in the sequel) and this automorphism defines sub-factor $\mathcal{N} \subset \mathcal{M}$ with a finite value of index $\mathcal{M} : \mathcal{N}$ [32]. Hence a promising idea is that finite Galois groups act as outer automorphisms of the associated hyper-finite factor of type II_1.

More precisely, sub-factors (containing Jones inclusions as a special case) $\mathcal{N} \subset \mathcal{M}$ are characterized by finite groups G acting on elements of $\mathcal{M}$ as outer automorphisms and leave the elements of $\mathcal{N}$ invariant whereas finite Galois group associated with the field extension K/L act as automorphisms of K and leave elements of L invariant. For finite groups the action as outer automorphisms is unique apart from a conjugation in von Neumann algebra. Hence the natural idea is that the finite subgroups of $Gal(\overline{Q}/Q)$ have outer automorphism action in group algebra of $Gal(\overline{Q}/Q)$ and that the hierarchies of inclusions provide a representation for the hierarchies of algebraic extensions. Amusingly, the notion of Jones inclusion was originally inspired by the analogy with field extensions [32]!

It must be emphasized that the groups defining sub-factors can be extremely general and can represent much more than number theoretical information understood in the narrow sense of the word. Even if one requires that the inclusion is determined by outer automorphism action of group G uniquely, one finds that any amenable, in particular compact [31], group defines a unique sub-factor by outer action [32]. It seems that practically any group works if uniqueness condition is given up.

The TGD inspired physical interpretation is that compact groups would serve as effective gauge groups defining measurement resolution by determining the measured quantum numbers. Hence the physical states differing by the action of $\mathcal{N}$ elements which are G singlets would not be indistinguishable from each other in the resolution used. The physical states would transform according to the finite-dimensional representations in the resolution defined by G.

The possibility of Lie groups as groups defining inclusions raises the question whether hyper-finite factors of type II_1 could mimic any gauge theory and one might think of interpreting gauge groups as Galois groups of the algebraic structure of this kind of theories. Also Kac-Moody algebras emerge naturally in this framework as will be discussed, and could also have an interpretation as Galois algebras for number theoretical dynamical systems obeying dynamics dictated by conformal field theory. The infinite hierarchy of infinite rationals in turn suggests a hierarchy of groups S_∞ so that even algebraic variants of Lie groups could be interpreted as Galois groups. These arguments would suggest that HFFs might be kind of Universal Math Machines able to mimic any respectable mathematical structure.

3.2.3 Number theoretic braids and unification of geometric and number theoretic Langlands programs

The notion of number theoretic braid has become central in the attempts to fuse real physics and p-adic physics to single coherent whole. Number theoretic braid leads to the discretization of quantum physics by replacing the stringy amplitudes defined over curves of partonic 2-surface with amplitudes involving only data coded by points of number theoretic braid. The discretization of quantum physics could have counterpart at the level of geometric Langlands program [7, 16], whose discrete version would correspond to number theoretic Galois groups associated with the points of number theoretic braid. The extension to braid group would mean that the global homotopic information is not lost.

1. Number theoretic braids belong to the intersection of real and p-adic partonic surface

The points of number theoretic braid belong to the intersection of the real and p-adic variant of partonic 2-surface consisting of rationals and algebraic points in the extension used for p-adic numbers. The points of braid have same projection on an algebraic point of the geodesic sphere of $S^2 \subset CP_2$ belonging to the algebraic extension of rationals considered (the reader willing to understand the details can consult [C1]).

The points of braid are obtained as solutions of polynomial equation and thus one can assign to them a Galois group permuting the points of the braid. In this case finite Galois group could be realized as left or right translation or conjugation in S_∞ or in braid group.

To make the notion of number theoretic braid more concrete, suppose that the complex coordinate w of $\delta M^4_\pm$ is expressible as a polynomial of the complex coordinate z of CP_2 geodesic sphere and the radial light-like coordinate r of $\delta M^4_\pm$ is obtained as a solution of polynomial equation $P(r, z, w) = 0$. By substituting w as a polynomial $w = Q(z, r)$ of z and r this gives polynomial equation $P(r, z, Q(z, r)) = 0$ for r for a given value of z. Only real roots can be accepted. Local Galois group (in a sense different as it is used normally in literature) associated with the algebraic point of S^2 defining the number theoretical braid is thus well defined.

If the partonic 2-surface involves all roots of an irreducible polynomial, one indeed obtains a braid for each point of the geodesic sphere $S^2 \subset CP_2$. In this case the action of Galois group is naturally a braid group action realized as the action on induced spinor fields and configuration space spinors.

The choice of the points of braid as points common to the real and p-adic partonic 2-surfaces would be unique so that the obstacle created by the fact that the finite Galois group as function of point of S^2 fluctuates wildly (when some roots become rational Galois group changes dramatically: the simplest example is provided by $y - x^2 = 0$ for which Galois group is Z_2 when y is not a square of rational and trivial group if y is rational).

2. Modified Dirac operator assigns to partonic 2-surface a unique prime p which could define l-adic representations of Galois group

The overall scaling of the eigenvalue spectrum of the modified Dirac operator assigns to the partonic surface a unique p-adic prime p which physically corresponds to the p-adic length scale which appears in the discrete coupling constant evolution [C1, C4]. One can solve the roots of the the resulting polynomial also in the p-adic number field associated with the partonic 2-surface by the modified Dirac equation and find the Galois group of the extension involved. The p-adic Galois group, known as local Galois group in literature, could be assigned to the p-adic variant of partonic surface and would have naturally l-adic representation, most naturally in the p-adic variant of the group algebra of S_∞ or B_∞ or equivalently in the p-adic variant of infinite-dimensional Clifford algebra. There are however physical reasons to believe that infinite-dimensional Clifford algebra does not depend on number field. Restriction to an algebraic number based group algebra therefore suggests itself. Hence, if one requires that the representations involve only algebraic numbers, these representation spaces might be regarded as equivalent.

3. Problems

There are however problems.

a) The triviality of the action of Galois group on the entire partonic 2-surface seems to destroy the hopes about genuine representations of Galois group.

b) For a given partonic 2-surface there are several number theoretic braids since there are several algebraic points of geodesic sphere S^2 at which braids are projected. What happens if the Galois groups are different? What Galois group should one choose?

A possible solution to both problems is to assign to each braid its own piece X_k^2 of the partonic 2-surface X^2 such that the deformations X^2 can be non-trivial only in X_k^2. This means separation of modular degrees of freedom to those assignable to X_k^2 and to "center of mass" modular degrees of freedom assignable to the boundaries between X_k^2. Only the piece X_k^2 associated with the k^{th} braid would be affected non-trivially by the Galois group of braid.

The modular invariance of the conformal field theory however requires that the entire quantum state is modular invariant under the modular group of X^2. The analog of color confinement would take place in modular degrees of freedom. Note that the region containing braid must contain single handle at least in order to allow representations of $SL(2,C)$ (or $Sp(2g,Z)$ for genus g).

As already explained, in the general case only the invariance under the subgroup $\Gamma_0(N)$ [14] of the modular group $SL(2,Z)$ can be assumed for automorphic representations of $GL(2,R)$ [9, 7, 5]. This is due to the fact that there is a finite set of primes (prime ideals in the algebra of integers), which are ramified [9]. Ramification means that their decomposition to a product of prime ideals of the algebraic extension of Q contains higher powers of these prime ideals: $p \rightarrow (\prod_k P_k)^e$ with $e > 1$. The congruence group is fixed by the integer $N = \prod_k p^{n_k}$ known as conductor coding the set of exceptional primes which are ramified.

The construction of modular forms in terms of representations of $SL(2,R)$ suggests that it is possible to replace $\Gamma_0(N)$ by the congruence subgroup $\Gamma(N)$, which is normal subgroup of $SL(2,R)$ so that $G_1 = SL(2,Z)/\Gamma$ is group. This would allow to assign to individual braid regions carrying single handle well-defined G_1 quantum numbers in such a manner that entire state would be G_1 singlet.

Physically this means that the separate regions of the partonic 2-surface each containing one braid strand cannot correspond to quantum states with full modular invariance. Elementary particle vacuum functionals [F1] defined in the moduli space of conformal equivalence classes of partonic 2-surface must however be modular invariant, and the analog of color confinement in modular degrees of freedom would take place.

3.2.4 Hierarchy of Planck constants and dark matter and generalization of imbedding space

Second hierarchy of candidates for Galois groups is based on the generalization of the notion of the imbedding space $H = M^4 \times CP_2$, or rather the spaces $H_\pm = M^4_\pm \times CP_2$ defining future and past light-cones inside H [A9]. This generalization is inspired by the quantization of Planck constant explaining dark matter as a hierarchy of macroscopically quantum coherent phases and by the requirement that sub-factors have a geometric representation at the level of the imbedding space and space-time (quantum-classical correspondence).

Galois groups could also correspond to finite groups $G_a \times G_b \subset SU(2) \times$

$SU(2) \subset$SL(2,C)$\times SU(3)$. These groups act as covering symmetries for the sectors of the imbedding space, which can be regarded as singular $H_{\pm} = M^4_{\pm} \times CP_2 \rightarrow H_{\pm}/G_a \times G_b$ bundles containing orbifold points (fixed points of $G_a \times G_b$ or either of them. The copies of H with same G_a or G_b are glued together along $M^4_{\pm}$ or CP_2 factor and along common orbifold points left fixed by G_b or G_a. The group $G_a \times G_b$ plays both the role of both Galois group and homotopy group.

There are good reasons to expect that both these Galois groups and those associated with number theoretic braids play a profound role in quantum TGD based description of dark matter as macroscopically quantum coherent phases. For instance, G_a would appear as symmetry group of dark matter part of bio-molecules in TGD inspired biology [3].

3.3 What could be the TGD counterpart for the automorphic representations?

Configuration space spinor fields would certainly transform according to a finite-dimensional and thus non-unitary representation of $SL(2,C)$ which is certainly the most natural group and should relate to the fact that Galois groups representable as subgroups of SU(2) acting as rotations of 3-dimensional space correspond to sub-factors with $\mathcal{M} : \mathcal{N} \leq 4$.

Also larger Lie groups can be considered and diagonal imbeddings of Galois groups would be naturally accompanied by diagonal imbeddings of compact and also non-compact groups acting on the decomposition of infinite-dimensional Clifford algebra Cl_∞ to an infinite tensor power of finite-dimensional sub-Clifford algebra of form $M(2,C)^n$. The basic difference between Galois group representation and corresponding Lie group representations is that the automorphisms in case of discrete groups are automorphisms of S_∞ or B_∞ whereas for Lie groups the automorphisms are in general automorphisms of group algebra of S_∞ or B_∞. This could allow to understand the correspondence between discrete groups and Lie groups naturally. Unitary automorphic representations are infinite-dimensional and require group algebra of $GL(n,F)$. Therefore configuration space spinors cannot realize them. TGD suggests two realizations of automorphic group representations.

3.3.1 Could Lorentz group realize automorphic representations?

There is obvious analogy with spinor fields in Minkowski space and with the unitary representations of Poincare group which correspond to finite-dimensional irreps of Lorentz group. One could indeed consider the possibility that Lorentz group acting on partonic 2-surfaces at light-cone boundary $\delta M^4_\pm$ could allow the automorphic representations.

In this case full modular invariance is not forced by any obvious physical reason. The group $\Gamma_0(N)$ would correspond to discrete Lorentz transformations, which leave invariant a light-like vector modulo N. At the limit $N \to \infty$ $\Gamma_0(N)$ would reduce to a sub-group of the little group of light-like vector consisting of upper diagonal integer valued matrices. For $N = 1$ full modular invariance would be predicted. $\Gamma_0(N)$ invariance would mean that wave functions for cm degrees of freedom of partonic 2-surface at light-cone boundary would be $\Gamma_0(N)$ invariant. Non-Euclidian lattice like symmetry of wave functions would be implied in cosmological length scale.

TGD based model for dark matter as a macroscopic quantum phase characterized by Planck constant which can have arbitrarily large value indeed predicts quantum coherence in astroscopic length scales [A9]. The original inspiration for the model came from the empirical findings suggesting that the orbits of planets and exoplanets seem to have quantized radii predicted by Bohr model [D6].

3.3.2 What about modular degrees of freedom?

A good guess is that modular degrees of freedom associated with the partonic 2-surface or with the regions X^2_k it decomposes and containing single strand of number theoretical braid could provide the degrees of freedom needed to have unitary representations of $SL(2,R)$ or more general non-compact groups. In the case of $SL(2,C)$ representations a reduction to representations in the space obtained by dividing upper plane H_u by $\Gamma_0(N)$ are obtained. $H_u/\Gamma_0(N = 1)$ corresponds to the space conformal equivalence classes of torus. $SL(2,C)$ acts in upper plane in a natural manner but not in $H_u/\Gamma_0(N = 1)$.

Physically and the space $H_u/\Gamma(N)$ looks more natural than $H_u/\Gamma_0(N)$.

a) The fact $\Gamma(N) \subset \Gamma_0(N)$ allows to consider the possibility that $\Gamma(N)$ might act as invariance group of automorphic representations. This would affect only the fundamental domain of upper plane in the case of modular forms.

b) $\Gamma(N)$ is a normal subgroup of the modular group so that also $G = SL(2,Z)/\Gamma(N)$ would be group and one could regard automorphic representations also as representation of this group. The invariance under $SL(2,Z)$ for the entire state consisting of several number theoretical braids would reduce to the requirement that the overall state is $SL(2,Z)/\Gamma(N)$ singlet.

3.3.3 Connection with elementary particle physics?

There might be a connection with elementary particle vacuum functionals discussed [F1].

a) The genus of the partonic 2-surface labels fermion generations and only 3 generations have been observed. The TGD based explanation relies on the observation that 2-surfaces having $g \leq 2$ are always hyper-elliptic and the fact that elementary particle vacuum functionals for $g \geq 3$ vanish for hyper-elliptic surfaces. This could decouple $g \leq 2$ and $g > 2$ worlds by making decays of the latter to the first ones very rare.

b) It could be that in $g > 2$ handles do not form $Sl(2, Z)$ invariant bound states but decompose to "many-particle" state formed from 2-handle bound states. This could mean that $g > 2$ partonic 2-surface decays rapidly to $g \leq 2$ partonic 2-surfaces. This could be due to a decomposition of the parton system to regions each of them containing a pair of handles forming a meson like bound state which is $SL(2, Z)$ singlet with the analog of color confinement occurring in $SL(2, Z)/\Gamma(N)$ degrees of freedom.

c) $g = 2$ elementary particle vacuum functionals might perhaps be regarded as bound state resulting from $g = 1$ elementary particle vacuum functionals for tori with holes glued together along hole boundary and containing number theoretic braid. The moduli space of $g = 2$ Riemann surfaces would be needed and $Sp(2, Z)$ would extend to $Sp(2g = 4, Z)$ as the group of modular symmetries for the bound states and having interpretation as cm vacuum functional assignable to the variable boundaries between braid regions X^2_k. At the level of configuration space spinors the tensor product of two $M(2, C)$ factors associated with M^4 degrees of freedom would define the basic unit and define a spinorial representation of $SL(4, C)$. For $g > 2$ the state would decompose into a product of meson like 2-handle bound states. $SL(2, Z)/\Gamma(N)$ singlets of three handles might not even exist.

d) Also the mere $\Gamma(N)$ modular invariance might force formation of meson like bound states of handles. According to [7] the discrete representations of $GL(2, R)$ decompose to direct sum of $k = 1$ and $k = -1$ representations of $SO(2)$ subgroup. These correspond to modular forms with opposite weights k. $\Gamma_0(N)$ invariance of the entire state expressed in terms of a product of modular forms would require even number of handles and meson like pair of $g = 1$ braids would be the simplest manner to achieve this symmetry. The transformation property under Γ_N is the same as for $\Gamma_0(N)$ for the modular forms f_π so that this argument could generalize.

3.4 Super-conformal invariance, modular invariance, and Langlands program

The geometric Langlands program [7, 8] deals with function fields, in particular the field of complex rational analytic functions on 2-dimensional surfaces. The sheaves in the moduli spaces of conformal blocks characterizing the n-point functions of conformal field theory replaces automorphic functions coding both arithmetic data and characterizing the modular representations of $GL(n)$ in number theoretic Langlands program [7]. These moduli spaces are labelled both by moduli characterizing the conformal equivalence class of 2-surface, in particular the positions of punctures, in TGD framework the positions of strands of number theoretic braids, as well as the moduli related to the Kac-Moody group involved.

3.4.1 Transition to function fields in TGD framework

According to [7] conformal field theories provide a very promising framework for understanding geometric Langlands correspondence.

a) That the function fields on 2-D complex surfaces would be in a completely unique role mathematically fits nicely with the 2-dimensionality of partons and well-defined stringy character of anticommutation relations for induced spinor fields. According to [7] there are not even conjectures about higher dimensional function fields.

b) There are very direct connections between hyper-finite factors of type II_1 and topological QFTs [21, 20], and conformal field theories. For instance, according to the review article [32] Ocneacnu has show that Jones inclusions correspond in one-one manner to topological quantum field theories and TGD can indeed be regarded as almost topological quantum field theory (metric is brought in by the light-likeness of partonic 3-surfaces). Furthermore, Connes has shown that the decomposition of the hierarchies of tensor powers $\mathcal{M} \otimes_{\mathcal{N}} .... \otimes_{\mathcal{N}} M$ as left and right modules to representations of lower tensor powers directly to fusion rules expressible in terms of 4-point functions of conformal field theories [32].

In TGD framework the transition from number fields to function fields would not be very dramatic.

a) Suppose that the representations of $SL(n, R)$ occurring in number theoretic Langlands program can indeed be realized in the moduli space for conformal equivalence classes of partonic 2-surface (or, by previous arguments, moduli space for regions of them with fixed boundaries). This means that representations of local Galois groups associated with number theoretic braids would involve global data about entire partonic 2-surface. This is physically very important since it otherwise discretization would lead to a loss of the information about dimension of partonic 2-surfaces.

b) In the case of geometric Langlands program this moduli space would be extended to the moduli space for n-point functions of conformal field theory defined at these 2-surfaces containing the original moduli space as a subspace. Of course, the extension could be present also in the number theoretic case. Thus it seems that number theoretic and geometric Langlands programs would utilize basic structures and would differ only in the sense that single braid would be replaced by several braids in the geometric case.

c) In TGD Kac-Moody algebras would be also present as well as the so called super-canonical algebra [C1] related to the isometries of "the world of classical worlds" (the space of light-like 3-surfaces) with generators transforming according to the irreducible representations of rotation group $SO(3)$ and color group $SU(3)$. It must be emphasized that TGD view about conformal symmetry generalizes that of string models since light-like 3-surfaces (orbits of partons) are the basic dynamical objects [C1].

3.4.2 What about more general reductive groups?

Langlands correspondence is conjectured to apply to all reductive Lie groups. The question is whether there is room for them in TGD Universe. There are good hopes.

1. Pairs formed by finite Galois groups and Lie groups containing them and defining sub-factors

Any amenable (in particular compact Lie) group acting as outer automorphism of $\mathcal{M}$ defines a unique sub-factor $\mathcal{N} \subset \mathcal{M}$ as a group leaving the elements of $\mathcal{N}$ invariant. The representations of discrete subgroups of compact groups extended to representations of the latter would define natural candidates for Langlands correspondence and would expand the repertoire of the Galois groups representable in terms of unique factors. If one gives up the uniqueness condition for the sub-factor, one can expect that almost any Lie group can define a sub-factor.

2. McKay correspondences and Langlands correspondence

The so called McKay correspondence assigns to the finite subgroups of SU(2) extended Dynkin diagrams of ADE type Kac-Moody algebras. McKay correspondence also generalizes to the discrete subgroups of other compact Lie groups [36]. The obvious question is how closely this correspondence between finite groups and Lie groups relates with Langlands correspondence.

The principal graphs representing concisely the fusion rules for Connes tensor products of $\mathcal{M}$ regarded as $\mathcal{N}$ bi-module are represented by the Dynkin diagrams of ADE type Lie groups for $\mathcal{M} : \mathcal{N} < 4$ (not all of them appear). For index $\mathcal{M} : \mathcal{N} = 4$ extended ADE type Dynkin diagrams labelling Kac-Moody algebras are assigned with these representations.

I have proposed that TGD Universe is able to emulate almost any ADE type gauge theory and conformal field theory involving ADE type Kac-Moody symmetry and represented somewhat misty ideas about how to construct representations of ADE type gauge groups and Kac-Moody groups using many particle states at the sheets of multiple coverings $H \rightarrow H/G_a \times G_b$ realizing the idea about hierarchy of dark matters already mentioned. Also vertex operator construction also distinguishes ADE type Kac-Moody algebras in a special position.

It is possible to considerably refine this conjecture picture by starting from the observation that the set of generating elements for Lie algebra corresponds to a union of triplets $\{J_i^{\pm}, J_i^3\}$, $i = 1, ..., n$ generating SU(2) sub-algebras. Here n is the dimension of the Cartan sub-algebra. The non-commutativity of quantum Clifford algebra suggests that Connes tensor product can induce deformations of algebraic structures so that ADE Lie algebra could result as a kind of deformation of a direct sum of commuting SU(2) Lie (Kac-Moody) algebras associated with a Connes tensor product. The physical interpretation might in terms of a formation of a bound state. The finite depth of $\mathcal{N}$ would mean that this mechanism leads to ADE Lie algebra for an n-fold tensor power, which then becomes a repetitive structure in tensor powers. The repetitive structure would conform with the diagonal imbedding of Galois groups giving rise to a representation in terms of outer automorphisms.

This picture encourages the guess that it is possible to represent the action of Galois groups on number theoretic braids as action of subgroups of dynamically generated ADE type groups on configuration space spinors. The connection between the representations of finite groups and reductive Lie groups would result from the natural extension of the representations of finite groups to those of Lie groups.

*3. What about Langlands correspondence for Kac-Moody groups?*vm

The appearance of also Kac-Moody algebras raises the question whether Langlands correspondence could generalize also to the level of Kac-Moody groups or algebras and whether it could be easier to understand the Langlands correspondence for function fields in terms of Kac-Moody groups as the transition from global to local occurring in both cases suggests.

3.4.3 Could Langlands duality for groups reduce to super-symmetry?

Langlands program involves dualities and the general structure of TGD suggests that there is a wide spectrum of these dualities.

a) A very fundamental duality would be between infinite-dimensional Clifford algebra and group algebra of S_∞ or of braid group B_∞. For instance, one can ask could it be possible to map this group algebra to the union of the moduli spaces of conformal equivalence classes of partonic 2-surfaces. HFFs consists of bounded operators of a separable Hilbert space. Therefore they are expected to have very many avatars: for instance there is an infinite number sub-factors isomorphic to the factor. This seems to mean infinite number of manners to represent Galois groups reflected as dualities.

b) Langlands program involves the duality between reducible Lie groups G and its Langlands dual having dual root lattices. The interpretation for this duality in terms of electric-magnetic duality is suggested by Witten [16]. TGD suggests an alternative interpretation. The super symmetry aspect of super-conformal symmetry suggests that bosonic and fermionic representations of Galois groups could be very closely related. In particular, the representations in terms of configuration space spinors and in terms of modular degrees of freedom of partonic 2-surface could be in some sense dual to each other. Rotation groups have a natural action on configuration space spinors whereas symplectic groups have a natural action in the moduli spaces of partonic 2-surfaces of given genus possessing symplectic and Kähler structure. Langlands correspondence indeed relates $SO(2g+1,R)$ realized as rotations of configuration space spinors and $Sp(2g,C)$ realized as transformations in modular degrees of freedom. Hence one might indeed wonder whether super-symmetry could be behind the Langlands correspondence.

3.5 What is the role of infinite primes?

Infinite primes primes at the lowest level of the hierarchy can be represented as polynomials and as rational functions at higher levels. These in turn define rational function fields. Physical states correspond in general to infinite rationals which reduce to unit in real sense but have arbitrarily complex number theoretical anatomy [E3, 1, 4].

3.5.1 Does infinite prime characterize the l-adic representation of Galois group associated with given partonic 2-surface

Consider first the lowest level of hierarchy of infinite primes [E3]. Infinite primes at the lowest level of hierarchy are in a well-defined sense composites of finite primes and correspond to states of super-symmetric arithmetic quantum field theory. The physical interpretation of primes appearing as composites of infinite prime is as characterizing of the p-adic prime p assigned by the modified Dirac action to partonic 2-surfaces associated with a given 3-surface [A6, C1].

This p-adic prime could naturally correspond to the possible prime associated with so called l-adic representations of the Galois group(s) associated with the p-adic counterpart of the partonic 2-surface. Also the Galois groups associated with the real partonic 2-surface could be represented in this manner. The generalization of moduli space of conformal equivalence classes must be generalized to its p-adic variant. I have proposed this generalization in context of p-adic mass calculations [F1].

It should be possible to identify configuration space spinors associated with real and p-adic sectors if anti-commutations relations for the fermionic oscillator operators make sense in any number field (that is involve only rational or algebraic numbers). Physically this seems to be the only sensible option.

3.5.2 Could one assign Galois groups to the extensions of infinite rationals?

A natural question is whether one could generalize the intuitions from finite number theory to the level of infinite primes, integers, and rationals and construct Galois groups and there representations for them. This might allow alternative very number theoretical approach to the geometric Langlands duality.

a) The notion of infinite prime suggests that there is entire hierarchy of infinite permutation groups such that the N_∞ at given level is defined as the product of all infinite integers at that level. Any group is a permutation group in formal sense. Could this mean that the hierarchy of infinite primes could allow to interpret the infinite algebraic sub-groups of Lie groups as Galois groups? If so one would have a unification of group theory and number theory.

b) An interesting question concerns the interpretation of the counterpart of hyper-finite factors of type II_1 at the the higher levels of hierarchy of infinite primes. Could they relate to a hierarchy of local algebras defined by HFF? Could these local algebras be interpreted in terms of direct integrals of HFFs so that nothing essentially new would result from von Neumann algebra point of view? Would this be a correlate for the fact that finite primes would be the irreducible building block of all infinite primes at the higher levels of the hierarchy?

c) The transition from number fields to function fields is very much analogous to the replacement of group with a local gauge group or algebra with local algebra. I have proposed that this kind of local variant based on multiplication by of HFF by hyper-octonion algebra could be the fundamental algebraic structure from which quantum TGD emerges. The connection with infinite primes would suggest that there is infinite hierarchy of localizations corresponding to the hierarchy of space-time sheets.

e) Perhaps it is worth of mentioning that the order of S_∞ is formally $N_\infty = \lim_{n\to\infty} n!$. This integer is very large in real sense but zero in p-adic sense for all primes. Interestingly, the numbers $N_\infty/n+n$ behave like normal integers in p-adic sense and also number theoretically whereas the numbers $N_\infty/n+1$ behave as primes for all values of n. Could this have some deeper meaning?

3.5.3 Could infinite rationals allow representations of Galois groups?

One can also ask whether infinite primes could provide representations for Galois groups. For instance, the decomposition of infinite prime to primes (or prime ideals) assignable to the extension of rationals is expected to make sense and would have clear physical interpretation. Also (hyper-)quaternionic and (hyper-)octonionic primes can be considered and I have proposed explicit number theoretic interpretation of the symmetries of standard model in terms of these primes. The decomposition of partonic primes to hyper-octonionic primes could relate to the decomposition of parton to regions, one for each number theoretic braid.

There are arguments supporting the view that infinite primes label the ground states of super-conformal representations [C1, E3]. The question is whether infinite primes could allow to realize the action of Galois groups. Rationality of infinite primes would imply that the invariance of ground states of super-conformal representations under the braid realization of $Gal(\overline{Q}/Q)$ of finite Galois groups. The infinite prime as a whole could indeed be invariant but the primes in the decomposition to a product of primes in algebraic extension of rationals need not be so. This kind of decompositions of infinite prime characterizing parton could correspond to the above described decomposition of partonic 2-surface to regions X_k^2 at which Galois groups act non-trivially. It could also be that only infinite integers are rational whereas the infinite primes decomposing them are hyper-octonionic. This would physically correspond to the decomposition of color singlet hadron to colored partons [E3].

3.6 Could Langlands correspondence, McKay correspondence and Jones inclusions relate to each other?

The understanding of Langlands correspondence for general reductive Lie groups in TGD framework seems to require some physical mechanism allowing the emergence of these groups in TGD based physics. The physical idea would be that quantum dynamics of TGD is able to emulate the dynamics of any gauge theory or even stringy dynamics of conformal field theory having Kac-Moody type symmetry and that this emulation relies on quantum deformations induced by finite measurement resolution described in terms of Jones inclusions of sub-factors characterized by group G leaving elements of sub-factor invariant. Finite measurement resolution would would result simply from the fact that only quantum numbers defined by the Cartan algebra of G are measured.

There are good reasons to expect that infinite Clifford algebra has the capacity needed to realize representations of an arbitrary Lie group. It is indeed known that that any quantum group characterized by quantum parameter which is root of unity or positive real number can be assigned to Jones inclusion [32]. For $q = 1$ this would gives ordinary Lie groups. In fact, all amenable groups define unique sub-factor and compact Lie groups are amenable ones.

It was so called McKay correspondence [36] which originally stimulated the idea about TGD as an analog of Universal Turing machine able to mimic both ADE type gauge theories and theories with ADE type Kac-Moody symmetry algebra. This correspondence and its generalization might also provide understanding about how general reductive groups emerge. In the following I try to cheat the reader to believe that the tensor product of representations of SU(2) Lie algebras for Connes tensor powers of $\mathcal{M}$ could induce ADE type Lie algebras as quantum deformations for the direct sum of n copies of $SU(2)$ algebras This argument generalizes also to the case of other compact Lie groups.

3.6.1 About McKay correspondence

McKay correspondence [36] relates discrete finite subgroups of $SU(2)$ ADE groups. A simple description of the correspondences is as follows [36].

a) Consider the irreps of a discrete subgroup $G \subset SU(2)$ which correspond to irreps of G and can be obtained by restricting irreducible representations of $SU(2)$ to those of G. The irreducible representations of $SU(2)$ define the nodes of the graph.

b) Define the lines of graph by forming a tensor product of any of the representations appearing in the diagram with a doublet representation which is always present unless the subgroup is 2-element group. The tensor product regarded as that for $SU(2)$ representations gives representations $j-1/2$, and $j+1/2$ which one can decompose to irreducibles of G so that a branching of the graph can occur. Only branching to two branches occurs for subgroups yielding extended ADE diagrams. For the linear portions of the diagram the spins of corresponding SU(2) representations increase linearly as $.., j, j+1/2, j+1, ...$

One obtains extended Dynkin diagrams of ADE series representing also Kac-Moody algebras giving A_n, D_n, E_6, E_7, E_8. Also A_∞ and $A_{-\infty,\infty}$ are obtained in case that subgroups are infinite. The Dynkin diagrams of non-simply laced groups B_n $(SO(2n+1))$, C_n (symplectic group $Sp(2n)$ and quaternionic group $Sp(n)$), and exceptional groups G_2 and F_4 are not obtained.

ADE Dynkin diagrams labelling Lie groups instead of Kac-Moody algebras and having one node less, do not appear in this context but appear in the classification of Jones inclusions for $\mathcal{M} : \mathcal{N} < 4$. As a matter fact, ADE type Dynkin diagrams appear in very many contexts as one can learn from John Baez's This Week's Finds [37].

a) The classification of integral lattices in Rn having a basis of vectors whose length squared equals 2

b) The classification of simply laced semisimple Lie groups.

c) The classification of finite sub-groups of the 3-dimensional rotation group.

d) The classification of simple singularities . In TGD framework these singularities could be assigned to origin for orbifold CP_2/G, $G \subset SU(2)$.

e) The classification of tame quivers.

3.6.2 Principal graphs for Connes tensor powers $\mathcal{M}$

The thought provoking findings are following.

a) The so called principal graphs characterizing $\mathcal{M}:\mathcal{N}=4$ Jones inclusions for $G=SU(2)$ are extended Dynkin diagrams characterizing ADE type affine (Kac-Moody) algebras. D_n is possible only for $n\geq 4$.

b) $\mathcal{M}:\mathcal{N}<4$ Jones inclusions correspond to ordinary ADE type diagrams for a subset of simply laced Lie groups (all roots have same length) A_n $(SU(n))$, D_{2n} $(SO(2n))$, and E_6 and E_8. Thus D_{2n+1} $(SO(2n+2))$ and E_7 are not allowed. For instance, for $G=S_3$ the principal graph is not D_3 Dynkin diagram.

The conceptual background behind principal diagram is necessary if one wants to understand the relationship with McKay correspondence.

a) The hierarchy of higher commutations defines an invariant of Jones inclusion $\mathcal{N}\subset\mathcal{M}$. Denoting by $\mathcal{N}'$ the commutant of $\mathcal{N}$ one has sequences of horizontal inclusions defined as $C=\mathcal{N}'\cap\mathcal{N}\subset\mathcal{N}'\cap\mathcal{M}\subset\mathcal{N}'\cap\mathcal{M}^1\subset ...$ and $C=\mathcal{M}'\cap\mathcal{M}\subset\mathcal{M}'\cap\mathcal{M}^1\subset ....$ There is also a sequence of vertical inclusions $\mathcal{M}'\cap\mathcal{M}^k\subset\mathcal{N}'\cap\mathcal{M}^k$. This hierarchy defines a hierarchy of Temperley-Lieb algebras [34] assignable to a finite hierarchy of braids. The commutants in the hierarchy are direct sums of finite-dimensional matrix algebras (irreducible representations) and the inclusion hierarchy can be described in terms of decomposition of irreps of k^{th} level to irreps of $(k-1)^{th}$ level irreps. These decomposition can be described in terms of Bratteli diagrams [36, 35].

b) The information provided by infinite Bratteli diagram can be coded by a much simpler bi-partite diagram having a preferred vertex. For instance, the number of $2k$-loops starting from it tells the dimension of k^{th} level algebra. This diagram is known as principal graph.

Principal graph emerges also as a concise description of the fusion rules for Connes tensor powers of $\mathcal{M}$.

a) It is natural to decompose the Connes tensor powers [36] $\mathcal{M}_k=\mathcal{M}\otimes_{\mathcal{N}}...\otimes_{\mathcal{N}}\mathcal{M}$ to irreducible $\mathcal{M}-\mathcal{M}$, $\mathcal{N}-\mathcal{M}$, $\mathcal{M}-\mathcal{N}$, or $\mathcal{N}-\mathcal{N}$ bi-modules. If $\mathcal{M}:\mathcal{N}$ is finite this decomposition involves only finite number of terms. The graphical representation of these decompositions gives rise to Bratteli diagram.

b) If $\mathcal{N}$ has finite depth the information provided by Bratteli diagram can be represented in nutshell using principal graph. The edges of this bipartite graph connect $\mathcal{M}-\mathcal{N}$ vertices to vertices describing irreducible $\mathcal{N}-\mathcal{N}$ representations resulting in the decomposition of $\mathcal{M}-\mathcal{N}$ irreducibles. If this graph is finite, $\mathcal{N}$ is said to have finite depth.

3.6.3 A mechanism assigning to tensor powers Jones inclusions ADE type gauge groups and Kac-Moody algebras

The proposal made for the first time in [A9] is that in $\mathcal{M}:\mathcal{N}<4$ case it is possible to construct ADE representations of gauge groups or quantum groups and in $\mathcal{M}:\mathcal{N}=4$ using the additional degeneracy of states implied by the multiple-sheeted cover $H \rightarrow H/G_a \times G_b$ associated with space-time correlates of Jones inclusions. Either G_a or G_b would correspond to G. In the following this mechanism is articulated in a more refined manner by utilizing the general properties of generators of Lie-algebras understood now as a minimal set of elements of algebra from which the entire algebra can be obtained by repeated commutation operator (I have often used " Lie algebra generator" as an synonym for "Lie algebra element"). This set is finite also for Kac-Moody algebras.

1. Two observations

The explanation to be discussed relies on two observations.

a) McKay correspondence for subgroups of G ($\mathcal{M}:\mathcal{N}=4$) *resp.* its variants ($\mathcal{M}:\mathcal{N}<4$) and its counterpart for Jones inclusions means that finite-dimensional irreducible representations of allowed $G \subset SU(2)$ label both the Cartan algebra generators and the Lie (Kac-Moody) algebra generators of t_+ and t_- in the decomposition $g = h \oplus t_+ \oplus t_-$, where h is the Lie algebra of maximal compact subgroup.

b) Second observation is related to the generators of Lie-algebras and their quantum counterparts (see Appendix for the explicit formulas for the generators of various algebras considered). The observation is that each Cartan algebra generator of Lie- and quantum group algebras, corresponds to a triplet of generators defining an SU(2) sub-algebra. The Cartan algebra of affine algebra contains besides Lie group Cartan algebra also a derivation d identifiable as an infinitesimal scaling operator L_0 measuring the conformal weight of the Kac-Moody generators. d is exceptional in that it does not give rise to a triplet. It corresponds to the preferred node added to the Dynkin diagram to get the extended Dynkin diagram.

2. Is ADE algebra generated as a quantum deformation of tensor powers of $SU(2)$ Lie algebras representations?

The ADE type symmetry groups could result as an effect of finite quantum resolution described by inclusions of HFFs in TGD inspired quantum measurement theory.

a) The description of finite resolution typically leads to quantization since complex rays of state space are replaced as $\mathcal{N}$ rays. Hence operators, which would commute for an ideal resolution cease to do so. Therefore the algebra $SU(2) \otimes ... \otimes SU(2)$ characterized by n mutually commuting triplets, where n is the number of copies of $SU(2)$ algebra in the original situation and identifiable as quantum algebra appearing in $\mathcal{M}$ tensor powers with $\mathcal{M}$ interpreted as $\mathcal{N}$ module, could suffer quantum deformation to a simple Lie algebra with $3n$ Cartan algebra generators. Also a deformation to a quantum group could occur as a consequence.

b) This argument makes sense also for discrete groups $G \subset SU(2)$ since the representations of G realized in terms of configuration space spinors extend to the representations of SU(2) naturally.

c) Arbitrarily high tensor powers of $\mathcal{M}$ are possible and one can wonder why only finite-dimensional Lie algebra results. The fact that $\mathcal{N}$ has finite depth as a sub-factor means that the tensor products in tensor powers of $\mathcal{N}$ are representable by a finite Dynkin diagram. Finite depth could thus mean that there is a periodicity involved: the kn tensor powers decomposes to representations of a Lie algebra with $3n$ Cartan algebra generators. Thus the additional requirement would be that the number of tensor powers of $\mathcal{M}$ is multiple of n.

3. Space-time correlate for the tensor powers $\mathcal{M} \otimes_{\mathcal{N}} ... \otimes_{\mathcal{N}} \mathcal{M}$

By quantum classical correspondence there should exist space-time correlate for the formation of tensor powers of $\mathcal{M}$ regarded as $\mathcal{N}$ module. A concrete space-time realization for this kind of situation in TGD would be based on n-fold cyclic covering of H implied by the $H \rightarrow H/G_a \times G_b$ bundle structure in the case of say G_b. The sheets of the cyclic covering would correspond to various factors in the n-fold tensor power of $SU(2)$ and one would obtain a Lie algebra, affine algebra or its quantum counterpart with n Cartan algebra generators in the process naturally. The number n for space-time sheets would be also a space-time correlate for the finite depth of $\mathcal{N}$ as a factor.

Configuration space spinors could provide fermionic representations of $G \subset SU(2)$. The Dynkin diagram characterizing tensor products of representations of $G \subset SU(2)$ with doublet representation suggests that tensor products of doublet representations associated with n sheets of the covering could realize the Dynkin diagram.

Singlet representation in the Dynkin diagram associated with irreps of G would not give rise to an SU(2) sub-algebra in ADE Lie algebra and would correspond to the scaling generator. For ordinary Dynkin diagram representing gauge group algebra scaling operator would be absent and therefore also the exceptional node. Thus the difference between ($\mathcal{M} : \mathcal{N} = 4$) and ($\mathcal{M} : \mathcal{N} < 4$) cases would be that in the Kac-Moody group would reduce to gauge group $\mathcal{M} : \mathcal{N} < 4$ because Kac-Moody central charge k and therefore also Virasoro central charge resulting in Sugawara construction would vanish.

4. Do finite subgroups of $SU(2)$ play some role also in $\mathcal{M} : \mathcal{N} = 4$ case?

One can ask wonder the possible interpretation for the appearance of extended Dynkin diagrams in ($\mathcal{M} : \mathcal{N} = 4$) case. Do finite subgroups $G \subset SU(2)$ associated with extended Dynkin diagrams appear also in this case. The formal analog for $H \rightarrow G_a \times G_b$ bundle structure would be $H \rightarrow H/G_a \times SU(2)$. This would mean that the geodesic sphere of CP_2 would define the fiber. The notion of number theoretic braid meaning a selection of a discrete subset of algebraic points of the geodesic sphere of CP_2 suggests that $SU(2)$ actually reduces to its subgroup G also in this case.

5. Why Kac-Moody central charge can be non-vanishing only for $\mathcal{M} : \mathcal{N} = 4$?

From the physical point of view the vanishing of Kac-Moody central charge for $\mathcal{M} : \mathcal{N} < 4$ is easy to understand. If parton corresponds to a homologically non-trivial geodesic sphere, space-time surface typically represents a string like object so that the generation of Kac-Moody central extension would relate directly to the homological non-triviality of partons. For instance, cosmic strings are string like objects of form $X^2 \times Y^2$, where X^2 is minimal surface of M^2 and Y^2 is a holomorphic sub-manifold of CP_2 reducing to a homologically non-trivial geodesic sphere in the simplest situation. A conjecture that deserves to be shown wrong is that central charge k is proportional/equal to the absolute value of the homology (Kähler magnetic) charge h.

6. More general situation

McKay correspondence generalizes also to the case of subgroups of higher-dimensional Lie groups [36]. The argument above makes sense also for discrete subgroups of more general compact Lie groups H since also they define unique sub-factors. In this case, algebras having Cartan algebra with nk generators, where n is the dimension of Cartan algebra of H, would emerge in the process. Thus there are reasons to believe that TGD could emulate practically any dynamics having gauge group or Kac-Moody type symmetry. An interesting question concerns the interpretation of non-ADE type principal graphs associated with subgroups of SU(2).

3.7 Technical questions related to Hecke algebra and Frobenius element

3.7.1 Frobenius elements

Frobenius element Fr_p is mapped to a conjugacy class of Galois group using the decomposition of prime p to prime ideals in the algebraic extension K/F.

a) At the level of braid group Frobenius element Fr_p corresponds to some conjugacy class of Galois group acting imbedded to S_n (only the conjugacy equivalence class is fixed) and thus can be mapped to an element of the braid group. Hence it seems possible to assign to Fr_p an element of infinitely cyclic subgroup of the braid group.

b) One can always reduce in given representation the element of given conjugacy class to a diagonal matrix so that it is possible to chose the representatives of Fr_p to be commuting operators. These operators would act as a spinor rotation on quantum Clifford algebra elements defined by Jones inclusion and identifiable as element of some cyclic group of the group G defining the sub-factor via the diagonal embedding.

c) Fr_p for a given finite Galois group G should have representation as an element of braid group to which G is imbedded as a subgroup. It is possible to chose the representatives of Fr_p so that they commute. Could one chose them in such a manner that they belong to the commuting subgroup defined by even (odd) generators e_i? The choice of representatives for Fr_p for various Galois groups must be also consistent with the hierarchies of intermediate extensions of rationals associated with given extension and characterized by subgroups of Galois group for the extension.

3.7.2 How the action of commutative Hecke algebra is realized in hyper-finite factor and braid group?

One can also ask how to imbed Hecke algebra to the braid algebra. Hecke algebra for a given value of prime p and group $GL(n,R)$ is a polynomial algebra in Hecke algebra generators. There is a fundamental difference between Hecke algebra and Frobenius element Fr_p in the sense that Fr_p has finite order as an element of finite Galois group whereas Hecke algebra elements do not except possibly for representations. This means that Hecke algebra cannot have a representation in a finite Galois groups.

Situation is different for braid algebra generators since they do not satisfy the condition $e_i^2 = 1$ and odd and even generators of braid algebra commute.

The powers of Hecke algebra generators would correspond to the powers of basic braiding operation identified as a π twist of neighboring strands. For unitary representations eigenvalues of e_i are phase factors. Therefore Hecke algebra might be realized using odd or even commuting sub-algebra of braid algebra and this could allow to deduce the Frobenius-Hecke correspondence directly from the representations of braid group. The basic questions are following.

a) Is it possible to represent Hecke algebra as a subalgebra of braid group algebra in some natural manner? Could the infinite cyclic group generated by braid group image of Fr_p belong represent element of Hecke algebra fixed by the Langlands correspondence? If this were the case then the eigenvalues of Frobenius element Fr_p of Galois group would correspond to the eigen values of Hecke algebra generators in the manner dictated by Langlands correspondence.

b) Hecke operators $H_{p,i}$, $i = 1, .., n$ commute and expressible as two-side cosets in group $GL(n, Q_p)$. This group acts in $\mathcal{M}$ and the action could be made rather explicit by using a proper representations of $\mathcal{M}$ (note however that physical situation can quite well distinguish between various representations). Does the action of the Hecke sub-algebra fixed by Hecke-Frobenius correspondence co-incide with the action of Frobenius element Fr_p identified as an element of braid sub-group associated with some cyclic subgroup of the Galois group identified as a group defining the sub-factor?

4 Appendix

4.1 Hecke algebra and Temperley-Lieb algebra

Braid group is accompanied by several algebras. For Hecke algebra, which is particular case of braid algebra, one has

$$\begin{aligned} e_{n+1}e_ne_{n+1} &= e_ne_{n+1}e_n \ , \\ e_n^2 &= (t-1)e_n + t \ . \end{aligned} \tag{4}$$

The algebra reduces to that for symmetric group for $t = 1$.

Hecke algebra can be regarded as a discrete analog of Kac Moody algebra or loop algebra with G replaced by S_n. This suggests a connection with Kac-Moody algebras and imbedding of Galois groups to Kac-Moody group. $t = p^n$ corresponds to a finite field. Fractal dimension $t = \mathcal{M} : \mathcal{N}$ relates naturally to braid group representations: fractal dimension of quantum quaternions might be appropriate interpretation. t=1 gives symmetric group. Infinite braid group could be seen as a quantum variant of Galois group for algebraic closure of rationals.

b) Temperley-Lieb algebra assignable with Jones inclusions of hyperfinite factors of type II_1 with $\mathcal{M}:\mathcal{N} < 4$ is given by the relations

$$\begin{aligned} e_{n+1}e_n en+1 &= e_{n+1} \\ e_n e_{n+1} e_n &= e_n \ , \\ e_n^2 &= te_n \ , \ \ , t = -\sqrt{\mathcal{M}:\mathcal{N}} = -2cos(\pi/n) \ \ , n = 3,4,... \end{aligned} \quad (5)$$

The conditions involving three generators differ from those for braid group algebra since e_n are now proportional to projection operators. An alternative form of this algebra is given by

$$\begin{aligned} e_{n+1}e_n en+1 &= te_{n+1} \\ e_n e_{n+1} e_n &= te_n \ , \\ e_n^2 &= e_n = e_n^* \ , \ \ , t = -\sqrt{\mathcal{M}:\mathcal{N}} = -2cos(\pi/n) \ \ , n = 3,4, \end{aligned} \quad (6)$$

This representation reduces to that for Temperley-Lieb algebra with obvious normalization of projection operators. These algebras are somewhat analogous to function fields but the value of coordinate is fixed to some particular values. An analogous discretization for function fields corresponds to a formation of number theoretical braids.

4.2 Some examples of bi-algebras and quantum groups

The appendix summarizes briefly the simplest bi- and Hopf algebras and some basic constructions related to quantum groups.

4.2.1 Simplest bi-algebras

Let $k(x_1,..,x_n)$ denote the free algebra of polynomials in variables x_i with coefficients in field k. x_i can be regarded as points of a set. The algebra $Hom(k(x_1,...,x_n),A)$ of algebra homomorphisms $k(x_1,...,x_n) \rightarrow A$ can be identified as A^n since by the homomorphism property the images $f(x_i)$ of the generators $x_1,...x_n$ determined the homomorphism completely. Any commutative algebra A can be identified as the $Hom(k[x],A)$ with a particular homomorphism corresponding to a line in A determined uniquely by an element of A.

The matrix algebra $M(2)$ can be defined as the polynomial algebra $k(a,b,c,d)$. Matrix multiplication can be represented universally as an algebra morphism Δ from from $M_2 = k(a,b,c,d)$ to $M_2^{\otimes 2} = k(a',a'',b',b'',c',c'',d',d'')$ to $k(a,b,c,d)$ in matrix form as

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} .$$

This morphism induces algebra multiplication in the matrix algebra $M_2(A)$ for any commutative algebra A.

$M(2)$, $GL(2)$ and $SL(2)$ provide standard examples about bi-algebras. $SL(2)$ can be defined as a commutative algebra by dividing free polynomial algebra $k(a,b,c,d)$ spanned by the generators a,b,c,d by the ideal $det-1 = ad-bc-1=0$ expressing that the determinant of the matrix is one. In the matrix representation μ and η are defined in obvious manner and μ gives powers of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

Δ, counit ϵ, and antipode S can be written in case of $SL(2)$ as

$$\begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

$$\begin{pmatrix} \epsilon(a) & \epsilon(b) \\ \epsilon(c) & \epsilon(d) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad-bc)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} .$$

Note that matrix representation is only an economical manner to summarize the action of Δ on the generators a,b,c,d of the algebra. For instance, one has $\Delta(a) = a \rightarrow a\otimes a + b\otimes c$. The resulting algebra is both commutative and co-commutative.

$SL(2)_q$ can be defined as a Hopf algebra by dividing the free algebra generated by elements a,b,c,d by the relations

$$\begin{aligned} ba &= qab \ , & db &= qbd \ , \\ ca &= qac \ , & dc &= qcd \ , \\ bc &= cb \ , & ad-da &= (q^{-1}-1)bc \ , \end{aligned}$$

and the relation

$$det_q = ad - q^{-1}bc = 1$$

stating that the quantum determinant of $SL(2)_q$ matrix is one.

$\mu, \eta, \Delta, \epsilon$ are defined as in the case of $SL(2)$. Antipode S is defined by

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = det_q^{-1} \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} \ .$$

The relations above guarantee that it defines quantum inverse of A. For q an n^{th} root of unity, $S^{2n} = id$ holds true which signals that these parameter values are somehow exceptional. This result is completely general.

Given an algebra, the R point of $SL_q(2)$ is defined as a four-tuple (A, B, C, D) in R^4 satisfying the relations defining the point of $SL_q(2)$. One can say that R-points provide representations of the universal quantum algebra $SL_q(2)$.

4.2.2 Quantum group $U_q(sl(2))$

Quantum group $U_q(sl(2))$ or rather, quantum enveloping algebra of $sl(2)$, can be constructed by applying Drinfeld's quantum double construction (to avoid confusion note that the quantum Hopf algebra associated with $SL(2)$ is the quantum analog of a commutative algebra generated by powers of a 2×2 matrix of unit determinant).

The commutation relations of $sl(2)$ read as

$$[X_+, X_-] = H \ \ , \ \ [H, X_\pm] = \pm 2X_\pm \ . \tag{7}$$

$U_q(sl(2))$ allows co-algebra structure given by

$$\begin{aligned} &\Delta(J) = J\otimes 1 + 1\otimes J \ , \quad S(J) = -J \ , \quad \epsilon(J) = 0 \ , \quad J = X_\pm, H \ , \\ &S(1) = 1 \ , \qquad\qquad\qquad \epsilon(1) = 1 \ . \end{aligned} \tag{8}$$

The enveloping algebras of Borel algebras $U(B_\pm)$ generated by $\{1, X_+, H\}$ $\{1, X_-, hH\}$ define the Hopf algebra H and its dual $H^\star$ in Drinfeld's construction. h could be called Planck's constant vanishes at the classical limit. Note that $H^\star$ reduces to $\{1, X_-\}$ at this limit. Quantum deformation parameter q is given by $exp(2h)$. The duality map $\star : H \rightarrow H^\star$ reads as

$$
\begin{aligned}
&a \rightarrow a^\star \ , \quad ab = (ab)^\star = b^\star a^\star \ , \\
&1 \rightarrow 1 \ , \qquad H \rightarrow H^\star = hH \ , \qquad X_+ \rightarrow (X_+)^\star = hX_- \ .
\end{aligned} \tag{9}
$$

The commutation relations of $U_q(sl(2)$ read as

$$
[X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}} \ , \quad [H, X_\pm] = \pm 2X_\pm \ . \tag{10}
$$

Co-product Δ, antipode S, and co-unit ϵ differ from those $U(sl(2))$ only in the case of $X_\pm$:

$$
\begin{aligned}
&\Delta(X_\pm) = X_\pm \otimes q^{H/2} + q^{-H/2} \otimes X_\pm \ , \\
&S(X_\pm) = -q^{\pm 1} X_\pm \ .
\end{aligned} \tag{11}
$$

When q is not a root of unity, the universal R-matrix is given by

$$
R = q^{\frac{H\otimes H}{2}} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_q!} q^{\frac{n(1-n)}{2}} q^{\frac{nH}{2}} X_+^n \otimes q^{-\frac{nH}{2}} X_-^n \ . \tag{12}
$$

When q is m:th root of unity the q-factorial $[n]_q!$ vanishes for $n \geq m$ and the expansion does not make sense.

For q not a root of unity the representation theory of quantum groups is essentially the same as of ordinary groups. When q is m^{th} root of unity, the situation changes. For $l = m = 2n$ n^{th} powers of generators span together with the Casimir operator a sub-algebra commuting with the whole algebra providing additional numbers characterizing the representations. For $l = m = 2n + 1$ same happens for m^{th} powers of Lie-algebra generators. The generic representations are not fully reducible anymore. In the case of $U_q(sl(2))$ irreducibility occurs for spins $n < l$ only. Under certain conditions on q it is possible to decouple the higher representations from the theory. Physically the reduction of the number of representations to a finite number means a symmetry analogous to a gauge symmetry. The phenomenon resembles the occurrence of null vectors in the case of Virasoro and Kac Moody representations and there indeed is a deep connection between quantum groups and Kac-Moody algebras [24].

One can wonder what is the precise relationship between $U_q(sl(2)$ and $SL_q(2)$ which both are quantum groups using loose terminology. The relationship is duality. This means the existence of a morphism $x \rightarrow \Psi(x)$

$M_q(2) \rightarrow U_q^\star$ defined by a bilinear form $\langle u, x\rangle = \Psi(x)(u)$ on $U_q \times M_q(2)$, which is bi-algebra morphism. This means that the conditions

$$\langle uv, x\rangle = \langle u\otimes v, \Delta(x)\rangle \ , \quad \langle u, xy\rangle = \langle \Delta(u), x\otimes y\rangle \ ,$$

$$\langle 1, x\rangle = \epsilon(x) \ , \qquad\qquad \langle u, 1\rangle = \epsilon(u)$$

are satisfied. It is enough to find $\Psi(x)$ for the generators $x = A, B, C, D$ of $M_q(2)$ and show that the duality conditions are satisfied. The representation

$$\rho(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \quad \rho(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \quad \rho(K = q^H) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \ ,$$

extended to a representation

$$\rho(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

of arbitrary element u of $U_q(sl(2)$ defines for elements in $U_q^\star$. It is easy to guess that $A(u), B(u), C(u), D(u)$, which can be regarded as elements of $U_q^\star$, can be regarded also as R points that is images of the generators a, b, c, d of $SL_q(2)$ under an algebra morphism $SL_q(2) \rightarrow U_q^\star$.

4.2.3 General semisimple quantum group

The Drinfeld's construction of quantum groups applies to arbitrary semi-simple Lie algebra and is discussed in detail in [24]. The construction relies on the use of Cartan matrix.

Quite generally, Cartan matrix $A = \{a_{ij}\}$ is $n \times n$ matrix satisfying the following conditions:

i) A is indecomposable, that is does not reduce to a direct sum of matrices.

ii) $a_{ij} \leq 0$ holds true for $i < j$.

iii) $a_{ij} = 0$ is equivalent with $a_{ij} = 0$.

A can be normalized so that the diagonal components satisfy $a_{ii} = 2$.

The generators e_i, f_i, k_i satisfying the commutations relations

$$\begin{array}{cc} k_ik_j = k_jk_i \ , & k_ie_j = q_i^{a_{ij}}e_jk_i \ , \\ k_if_j = q_i^{-a_{ij}}e_jk_i \ , & e_if_j - f_je_i = \delta_{ij}\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \ , \end{array} \tag{13}$$

and so called Serre relations

$$
\begin{array}{l}
\sum_{l=0}^{1-a_{ij}}(-1)^l \left[\begin{array}{c} 1-a_{ij} \\ l \end{array} \right]_{q_i} e_i^{1-a_{ij}-l} e_j e_i^l = 0, \ \ i \neq j \ \ , \\
\sum_{l=0}^{1-a_{ij}}(-1)^l \left[\begin{array}{c} 1-a_{ij} \\ l \end{array} \right]_{q_i} f_i^{1-a_{ij}-l} f_j f_i^l = 0 \ \ , \ \ i \neq j \ \ .
\end{array} \tag{14}
$$

Here $q_i = q^{D_i}$ where one has $D_i a_{ij} = a_{ij} D_i$. $D_i = 1$ is the simplest choice in this case.

Comultiplication is given by

$$
\begin{aligned}
\Delta(k_i) &= k_i \otimes k_i \ \ , & (15)\\
\Delta(e_i) &= e_i \otimes k_i + 1 \otimes e_i \ \ , & (16)\\
\Delta(f_i) &= f_i \otimes 1 + k_i^{-1} \otimes 1 \ \ . & (17)\\
& & (18)
\end{aligned}
$$

The action of antipode S is defined as

$$
S(e_i) = -e_i k_i^{-1} \ \ , \ \ S(f_i) = -k_i f_i \ \ , \ \ S(k_i) = -k_i^{-1} \ \ . \tag{19}
$$

4.2.4 Quantum affine algebras

The construction of Drinfeld and Jimbo generalizes also to the case of untwisted affine Lie algebras, which are in one-one correspondence with semisimple Lie algebras. The representations of quantum deformed affine algebras define corresponding deformations of Kac-Moody algebras. In the following only the basic formulas are summarized and the reader not familiar with the formalism can consult a more detailed treatment can be found in [24].

1. Affine algebras

The Cartan matrix A is said to be of affine type if the conditions $det(A) = 0$ and $a_{ij}a_{ji} \geq 4$ (no summation) hold true. There always exists a diagonal matrix D such that $B = DA$ is symmetric and defines symmetric bilinear degenerate metric on the affine Lie algebra.

The Dynkin diagrams of affine algebra of rank l have $l+1$ vertices (so that Cartan matrix has one null eigenvector). The diagrams of semisimple Lie-algebras are sub-diagrams of affine algebras. From the $(l+1) \times (l+1)$

Cartan matrix of an untwisted affine algebra $\hat{A}$ one can recover the $l \times l$ Cartan matrix of A by dropping away 0:th row and column.

For instance, the algebra A_1^1, which is affine counterpart of $SL(2)$, has Cartan matrix a_{ij}

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

with a vanishing determinant.

Quite generally, in untwisted case quantum algebra $U_q(\hat{G}_l)$ as $3(l+1)$ generators e_i, f_i, k_i $(i = 0, 1, .., l)$ satisfying the relations of Eq. 14 for Cartan matrix of $\mathcal{G}^{(1)}$. Affine quantum group is obtained by adding to $U_q(\hat{\mathcal{G}}_l)$ a derivation d satisfying the relations

$$[d, e_i] = \delta_{i0} e_i \ , \quad [d, f_i] = \delta_{i0} f_i, \quad [d, k_i] = 0 \ . \tag{20}$$

with comultiplication $\Delta(d) = d \otimes 1 + 1 \otimes d$.

2. Kac Moody algebras

The undeformed extension $\hat{\mathcal{G}}_l$ associated with the affine Cartan matrix $\mathcal{G}_l^{(1)}$ is the Kac Moody algebra associated with the group G obtained as the central extension of the corresponding loop algebra. The loop algebra is defined as

$$L(\mathcal{G}) = \mathcal{G} \otimes C\left[t, t^{-1}\right] \ , \tag{21}$$

where $C\left[t, t^{-1}\right]$ is the algebra of Laurent polynomials with complex coefficients. The Lie bracket is

$$[x \times P, y \otimes Q] = [x, y] \otimes PQ \ . \tag{22}$$

The non-degenerate bilinear symmetric form $(,)$ in $\mathcal{G}_l$ induces corresponding form in $L(\mathcal{G}_l)$ as $(x \otimes P, y \otimes Q) = (x, y)PQ$.

A two-cocycle on $L(\mathcal{G}_l)$ is defined as

$$\Psi(a, b) = Res(\frac{da}{dt}, b) \ , \tag{23}$$

where the residue of a Laurent is defined as $Res(\sum_n a_n t^n) = a_{-1}$. The two-cocycle satisfies the conditions

$$\Psi(a,b) = -\Psi(b,a) \ ,$$
$$\Psi([a,b]\,,c) + \Psi([b,c]\,,a) + \Psi([c,a]\,,b) = 0 \ . \tag{24}$$

The two-cocycle defines the central extension of loop algebra $L(\mathcal{G}_l)$ to Kac Moody algebra $L(\mathcal{G}_l)\otimes Cc$, where c is a new central element commuting with the loop algebra. The new bracket is defined as $[,]+\Psi(,)c$. The algebra $\tilde{L}(\mathcal{G}_l)$ is defined by adding the derivation d which acts as td/dt measuring the conformal weight.

The standard basis for Kac Moody algebra and corresponding commutation relations are given by

$$\begin{aligned} J_n^x &= x\otimes t^n \ , \\ [J_n^x, J_m^y] &= J_{n+m}^{[x,y]} + n\delta_{m+n,0}c \ . \end{aligned} \tag{25}$$

The finite dimensional irreducible representations of G defined representations of Kac Moody algebra with a vanishing central extension $c=0$. The highest weight representations are characterized by highest weight vector $|v\rangle$ such that

$$\begin{aligned} J_n^x|v\rangle &= 0, \ \ n>0 \ , \\ c|v\rangle &= k|v\rangle \ . \end{aligned} \tag{26}$$

3. Quantum affine algebras

Drinfeld has constructed the quantum affine extension $U_q(\mathcal{G}_l)$ using quantum double construction. The construction of generators uses almost the same basic formulas as the construction of semi-simple algebras. The construction involves the automorphism $D_t : U_q(\tilde{\mathcal{G}}_l)\otimes C\left[t,t^{-1}\right] \rightarrow U_q(\tilde{\mathcal{G}}_l)\otimes C\left[t,t^{-1}\right]$ given by

$$\begin{array}{ll} D_t(e_i) = t^{\delta_{i0}}e_i \ , & D_t(f_i) = t^{\delta_{i0}}f_i \ , \\ D_t(k_i) = k_i & D_t(d) = d \ , \end{array} \tag{27}$$

and the co-product

$$\Delta_t(a) = (D_t\otimes 1)\Delta(a) \ , \quad \Delta_t^{op}(a) = (D_t\otimes 1)\Delta^{op}(a) \ , \tag{28}$$

where the $\Delta(a)$ is the co-product defined by the same general formula as applying in the case of semi-simple Lie algebras. The universal R-matrix is given by

$$\mathcal{R}(t) = (D_t \otimes 1)\mathcal{R} \ , \tag{29}$$

and satisfies the equations

$$\begin{aligned}
&\mathcal{R}(t)\Delta_t(a) = \Delta_t^{op}(a)\mathcal{R} \ , \\
&(\Delta_z \otimes id)\mathcal{R}(u) = \mathcal{R}_{13}(zu)\mathcal{R}_{23}(u) \ , \\
&(id \otimes \Delta_u)\mathcal{R}(zu) = \mathcal{R}_{13}(z)\mathcal{R}_{12}(zu) \ , \\
&\mathcal{R}_{12}(t)\mathcal{R}_{13}(tw)\mathcal{R}_{23}(w) = \mathcal{R}_{23}(w)\mathcal{R}_{13}(tw)\mathcal{R}_{12}(t) \ .
\end{aligned} \tag{30}$$

The infinite-dimensional representations of affine algebra give representations of Kac-Moody algebra when one restricts the consideration to generations e_i, f_i, k_i, $i > 0$.

References

[TGDview] M. Pitkänen (2006), *Topological Geometrodynamics: Overview.* http://www.helsinki.fi/~matpitka/tgdview/tgdview.html.

[TGDgeom] M. Pitkänen (2006), *Quantum Physics as Infinite-Dimensional Geometry.* http://www.helsinki.fi/~matpitka/tgdgeom/tgdgeom.html.

[TGDquant] M. Pitkänen (2006), *Quantum TGD.* http://www.helsinki.fi/~matpitka/tgdquant/tgdquant.html.

[TGDclass] M. Pitkänen (2006), *Physics in Many-Sheeted Space-Time.* http://www.helsinki.fi/~matpitka/tgdclass/tgdclass.html.

[TGDnumber] M. Pitkänen (2006), *TGD as a Generalized Number Theory.* http://www.helsinki.fi/~matpitka/tgdnumber/tgdnumber.html.

[TGDpad] M. Pitkänen (2006), *p-Adic length Scale Hypothesis and Dark Matter Hierarchy.* http://www.helsinki.fi/~matpitka/paddark/paddark.html.

[TGDfree] M. Pitkänen (2006), *TGD and Fringe Physics*.
http://www.helsinki.fi/∼matpitka/freenergy/freenergy.html.

[TGDconsc] M. Pitkänen (2006), *TGD Inspired Theory of Consciousness*.
http://www.helsinki.fi/∼matpitka/tgdconsc/tgdconsc.html.

[TGDselforg] M. Pitkänen (2006), *Bio-Systems as Self-Organizing Quantum Systems*.
http://www.helsinki.fi/∼matpitka/bioselforg/bioselforg.html.

[TGDware] M. Pitkänen (2006), *Quantum Hardware of Living Matter*.
http://www.helsinki.fi/∼matpitka/bioware/bioware.html.

[TGDholo] M. Pitkänen (2006), *Bio-Systems as Conscious Holograms*.
http://www.helsinki.fi/∼matpitka/hologram/hologram.html.

[TGDgeme] M. Pitkänen (2006), *Mathematical Aspects of Consciousness Theory*.
http://www.helsinki.fi/∼matpitka/genememe/genememe.html.

[TGDeeg] M. Pitkänen (2006), *TGD and EEG*.
http://www.helsinki.fi/∼matpitka/tgdeeg/tgdeeg/tgdeeg.html.

[TGDmagn] M. Pitkänen (2006), *Magnetospheric Consciousness*.
http://www.helsinki.fi/∼matpitka/magnconsc/magnconsc.html.

[1] M. Pitkänen (2006), *Topological Geometrodynamics: an Overall View*, http://www.helsinki.fi/ matpitka/TGDbrief.pdf.

[2] M. Pitkänen (2006), *TGD Inspired Theory of Consciousness*, http://www.helsinki.fi/ matpitka/tgdconsc.pdf.

[3] M. Pitkänen (2006), *TGD Inspired Quantum Model of Living Matter*, http://www.helsinki.fi/ matpitka/quantumbio.pdf.

[4] M. Pitkänen (2007), *About Correspondence Between Infinite Primes, Space-time Surfaces, and Configuration Space Spinor Fields.* http://www.helsinki.fi/ matpitka/brahma.pdf.

[5] E. Frenkel (2004), *Representation Theory: Its rise and Its Role in Number Theory*, http://www.sunsite.ubc.ca/DigitalMathArchive/Langlands/pdf/gibbs-ps.pdf.

[6] *Langlands program*,http://en.wikipedia.org/wiki/Langlands_program.

[7] E. Frenkel (2005), *Lectures on Langlands program and conformal field theory*, arXiv:hep-th/0512172.

[8] E. Frenkel (2004), *Recent Advances in Langlands program*, Bulletin of the American Mathematical Society, Vol. 41, Number 2, Pages 151-184.

[9] *Ramification*, http://en.wikipedia.org/wiki/Unramified.

[10] *Zeta function*, http://en.wikipedia.org/wiki/Zeta_function.

[11] *A directory of all known zeta functions*, http://secamlocal.ex.ac.uk/∼ mwatkins/zeta/directoryofzetafunctions.htm.

[12] *Weil conjectures*, http://en.wikipedia.org/wiki/Weil_conjectures.

[13] *Mellin transform*, http://en.wikipedia.org/wiki/Mellin_transform.

[14] *Modular forms*, http://en.wikipedia.org/wiki/Modular_forms.

[15] *Adelic ring*, http://en.wikipedia.org/wiki/Adelic.

[16] A. Kapustin and E. Witten (2006), *Electric-Magnetic Duality And The Geometric Langlands Program*, arXiv:hep-th/060451.

[17] M. Freedman, A. Kitaev, M. Larson, Z. Wang (200?), www.arxiv.of/quant-ph/0101025.

[18] M. Freedman, H. Larsen, and Z. Wang (2002), *A modular functor which is universal for quantum computation*, Found. Comput. Math. 1, no 2, 183-204. Comm. Math. Phys. 227, no 3, 605-622. quant-ph/0001108.

[19] L. H. Kauffmann and S, J. Lomonaco Jr. (2004), *Braiding operations are universal quantum gates*, arxiv.org/quant-ph/0401090.

[20] S. Sawin (1995), *Links, Quantum Groups, and TQFT's*, q-alg/9506002.

[21] E. Witten 1989), *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. 121 , 351-399.

[22] V. F. R. Jones (1987), *Hecke algebra representations of braid groups and link polynomial*, Ann. Math., 126, 335-388.

[23] C. Kassel (1995), *Quantum Groups*, Springer Verlag.

[24] C. Gomez, M. Ruiz-Altaba, G. Sierra (1996), *Quantum Groups and Two-Dimensional Physics*, Cambridge University Press.

[25] C. N. Yang, M. L. Ge (1989), *Braid Group, Knot Theory, and Statistical Mechanics*, World Scientific.

[26] F. Schaflitzer(1997), II_1-factors associated with the C^*-Tensor Category of a a Finite Group.
http://arxiv.org/PS_cache/funct-an/pdf/9701/9701008.pdf.

[27] B.I. Plotkin (1972), *Groups of automorphisms of algebraic systems*, Wolters-Noordhoff.
See also *Symmetric Group*, http://eom.springer.de/s/s091670.htm.

[28] *Yang tableau*, http://en.wikipedia.org/wiki/Young_tableau.

[29] *Representation theory of the symmetric group*,
http://en.wikipedia.org/wiki/Representation_theory_of_the_symmetric_group.

[30] *Reductive group*, http://en.wikipedia.org/wiki/Reductive_group.

[31] *Amenable Group*,
http://en.wikipedia.org/wiki/Amenable_group.

[32] D. Evans and Y. Kawahigashi(1998), *Quantum symmetries on operator algebras*, Oxford University Press, New York. The review of the book can be found in Bulletin of the American Mathematical Society, Vol. 28, Number 3, Pages 369-377.
http://www.ams.org/bull/2001-38-03/S0273-0979-01-00906-5/S0273-0979-01-00906-5.pdf.

[33] *Hyperfinite type II factor*,
http://en.wikipedia.org/wiki/Hyperfinite_type_II-1_factor.

[34] N. H. V. Temperley and E. H. Lieb (1971), *Relations between the percolation and colouring problem and other graph-theoretical problems associated with regular planar lattices:some exact results for the percolation problem*, Proc. Roy. Soc. London 322 (1971), 251-280.

[35] O. Bratteli and D. W. Robinson (1979-1981), *Operator Algebras and Quantum Statistical Mechanics*, Volumes 1-2, New York: Springer-Verlag.

[36] V. Jones (2003), *In and around the origin of quantum groups*, arXiv:math.OA/0309199.

[37] *This Week's Finds in Mathematical Physics: Week 230*, http://math.ucr.edu/home/baez/week230.html.

[38] V. F. Jones(1980), *Actions of finite groups on hyperfinite factor type II_1*. Memoirs of the American Mathematical Society, no 237, volume 28.

[39] R. Schaflitzel (1997), *II_1-sub-factors associated with the C^*-Tensor Category of a Finite Group*, arXiv:funct-an/9701008.

[A6] The chapter *Configuration Space Spinor Structure* of [TGDgeom]. http://www.helsinki.fi/~matpitka/tgdview/tgdview.html#cspin.

[A9] The chapter *Does TGD Predict the Spectrum of Planck Constants?* of [TGDview]. http://www.helsinki.fi/~matpitka/tgdview/tgdview.html#Planck.

[C1] The chapter *Construction of Quantum Theory: Symmetries* of [TGDquant]. http://www.helsinki.fi/~matpitka/tgdquant/tgdquant.html#quthe.

[C2] The chapter *Construction of S-matrix* of [TGDquant]. http://www.helsinki.fi/~matpitka/tgdquant/tgdquant.html#smatrix.

[C2] The chapter *Construction of Quantum Theory: S-matrix* of [TGDquant]. http://www.helsinki.fi/~matpitka/tgdquant/tgdquant.html#towards.

[C4] The chapter *Is it Possible to Understand Coupling Constant Evolution at Space-Time Level?* of [TGDquant]. http://www.helsinki.fi/~matpitka/tgdquant/tgdquant.html#rgflow.

[C5] The chapter *Equivalence of Loop Diagrams with Tree Diagrams and Cancellation of Infinities in Quantum TGD* of [TGDquant]. http://www.helsinki.fi/~matpitka/tgdquant/tgdquant.html#bialgebra.

[C6] The chapter *Was von Neumann Right After All* of [TGDquant]. http://www.helsinki.fi/~matpitka/tgdquant/tgdquant.html#vNeumann.

[D1] The chapter *Basic Extremals of Kähler Action* of [TGDclass]. http://www.helsinki.fi/~matpitka/tgdclass/tgdclass.html#class.

[D6] The chapter *TGD and Astrophysics* of [TGDclass]. http://www.helsinki.fi/~matpitka/tgdclass/tgdclass.html#astro.

[E10] The chapter *Intentionality, Cognition, and Physics as Number theory or Space-Time Point as Platonia* of [TGDnumber].
http://www.helsinki.fi/~matpitka/tgdnumber/tgdnumber.html#intcognc.

[E1] The chapter *TGD as a Generalized Number Theory: p-Adicization Program* of [TGDnumber].
http://www.helsinki.fi/~matpitka/tgdnumber/tgdnumber.html#visiona.

[E2] The chapter *TGD as a Generalized Number Theory: Quaternions, Octonions, and their Hyper Counterparts* of [TGDnumber].
http://www.helsinki.fi/~matpitka/tgdnumber/tgdnumber.html#visionb.

[E3] The chapter *TGD as a Generalized Number Theory: Infinite Primes* of [TGDnumber].
http://www.helsinki.fi/~matpitka/tgdnumber/tgdnumber.html#visionc.

[E8] The chapter *Riemann Hypothesis and Physics* of [TGDnumber].
http://www.helsinki.fi/~matpitka/tgdnumber/tgdnumber.html#riema.

[E9] The chapter *Topological Quantum Computation in TGD Universe* of [TGDnumber].
http://www.helsinki.fi/~matpitka/tgdnumber/tgdnumber.html#tqc.

[F1] The chapter *Elementary Particle Vacuum Functionals* of [TGDpad].
http://www.helsinki.fi/~matpitka/paddark/paddark.html#elvafu.

[F6] The chapter *Topological Condensation and Evaporation* of [TGDpad].
http://www.helsinki.fi/~matpitka/paddark/paddark.html#padaelem.