On the large N limit of Exceptional Jordan Matrix Models, Chern-Simons Foliations and M, F theory

Carlos Castro
Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, GA. 30314, castro@ctsps.cau.edu

July, 2005

Abstract

The large $N \rightarrow \infty$ limit of the Exceptional F_4, E_6 Jordan Matrix Models of Smolin-Ohwashi leads to a *novel* Chern-Simons Membrane Lagrangian which is the suitable candidate Lagrangian for non-perturbative *bosonic* M Theory in $D = 27$ real/complex dimensions, respectively, and whose degrees of freedom encode the global dynamics of membranes beginning/ending on $D16$ branes. We rely on the seminal work of $foliations$ by Zois who has studied the dynamics of multiple "parallel" D-branes as leaves of foliations of the underlying bulk space time with the purpose of understanding the Non-commutative topology of M theory. The presence of Octonions will lead naturally to a Non-commutative and Non-associative topology . Bosonic F theory Lagrangians involve the Chern-Simons-Zaikov 7-branes construction and which are based on the $\mathbf{56}, \mathbf{112}$ dim Freudenthal algebras $Fr[O], Fr[C \times O]$. The latter 7-branes actions construction requires the definition of *triple* Freudenthal products in order to build the appropriate *quartic* E_7, E_8 invariants, instead of the cubic F_4, E_6 invariants of Smolin-Ohwashi that are related to bosonic M theory. We conclude with a discussion of future avenues of research.

Keywords: Jordan and Division Algebras, Chern-Simons, Exceptional Groups, Strings, Branes, Matrix Models.

1 Introduction : Chern-Simons Branes and the large N limit

In this section we will discuss the interplay between Chern Simons branes [1] and Chern Simons Topological Matrix models [19] in the large N limit. Such topological Chern Simons brane actions have very interesting properties and were constructed by Zaikov [1]. Let us start, for example, with the action for a spacetime-filling p-brane, whose world-volume $p+1=D$ saturates the target spacetime dimensions:

$$S = T \int [d^{p+1}\sigma] \ \partial_{\sigma^1} X^{\mu_1} \wedge .... \wedge \partial_{\sigma^{p+1}} X^{\mu_D}. \qquad (1.1)$$

where T is the p-brane (extendon) tension.

Zaikov noticed that in some instances the world-volume of a boundary may coincide with the boundary of a world-volume. For example, if one has a p'-brane whose $p'+1$-dim world-volume can be identified with the natural *boundary* of an open-domain associated with a D-dim bulk-region, given by the world-volume of a spacetime filling p-brane ($p+1=D$), then an integration (Gauss law) of (1-1) yields :

$$S_{CS} = T \int_{\partial V} [d^{D-1}\Sigma] \ X^{\mu_1} \wedge \partial_{\sigma^2} X^{\mu_2} \wedge .... \wedge \partial_{\sigma^D} X^{\mu_D}. \qquad (1.2)$$

and one then recovers the Zaikov action for the Chern-Simons p' brane whose $p'+1$-dimensional world-volume spans over the $D-1$-dim boundary ∂V of the D-dim domain V associated with the world-volume of the spacetime-filling open p-brane. The value of p' must be such that $p'+1=p=D-1$. Zaikov concluded that these topological Chern Simons p'-branes exit only in target spacetimes of dimensionality $D=p'+2$, they are codimension *two* objects (like knots). To ensure *translational* invariance of CS branes, the variables X^μ must be *understood* as those variables defined relative to an origin $X^\mu - X^\mu(0)$.

What perhaps is the most significant and salient feature of Chern-Simons p'-branes which live in the $p'+1$-dim *boundary* of a D-dim bulk region, such that $p'+1=D-1$, is the fact that they admit an infinite number of secondary constraints which form an infinite dimensional closed algebra with respect to the Poisson bracket [1] . Such algebra *contains* precisely the clasical $w_{1+\infty}$ as a subalgebras. The latter algebra corresponds to the area-preserving diffeomorphisms of a cylinder.; the w_∞ algebra corresponds to the area-preserving diffs of a plane; while the $su(\infty)$ is the area-preserving diffs of a sphere [2], [5]

In particular, when the dimensionality of the target spacetime is saturated, $D=p+1$, one can construct, in addition, self-dual p-brane (extendon) solutions obeying the equations of motion and constraints (resulting from $p+1$ reparametrization invariance of the world-volume) that are directly related to these topological Chern-Simons p'-branes. This holds provided $p'+1=p$ and the embedding manifold is Euclidean . Furthermore, when $D=p+1=2n$ = even, one has *conformal* invariance as well. It is in this fashion how the relationship between the self dual p-branes and Chern Simons $p'=p-1$ branes emerges. This is roughly the analogy with Witten's discovery of the one-to-one relationship between $3D$ Nonabelian Chern-Simons theories and $2D$ rational CFT, [65]

Chern-Simons p'-branes have codimension *two* and for this reason they are the higher-dimensional extensions of Knots (embeddings of loops S^1 into three-dimensions). Because Chern-Simons branes have codimension 2 , one can have two different CS branes living in two complementary dimensions , d_1, d_2, such that $(p_1 + 2) + (p_2 + 2) = p_1 + p_2 + 4 = d_1 + d_2 = D$. The latter relation is exactly the same one between a p_1-brane and its EM dual p_2-brane living in D dimensions. In this sense, these two CS branes (high dim knots) intersect *transversely* from the $D = d_1 + d_2$ perspective and can be seen as EM duals of each-other.

Since Zaikov's Chern-Simons branes **are** high-dimensional Knots , its relation to algebraic **K, L** theory has to be explored deeper. Zaikov has also pointed out that in the $D = \infty$ limit, Chern-Simons p-branes acquire true **local** dynamics ! Infinite dimensions based on a hierarchy of infinitely nested spaces of increasing dimensions, " Russian Doll", will give us a unique vantage point in the sense that one master gauge field in infinite dimensions enocodes the dynmics of all the infinite number of massless fields in lower dimensions.

Having discussed CS branes we turn our attention to those topological matrix models which are intricately related to CS branes. Matrix Chern-Simons models in *odd* dimensions, $D = n = odd$, are defined by the Lagrangians consisting of n Hermitian $N \times N$ matrices X^μ [19] :

$$L = \epsilon_{\mu_1 \mu_2 .... \mu_n} Trace \ [X^{\mu_1} X^{\mu_2} .... X^{\mu_n}]. \tag{1.3}$$

where X^μ is a $N \times N$ matrix

Due to the cyclic property of the trace, one can permute the X^{μ_1} matrix-factor inside the trace past all the other matrix-factors until it is placed at the end of the sequence without changing the value of the trace. A subsequent permutation of $\mu_1$index of the epsilon factor past all the other indices, until it reaches the end of the sequence, it brings an overall factor of $(-1)^{n-1}$, leaving the relation : $L = (-1)^{n-1} L = (-1)^{D-1} L$. Thus in order to have a non-vanishing action (1-3), we require to have $(-1)^{D-1} = 1$, which implies that that $D = n$ must be *odd* or else the action is trivially zero, $L = -L \Rightarrow L = 0$. Since D is odd, and CS branes require co-dimension two-embeddings, this means that $p' = D - 2$ is odd. Hence the $p' + 1$-dim worldvolume of the corresponding CS brane is even.

To show how the large N limit of Topological Chern-Simons Matrix models [19] are related to Chern-Simons branes [78] it is essential to explain the derivation of how Hadronic Bags (branes) and Chern-Simons Branes can be obtained from the Large N limit of Yang-Mills and Generalized Yang-Mills theories in Flat Backgrounds. A Moyal deformation quantization was instrumental in the construction of p-brane actions and Chern-Simons branes from the large N limit of $SU(N)$ YM in flat backgrounds. $SU(N)$ reduced-quenched gauge theories have been shown by us to be related to Hadronic Bags and Chern Simons Membranes in the large N limit [8] .

This is reminiscent of the chiral model approaches to Self Dual Gravity based on Self Dual Yang Mills theories [29] . A Moyal deformation quantization of the Nahm equations associated with a $SU(2)$ YM theory yields the *classical* $N \rightarrow \infty$ limit of the $SU(N)$ YM Nahm equations *directly*, without ever having to use $\infty \times \infty$ matrices in the large N matrix models. By simply taking the classical $\hbar =$

0 limit of the Moyal brackets, the ordinary Poisson bracket algebra associated with area-preserving diffs algebra $SU(\infty)$ [2] is automatically recovered.

This Moyal deformation approach also furnishes dynamical membranes as well [8] when one uses the spatial quenching approximation to a *line* (one dimension) , instead of quenching to a point. In this fashion we constructed what is called a QCD membrane. Basically, a Moyal quantization takes the operator $\hat{A}_\mu(x^\mu)$ into $A_\mu(x^\mu; q, p)$ and commutators into Moyal brackets. A dimensional reduction to one temporal dimension (quenching to a line) brings us to functions of the form $A_\mu(t, q, p)$, which precisely corresponds to the membrane coordinates $X_\mu(t, \sigma^1, \sigma^2)$ after identifying the σ^a variables with q, p. The $\hbar = 0$ limit turns the Moyal bracket into a Poisson one. Upon the identification of $\hbar = 2\pi/N$, the classical $\hbar = 0$ limit is tantamount to the $N = \infty$ limit and it is in this fashion how the large N $SU(N)$ matrix model bears a direct relation to the physics of membranes. The Moyal quantization explains this in a straightforward fashion without having to use $\infty \times \infty$ matrices ! The large N limit of Nonabelian $SU(N)$ Born-Infeld models and its relation to Nambu-Goto-Dirac string actions was also achieved in [8].

We will briefly review [8] how a $4D$ Yang-Mills theory reduced and quenched to a point, and supplemented by a topological *theta* term can be related through a Weyl-Wigner Groenowold Moyal (WWGM) quantization procedure to an *open* domain of the 3-dim disk D^3. The bulk $D^3 \times R^1$ is the interior of a hadronic bag and the (lateral) boundary is the Chern-Simons world volume $S^2 \times R^1$ of a membrane of topology S^2 (a codimension two object). Hence, we have an example where the world-volume of a boundary $S^2 \times R^1$ is the lateral-boundary of the world-volume of an open 3-brane of topology D^3 : $\partial(D^3 \times R^1) = S^2 \times R^1$ (setting asside the points at infinity). The boundary dynamics is *not* trivial despite the fact that there are no transverse bulk dynamics associated with the interior of the bag. This is due to the fact that the 3-brane is spacetime filling : $3 + 1 = 4$ and therefore has no transverse physical degrees of freedom.

The reduced-quenched action to a point in $D = 4$ is:

$$S = -\frac{1}{4}(\frac{2\pi}{a})^4 \frac{N}{g^2_{YM}} Tr(F_{\mu\nu}F^{\mu\nu}).$$
$$F_{\mu\nu} = [iD_\mu, iD_\nu]. \tag{1.4}$$

Notice that the reduced-quenched action is defined at a " point " x_o . The quenched approximation is based essentially by replacing the field strengths by their commutator dropping the ordinary derivative terms . For simplicity we have omitted the matrix $SU(N)$ indices in (1-4). The *theta* term is:

$$S_\theta = -\frac{\theta N g^2_{YM}}{16\pi^2}(\frac{2\pi}{a})^4 \epsilon^{\mu\nu\rho\sigma} Tr(F_{\mu\nu}F_{\rho\sigma}). \tag{1.5}$$

The WWGM quantization establishes a one-to-one correspondence between a linear operator $D_\mu = \partial_\mu + A_\mu$ acting on the Hilbert space $\mathcal{H}$ of square integrable functions in R^D and a smooth function $\mathcal{A}_\mu(x, y)$ which is the Fourier transform of $\mathcal{A}_\mu(q, p)$. The latter quantity is obtained by evaluating the trace of the

$D_\mu = \partial_\mu + A_\mu$ operator summing over the diagonal elements with respect to an orthonormal basis in the Hilbert space. Under the WWGM correspondence , in the quenched-reduced approximation, the matrix product $A_\mu.A_\nu$ is mapped into the *noncommutative* Moyal star product of their symbols $\mathcal{A}_\mu * \mathcal{A}_\nu$ and the commutators are mapped into their Moyal brackets:

$$\frac{1}{i\hbar}[A_\mu, A_\nu] \Rightarrow \frac{1}{i\hbar}\{\mathcal{A}_\mu, \mathcal{A}_\nu\}_{MB} \rightarrow \{\mathcal{A}_\mu, \mathcal{A}_\nu\}_{PB} \ when\ \hbar \rightarrow 0. \tag{1.6}$$

Replacing the Trace operation with an integration w.r.t the internal phase space variables , $\sigma \equiv q^i, p^i$ gives:

$$\frac{(2\pi)^4}{N^4} Trace \rightarrow \int d^4\sigma. \tag{1.7}$$

The WWGM deformation quantization of the quenched-reduced orginal actions is:

$$S^* = -\frac{1}{4}(\frac{2\pi}{a})^4 \frac{N}{g^2_{YM}} \int d^4\sigma \mathcal{F}_{\mu\nu}(\sigma) * \mathcal{F}^{\mu\nu}(\sigma).$$
$$\mathcal{F}_{\mu\nu} = \{i\mathcal{A}_\mu, i\mathcal{A}_\nu\}. \tag{1.8}$$

And the corresponding WWGM deformation of the *theta* term:

$$S^*_\theta = -\frac{\theta N g^2_{YM}}{16\pi^2}(\frac{2\pi}{a})^4 \epsilon^{\mu\nu\rho\sigma} \int d^4\sigma \mathcal{F}_{\mu\nu}(\sigma) * \mathcal{F}_{\rho\sigma}(\sigma). \tag{1.9}$$

By performing the following gauge fields/coordinate correspondence:

$$\mathcal{A}_\mu(\sigma) \rightarrow (\frac{2\pi}{N})^{1/4} X_\mu(\sigma)$$
$$\mathcal{F}_{\mu\nu}(\sigma) \rightarrow (\frac{2\pi}{N})^{1/2}\{X_\mu(\sigma), X_\nu(\sigma)\}. \tag{1.10}$$

And, finally, by setting the Moyal deformation parameter "$\hbar$" $= 2\pi/N$ of the WWGM deformed action , to zero ; i.e by taking the classical $\hbar = 0$ limit, which is tantamount to taking the $N = \infty$ limit, one can see that the quenched-reduced YM action in the large N limit will become the Dolan-Tchrakian action for a 3-brane, in the conformal gauge [27], moving in a *flat* $D = 4$-dim background [8]:

$$S = -\frac{1}{4g^2_{YM}}(\frac{2\pi}{a})^4 \int d^4\sigma \{X^\mu, X^\nu\}_{PB}\{X^\rho, X^\tau\}_{PB}\ \eta_{\mu\rho}\eta_{\nu\tau}. \tag{1.11}$$

due to the fact that the $\frac{1}{\hbar}$ times the Moyal brackets collapse to the ordinary Poisson brackets in the $\hbar = 2\pi/N = 0$ limit (large N limit).

Whereas the action corresponding to the *theta* term will become in the $N = \infty$ limit, the Chern-Simons Zaikov action for a closed membrane embedded in a four-dimensional (pseudo) Euclidean background and whose 3-dim worldvolume is the boundary of the four-dim hadronic bag . The Chern-Simons membrane has *nontrivial* boundary dynamics compared with the trivial bulk dynamics of the spacetime filling 3-brane. This introductory review is essential to understand how to related the large N limit of the Exceptional Matrix Models [39], [38] to a novel version of Chern-Simons branes .

2 The large N limit of Exceptional Jordan Matrix Models

2.1 Exceptional Chern-Simons Branes

Ohwashi [38] defined his E_6 Matrix model by starting with the Matrix $\mathcal{M}^A$ elements of the algebra $\mathcal{J} \times \mathcal{G}$

$$\begin{pmatrix} \mathcal{A}_1^A & \Phi_3^A & \bar{\Phi}_2^A \\ \bar{\Phi}_3^A & \mathcal{A}_2^A & \Phi_1^A \\ \Phi_2^A & \bar{\Phi}_1^A & \mathcal{A}_3^A \end{pmatrix}$$

where $\mathcal{J}$ is the complexified Jordan algebra of degree three $J_3[C \times O]$ and $\mathcal{G}$ is the $u(N)$ Lie algebra corresponding to the $U(N)$ group with structure constants f_{ABC}. $[T_A, T_B] = f_{ABC} T_C$. $\mathcal{A}_I, (I = 1, 2, 3)$ are complex-valued numbers and Φ_I are elements of the *complex* Graves-Cayley octonion algebra comprised of complex octonions $(x_o + \mathbf{i} y_o) e_o + (x_i + \mathbf{i} y_i) e_i$; $i = 1, 2, 3, ..., 7$. The bar operation $\bar{\Phi}$ denotes the octonionic-conjugation $(x_o + \mathbf{i} y_o) e_o - (x_i + \mathbf{i} y_i) e_i$ that must not be confused with complex conjugation $(x_o - \mathbf{i} y_o) e_o + (x_i - \mathbf{i} y_i) e_i$. The action of Ohwashi was based on the cubic form

$$S = (\ \rho^2(\mathcal{M}^{[A}),\ \rho(\mathcal{M}^B),\ \mathcal{M}^{C]}\ )\ f_{ABC} \qquad (X, Y, Z) = tr\ (\ X\ \cdot\ (Y \times_F Z)\ ) \quad (2.1)$$

$\rho,\ \rho^3 = 1$ is the cycle mapping (based on the *triality* symmetry of $SO(8)$) that takes the index $I \rightarrow I + 1$, modulo 3. It is essential to introduce the cycle mapping in (2.1) otherwise the expression would have been identically equal to zero due to the fact that the cubic form is symmetric in its three entries while f_{ABC} is antisymmetric . The product $Y \times_F Z$ is the *symmetric* Freudenthal product

$$Y \times_F Z = Y \cdot Z - \frac{1}{2} tr\ (Y)\ Z - \frac{1}{2} tr\ (Z)\ Y + \frac{1}{2} tr\ (Y)\ tr\ (Z) - \frac{1}{2} tr\ (Y \cdot Z)\ \mathbf{1}. \quad (2.2)$$

and $X \cdot Y$ is the commutative but non-associative Jordan product given by the anti-commutator $\frac{1}{2}(XY + YX)$ obeying the Jordan identity $(X \cdot Y) \cdot X^2 = X \cdot (Y \cdot X^2)$. The cubic form (2.1) is very different from the trilinear form $tr\ (X \cdot (Y \cdot Z))$ used by Smolin [39] to construct the F_4 matrix model based on

$J_3[O]$ rather than $J_3[C \times O]$. The action of Ohwashi is complex-valued while that of Smolin is real-valued. The explicit evaluation of the expression (2.1) can be found in [38] where he includes a detailed appendix with numerous important formulae that are indispensable to be able to write down all the explicit terms of the cubic form.

We shall prove now why the $N \rightarrow \infty$ limit of Ohwashi E_6 matrix model is given by the *novel* Chern-Simons membrane model (not to be confused with the Zaikov's Chern-Simons membrane [1] of the previous section) :

$$S = \int_V [d^3V]\ \epsilon^{abc}\ (\ \partial_a \mathbf{J},\ \partial_b\ \rho(\mathbf{J}),\ \partial_c\ \rho^2(\mathbf{J})\ ). \tag{2.3}$$

The action, omitting numerical factors, is explicitly given by

$$S = \int_V d^3V\ \epsilon^{abc}\ [\ \epsilon^{IJK}(\ \partial_a \mathcal{A}_I\ \partial_b \mathcal{A}_J\ \partial_c \mathcal{A}_K + \eta^{ij}\ \partial_a \Phi_{0I}\ \partial_b \Phi_{iJ}\ \partial_c \Phi_{jK} +$$

$$\sigma^{ijk}\ \partial_a \Phi_{iI}\ \partial_b \Phi_{jJ}\ \partial_c \Phi_{kK}\ + \eta^{ij}\ \partial_a \mathcal{A}_I\ \partial_b \Phi_{iJ}\ \partial_c \Phi_{jK}\ + \partial_a \mathcal{A}_I\ \partial_b \Phi_{0J}\ \partial_c \Phi_{0K} +$$

$$\partial_a \Phi_{0I}\ \partial_b \Phi_{0J}\ \partial_c \Phi_{0K}\ )\ +\ \sigma^{ijk}\ \sum_{I=1}^{I=3} \partial_a \Phi_{iI}\ \partial_b \Phi_{jI}\ \partial_c \Phi_{kI}\ ] \tag{2.4}$$

The above action can be recast in terms of Nambu-Poisson brackets as

$$\int d^3\sigma\ [\ \epsilon^{IJK}(\ \{\ \mathcal{A}_I,\ \mathcal{A}_J,\ \mathcal{A}_K\ \}\ +\ \eta^{ij}\ \{\ \Phi_{0I},\ \Phi_{iJ},\ \Phi_{jK}\ \} +$$

$$\sigma^{ijk}\ \{\ \Phi_{iI},\ \Phi_{jJ},\ \Phi_{kK}\ \}\ + \eta^{ij}\ \{\ \mathcal{A}_I,\ \Phi_{iJ},\ \Phi_{jK}\ \}\ + \{\ \mathcal{A}_I,\ \Phi_{0J},\ \Phi_{0K}\ \} +$$

$$\{\ \Phi_{0I},\ \Phi_{0J},\ \Phi_{0K}\ \}\ )\ +\ \sigma^{ijk}\ \sum_{I=1}^{I=3}\ \{\ \Phi_{iI},\ \Phi_{jI},\ \Phi_{kI}\}\ ]. \tag{2.5}$$

The integrand is a total derivative that can be integrated over a two-dim boundary domain $\Sigma \equiv \partial V$ giving

$$S = \int_{\partial V} [d^2\Sigma]_a\ \epsilon^{abc}\ [\ \epsilon^{IJK}(\ \mathcal{A}_I\ \partial_b \mathcal{A}_J\ \partial_c \mathcal{A}_K + \eta^{ij}\ \Phi_{0I}\ \partial_b \Phi_{iJ}\ \partial_c \Phi_{jK} +$$

$$\sigma^{ijk}\ \Phi_{iI}\ \partial_b \Phi_{jJ}\ \partial_c \Phi_{kK}\ + \eta^{ij}\ \mathcal{A}_I\ \partial_b \Phi_{iJ}\ \partial_c \Phi_{jK}\ + \mathcal{A}_I\ \partial_b \Phi_{0J}\ \partial_c \Phi_{0K} +$$

$$\Phi_{0I}\ \partial_b \Phi_{0J}\ \partial_c \Phi_{0K}\ )\ +\ \sigma^{ijk}\ \sum_{I=1}^{I=3} \Phi_{iI}\ \partial_b \Phi_{jI}\ \partial_c \Phi_{kI}\ ]. \tag{2.6}$$

The novel Chern-Simons action (2.6) is the candidate action for non-perturbative *bosonic* M theory Lagrangian in $D = 27$ dim.

We will see now how that the large N limit of the E_6 Exceptional Matrix Model action described in [38] is given by the action (2-6) . To achieve this one needs to follow similar steps as those taken in the previous section to relate the large N limit of quenched-reduced $SU(N)$ Yang-Mills actions to strings, membranes and 3-branes (bags) actions [8]. Upon doing so one arrives at the following correspondence among the Matrix Model action [38] with our Chern-Simons action (2.6)

$$
lim_{N\rightarrow\infty} \quad \epsilon^{IJK} \ tr_{N\times N} \ (\ \mathcal{A}_I^A T_A \ [\mathcal{A}_J^B T_B, \ \mathcal{A}_K^C T_C] \ ) \ \rightarrow
$$

$$
\int [d^2\Sigma]_a \ \epsilon^{abc} \mathcal{A}_I \ (\partial_b \mathcal{A}_J) \ (\partial_c \mathcal{A}_K) \ \epsilon^{IJK}. \tag{2.7}
$$

where the N^2 matrices $T_A, T_B, T_C...$ are $N \times N$ Hermitian matrices associated with the Lie algebra $u(N)$ generators corresponding to the group $U(N) = SU(N) \times U(1)$. The $U(1)$ piece corresponds to the center of mass mode since the variables in the Chern-Simons brane actions must be understood in terms of $X - X(0)$; i.e. relative to an origin in order to preserve translational invariance. The indices I, J, K run over $1, 2, 3$. The indices $a, b.c$ run over $1, 2, 3$, the three degrees of freedom of the world-volume of a membrane. The surface boundary element of the 3 dim world-volume V is $d^2\vec{S} = d^2\Sigma_a n^a$ where n^a is a unit vector pointing in the outwards normal direction.

The remaining terms in the E_6 Exceptional Matrix Model/Chern-Simons branes correspondence (in the large N limit) goes as follows :

$$
lim_{N\rightarrow\infty} \quad \epsilon^{IJK} \ tr_{N\times N} \ (\ \eta^{ij} \ \Phi_{iI}^A T_A \ [\mathcal{A}_J^B T_B, \ \Phi_{jK}^C T_C] \ ) \ \rightarrow
$$

$$
\int [d^2\Sigma]_a \ \epsilon^{abc} \ \eta^{ij} \ \Phi_{iI} \ (\partial_b \mathcal{A}_J) \ (\partial_c \Phi_{jK}) \ \epsilon^{IJK}. \tag{2.8}
$$

$$
lim_{N\rightarrow\infty} \quad \epsilon^{IJK} \ tr_{N\times N} \ (\ \sigma^{ijk} \ \Phi_{iI}^A T_A \ [\Phi_{jJ}^B T_B, \ \Phi_{kK}^C T_C] \ ) \ \rightarrow
$$

$$
\int [d^2\Sigma]_a \ \epsilon^{abc} \ \sigma^{ijk} \ \Phi_{iI} \ (\partial_b \Phi_{jJ}) \ (\partial_c \Phi_{kK}) \ \epsilon^{IJK}. \tag{2.9}
$$

etc..................

In this way we can show that the large N limit of the E_6 Exceptional Matrix Model [38] is given by a Chern-Simons brane action (2.6) . The crux of this large $N \rightarrow \infty$ limit correspondence relies on the fact that $N \times N$ matrices $\mathbf{A} \rightarrow \mathcal{A}(\sigma^1, \sigma^2, \sigma^3)$ become the membrane coordinates in the continuum limit; the $trace_{N\times N} \rightarrow \int$; *commutators* $\rightarrow$ *brackets* and the Jordan algebra non-associator $[X, Y, Z] = X \cdot (Y \cdot Z) - (X \cdot Y) \cdot Z$ has a correspondence with the Nambu-Poisson brackets $\{X(\sigma^a), Y(\sigma^a), Z(\sigma^a)\}$ as discussed by [56]. Similar results can be obtained in the large N limit of the F_4 Matrix Models of [39], with the only difference that one must use the trilinear form based on Jordan products instead of the cubic form (based on the Jordan and Freudenthal product).

The action in the form (2-5) is clearly invariant under volume-preserving reparametrizations of the three-dim world-volume that leave invariant the Nambu-Poisson brackets. One may view the membrane as an incompressible fluid that can change its shape while maintaining its volume. The true dynamics of (2-5) reside in the two-dim *boundary* captured by the two-dim boundary action $S(\Sigma)$ (2.6).

There is invariance under the *global* rigid E_6 (simply connected compact group) mappings which are encoded as automorphims of the $J_3[C \times O]$ algebra under the transformations $\mathbf{J} \rightarrow \alpha\ \mathbf{J}$, where α is a 3×3 matrix whose entries are numerical constants, and which leave invariant the cubic form

$$S(\alpha \mathbf{J}) = \int_V\ [d^3V]\ \epsilon^{abc}\ (\ \partial_a\ (\alpha \mathbf{J}),\ \partial_b\ (\alpha \rho(\mathbf{J})),\ \partial_c\ (\alpha \rho^2(\mathbf{J}))\ ) =$$

$$S(\mathbf{J}) = \int_V\ [d^3V]\ \epsilon^{abc}\ (\ \partial_a\ \mathbf{J},\ \partial_b\ \rho(\mathbf{J}),\ \partial_c\ \rho^2(\mathbf{J})\ ). \tag{2.10}$$

these E_6 global (rigid) mappings α also leave invariant the Hermitian product

$$< \alpha \mathbf{X}, \alpha \mathbf{Y} >=< \mathbf{X}, \mathbf{Y} >= (\mathbf{X}^*, \mathbf{Y}) = tr\ (\mathbf{X}^*\ \cdot\ \mathbf{Y}). \tag{2.11}$$

There is also symmetry under the cycle mapping ρ by construction. See [38] for futher details.

2.2 Chern-Simons Foliations and M Theory

Next we will show that our Chern-Simons Brane Lagrangian is a suitable candidate for the *bosonic* M Theory Lagrangians in $D = 27$ real and complex dimensions (54 real dim), respectively, whose degrees of freedom encode the global dynamics of membranes beginning/ending on $D16$ branes. We will rely on the seminal work of foliations by Zois [67] who has studied the dynamics of multiple "parallel" D-branes from the perspective of leaves of foliations of the underlying bulk spacetime with the purpose of understanding the Non-commutative topology of M theory. The presence of Octonions will lead naturally to a Non-commutative and Non-associative topology .

One may interpret the 27 functions $\mathcal{A}_I(\sigma^a), \Phi_{0I}(\sigma^a), \Phi_{iI}(\sigma)$ associated with the Chern-Simons action (2-5, 2-6) as **Foliations**-maps X from the 3-dim world-volume of the membrane (and its boundary) to the target space M^{27}. This, of course, requires to show that these foliations-maps indeed correspond to a certain class of solutions to the equations of motion associated with the boundary dynamics of the Chern-Simons action. Let us assume there are such foliations-maps that solve the equations of motion. Upon foliating the 3-dim world-volume V into $\Sigma \times R$, where the leaves Σ are two-dim Riemann surfaces, and R is a world-volume clock, yields the maps :

$$X\ :\ \Sigma \times R\ \rightarrow\ \Upsilon^{16} \times M^{11}. \tag{2.12}$$

where the "horizontal" leaves Υ^{16} foliate the underlying space M^{27} along the "vertical" directions M^{11} . Let us study now the Membranes ending on D-branes scenario [68]. Given a $finite$ size tubular region V with two boundary-components $\partial V = \Sigma_1 \cup \Sigma_2$ one may interpret the first and last leaves of the foliations of M^{27} as two $D16$ branes at the "end of the world scenario" [66]

$$X\ :\ V_{bulk}\ \rightarrow\ M^{27}.\quad X\ :\ \partial V = \Sigma_1 \cup \Sigma_2\ \rightarrow \partial M^{27} = D16 \cup D16. \tag{2.13}$$

The foliation into a *continuum* of multi "parallel" $D16$-branes [67] (what the $N \to \infty$ continuum limit truly represents) will break the $SO(26,1)$ symmetry of the bulk Minkowski spacetime M^{27}, leaving the symmetry group of $SO(10,1) \times$ *isometries* (Υ^{16}) unbroken. In the case of bosonic F theory one has a 28 dim bulk spacetime, instead.

When the coordinates X belong to the $J_3[O]$ algebra, the foliations of M^{27} into $\Upsilon^{16} \times M^{11}$ are the maps

$$X(\sigma^1, \sigma^2, \sigma^3) \; : \; V_{bulk} \; \to \; M^{11} \times \Upsilon^{16}. \tag{2.14}$$

and the boundary maps obey

$$X(\partial V) \; : \; \partial V = \Sigma_1 \cup \Sigma_2 \; \to (R \times O)P^2 \cup (R \times O)P^2. \tag{2.15}$$

with

$$(R \times O)P^2 \equiv \frac{F_4}{Spin(9)}. \tag{2.16}$$

Since ∂V is made out of *compact* regions (Riemann surfaces) Σ_1, Σ_2 the continous maps $X(\partial V)$ should map compact sets into compact sets that is indeed consistent by viewing the projective planes as compact domains of the $D16$ branes (noncompact hypersurfaces in general) .

Because the isometries of $(R \times O)P^2 =$ Automorphism group of the Jordan algebra $J_3[O] = F_4$, this picture of foliations is consistent at the boundaries since F_4 acts naturally on $(R \times O)P^2$. The split of $27 = 3 + 8 + 16$ can be understood as follows : The three functions $\mathcal{A}_1(\sigma^a), \mathcal{A}_2(\sigma^a), \mathcal{A}_3(\sigma^a)$ in (2-4, 2-5, 2-6) are associated with the three longitudinal directions corresponding to the $2 + 1 = 3$ dim world-volume of the membrane. The 8 functions Φ_{01}, Φ_{i1} ($i = 1, 2, 3, ...7$) represent the 8 transverse-directions of the membrane with *respect* to an 11-dim domain region inside the 27-dim bulk of M^{27}. The 16 (real) functions $\Phi_{02}, \Phi_{i2}, \Phi_{03}, \Phi_{i3}$ correspond to the coordinates of the 16 real-dim $(R \times O)P^2$ projective plane.

A word of *caution* is warranted at this point. Our interpretation of the variables in $J_3[O]$ differs from the $D = 10$ strings picture of [45] where they view the variables of the Jordan algebra $J_3[O] \equiv R \oplus J_2[O] \oplus O^2$ as follows : they treat the variable $\mathcal{A}_3$ as scalar (like a dilaton) ; the variables $\mathcal{A}_1, \mathcal{A}_2$ are seen as the two light-cone directions in $D = 10$; the variables Φ_{01}, Φ_{i1} are 8 transverse directions (vectors) ; and the $8 + 8 = 16$ variables $\Phi_{02}, \Phi_{i2}, \Phi_{03}, \Phi_{i3}$ are two $D = 10$ Majorana-Weyl *commuting* spinors. The F_4 invariant *cubic* norm in $D = 10$ is defined as

$$N(J) = a\mathcal{A}_3 X^\mu X^\nu \eta_{\mu\nu} \; + \; bX^\mu \bar{\psi} \gamma_\mu \psi. \quad a^2 = 2. \; b = 1. \tag{2.17}$$

Exceptional strings based on Jordan algebras have also been studied by several authors [46] and the interpretation of the dimensions $D = 26, 27, 28$ in terms of the traceless $J_3[O]$ (26-dim algebra) , ordinary $J_3[O]$ (27-dim algebra) and $J_4[H]$ (28-dim algebra) Jordan algebras have been provided by [53].

However, in our work, we pose a different interpretation of the coordinate variables X of $J_3[O]$, such that the large N limit of Smolin's F_4 matrix model

[39] describes the global dynamics of a 3 real-dim world-volume of the membrane whose 2 real-dim boundary-regions $\Sigma = \partial V$ ("endpoints") are situated on the $D16$ branes ($(R \times O)P^2$ planes of 16 real dimensions) embedded in a $D = 27$ real-dim bulk spacetime.

When the variables X belong to the complexified Jordan algebra $J_3[C \times O]$, the large N limit of Ohwashi's E_6 matrix model [38], given by our Chern-Simons action (2.6), describes the global dynamics of a 3 complex-dim world-volume of the membrane whose 2 complex-dim boundary-regions $\Sigma = \partial V$ ("endpoints") are situated on the $(C \times O)P^2$ planes; i.e. we have complexified-membranes beginining/ending on the complexified $D16$-branes (of real dimensions $2 \times 16 = 32$) embedded in a $D = 27$ complex-dim bulk spacetime. The Automorphism group of $J_3[C \times O]$ coincides with the isometry group of $(C \times O)P^2$ given by E_6. This projective plane can be described as the coset

$$(C \times O)P^2 \equiv \frac{E_6}{(Spin(10) \times U(1))/Z_4}. \tag{2.18}$$

To finalize this section we should add the importance of having membranes on *curved* backgrounds, in particular, on symmetric spaces obtained from the quotients of the conformal group of the corresponding Jordan algebra by their maximal compact subgroups [42] . For instance, there is a whole family of symmetric spaces associated with the four Jordan magical algebras $J_3[R, C, H, O]$ which appear in compactifications of $N = 2$ Maxwell-Einstein Supergravities from $5D$ to $4D$ and $3D$ [42]. The following $2 \times 27 = 54$ dim symmetric space (resulting from the compactification from $5D \rightarrow 4D$) $E_{7(-25)}/E_6 \times U(1)$ is one of them. Therefore, once can naturally identify the $2 \times 27 = 54$ real degrees of freedom of the $J_3[C \times O]$ algebra in the large N limit of Ohwashi E_6 matrix model as the coordinates of the 54-dim (curved) symmetric space where the complexified-membrane is living. The Chern-Simons action foliations picture works also in curved spaces. The E_6 can be identified now as part of the maximal compact stability subgroup of the non-compact $E_{7(-25)}$ group.

3 On F theory, 7-branes and Freudenthal Triple Systems

We begin this last section by replacing the Jordan algebras $J_3[O], J_3[C \times O]$ with the $\mathbf{56}, \mathbf{112}$ dim Freudenthal algebras $Fr[O], Fr[C \times O]$, respectively. We will see that the complexified F theory in $D = 2 \times 28 = 56$ dim [70] stands now for Freudenthal ! In the first part of this section we identify the algebras and their associated spaces while in the second part we construct the 7-brane actions that would require the triple Freudenthal product in order to construct a quartic invariant (X, Y, Z, W) [42] .

When the coordinates J belong to the real Freudenthal algebra :

$$\begin{pmatrix} a & J_3[O] \\ J_3[O] & b \end{pmatrix}$$

the two real variables a, b along the diagonal parametrize the two-torus T^2 fibration over each point of the complexified 11-dim spacetime M_C^{11}. The dimension of the algebra $Fr\ [O]$ is $2+2\times 27 = 2+54 = 2\times 28 = 56$. The complexified M_C^{28} could be foliated as $M_C^{11} \times_f T^2 \times \Upsilon_C^{16}$. The Automorphims group of $Fr[O] = E_6$.

The reason we may have a 7-brane whose world-volume is 8-dim can be seen as follows : The $Fr[O]$ has two copies of $J_3[O]$, this entails that we have 6 coordinates $\mathcal{A}_I, \mathcal{B}_I,\ I = 1,2,3$. There are the additional 2 coordinates a, b giving a total of $6+2 = 8$ coordinates that agree with the number of longitudinal degrees of freedom associated with the 8 dim world-volume of a 7-brane.

When the coordinates X belong to the *complexified* Freudenthal algebra :

$$\begin{pmatrix} a_1 + ia_2 & J_3[C \times O] \\ J_3[C \times O] & b_1 + ib_2 \end{pmatrix}$$

The four real variables a_1, a_2, b_1, b_2 along the diagonal parametrize the four-torus T^4 fibration over each point of the quaternified 11-dim spacetime M_H^{11}. The dimension of the algebra $Fr\ [C \times O]$ is $2 \times (2 + 54) = 4 \times 28 = 112$ that suggests that a quaternionic version of F theory may exist since the $M_H^{28} = H^{28}$ space may be foliated into $M_H^{11} \times_f T^4 \times \Upsilon_H^{16}$. The Automorphism group of $Fr[C \times O] = E_7$.

In the compactifications of $N = 2$ Maxwell-Einstein Supergravities from $5D$ to $3D$ [42] there is a $4 \times 28 = 112$ real-dim symmetric space $\mathcal{M}(J) = E_{8(-24)}/E_7 \times SU(2)$. Therefore, for this reason, one can naturally identify the $4 \times 28 = 2 \times 56 = 112$ real degrees of freedom of the $Fr\ [C \times O]$ algebra as the *coordinates* of the 112 dim (curved) symmetric space $\mathcal{M}(J)$ where the 7-brane is living. In this case (as we shall see below) one may have an 8 complex-dim world-volume of the complexified 7-brane living in 56 complex-dim. The E_7 automorphism group of the $Fr[C\times O]$ algebra (also known as the ternary Brown algebra [48]) can be identified now as part of the maximal compact subgroup of the non-compact $E_{8(-24)}$ group.

The complexification of the 112-dim symmetric space $E_{8(-24)}/E_7\times SU(2)$ (if this symmetric space admits an integrable complex structure) has $2 \times 4 \times 28 = 8\times 28 = 4\times 56 = 224$ real dim. In this case we may have have an 8 quaternionic-dim world-volume of the quaternionic 7-brane living in 56 quaternionic-dim (28 octonionic-dim) that may correspond to an octonionic version of F theory. There is a 224 dim space that has been coined as the *tesserat* [53] and is related to the 248-dim E_8 algebra and the 24-dim Chevalley algebra ($224 = 248 - 24$). The latter is obtained by deleting the diagonal part of the 3×3 matrix elements of the algebra $J_3[O]$ and in the Jordan products $X \bullet_{Ch} Y = (X \cdot Y)_{offdiagonal}$. Therefore, the *tesserat* construction may provide the algebraic structures behind a putative octonionic bosonic F theory in 28 octonionic-dim (224 real-dim).

The reason one cannot use $h_3[H\times O]$ and $h_3[O\times O]$ entries in the above Zorn 2×2 block matrix descriptions of the Freudenthal algebras is because the former matrices $don't$ belong to a Jordan algebra [47] . In order for the matrix algebra to close one has to add the matrix commutators. All of these structures, whether they correspond to Jordan algebras or not, can be embedded in the polyvector-geometry of C-spaces (an Extended Relativity theory in Clifford-spaces) [76], [77] because there are octonionic realizations of Clifford algebras in $D = 8$ despite the fact that octonions are non-associative [52], by introducting left and right products [52].

Having discusses the algebraic details, we go back to the work of Gunaydin et al [42] who have constructed the conformal (related to $SO(D, 2)$, "two times ") and quasi-conformal (related to $SO(D + 2, 4)$, "four times") realizations of Exceptional Lie groups based on the 3-grading and 5-grading decompositions of the noncompact groups $E_{7(7)}$ and $E_{8(8)}$ respectively. The **56** dim representation of $E_{7(7)}$ admits the 3-grading decomposition under the $E_{6(6)} \times \mathcal{D}(dilations)$ subgroup as

$$\mathbf{1} \oplus (\bar{\mathbf{27}} \oplus \mathbf{27}) \oplus \mathbf{1}. \tag{3.6}$$

The $E_{8(8)}$ admits the 5-grading decomposition under the $E_{7(7)} \times \mathcal{D}$ subgroup as :

$$\mathbf{1} \oplus \mathbf{56} \oplus (\mathbf{133} \oplus \mathbf{1}) \oplus \mathbf{56} \oplus \mathbf{1}. \tag{3.7}$$

The physical significance of this "mirror" symmetry in these graded decompositions is that one has a phase-space structure of coordinates and momenta with an underlying conformal (quasi-conformal) structure. In particular, a Dual-Phase Space Relativity in Clifford spaces has been constructed in [77], implementing the work of Max Born who many years ago suggested that , by duality of $q \leftrightarrow p$, there should be a maximal bound on the proper four-forces in Nature, acting on a $fundamental$ elementary particle, that one may assume to be $\mathcal{F} = m_{Planck} c^2 / L_{Planck}$, in the same vein there is a velocity bound given by the speed of light.

Gunaydin et al [42] have shown that there are *no* quadratic $E_{7(7)}$ invariants in the **56** representation but instead a real *quartic* invariant I_4 can be constructed by means of the Freudenthal *ternary* product $X \times Y \times Z \rightarrow W$ and a skew-symmetric bilinear form $< X, Y >$:

$$\begin{aligned} I_4 = \frac{1}{48} < (X, X, X),\ X > \ &= X^{ij} X_{jk} X^{kl} X_{li} - \frac{1}{4} X^{ij} X_{ij} X^{kl} X_{kl} + \\ &\frac{1}{96} \epsilon^{ijklmnpq} X_{ij} X_{kl} X_{mn} X_{pq} + \frac{1}{96} \epsilon_{ijklmnpq} X^{ij} X^{kl} X^{mn} X^{pq}. \end{aligned} \tag{3.8}$$

where the symplectic invariant of two **56** representations, like the area element in phase space $\int dp \wedge dq$, is given by :

$$< X, Y >= X^{ij} Y_{ij} - X_{ij} Y^{ij} \tag{3.9}$$

where the fundamental **56** dimensional representation of $E_{7(7)}$ is spanned by the anti-symmetric real tensors (bi-vectors) X^{ij}, X_{ij} built from $SL(8, R)$ indices $1 \leq i, j \leq 8$ so that $56 = 28 + 28$ since an $SL(8, R)$ bi-vector has 28 independent components.

The next step is to construct $E_{7(7)} \times U(N)$ invariants in the large N limit. This is straightforward once we follow the steps in the previous sections after defining the matrix coordinates $\mathcal{M}^A T_A = X^{ijA} T_A$ which take values in the Lie algebra $e_{7(7)} \times u(N)$. The quartic invariant reads

$$I_4 = \rho_{ABCD} \ [\ X^{ijA} X^B_{jk} X^{klC} X^D_{li} - \frac{1}{4} X^{ijA} X^B_{ij} X^{klC} X^D_{kl} +$$

$$\frac{1}{96} \epsilon^{ijklmnpq} X^A_{ij} X^B_{kl} X^C_{mn} X^D_{pq} + \frac{1}{96} \epsilon_{ijklmnpq} X^{ijA} X^{klB} X^{mnC} X^{pqD} \ ]. \qquad (3.10)$$

where $\rho_{ABCD} = f^E_{[AB} \ f_{CD]E}$ is fully antisymmetric and is built from the structure constants f_{ABC} of the $u(N)$ algebra.

The large N limit of the expression I_4 in (3-10) is not unique [8] and admits the following actions :

• Maps of bivector-valued world-volumes onto bivector-valued target spaces $X^{ij}(\sigma^{ab}), X_{ij}(\sigma^{ab})$:

$$S_8 = \int \ [\mathcal{D}\sigma^{ab}] \ [\ \{ \ X^{ij}, \ X_{jk}, \ X^{kl}, \ X_{li} \ \} \ - \ \frac{1}{4} \{ \ X^{ij}, \ X_{ij}, \ X^{kl}, \ X_{kl} \ \} +$$

$$\frac{1}{96} \epsilon^{ijklmnpq} \ \{ \ X_{ij}, \ X_{kl}, \ X_{mn}, \ X_{pq} \ \} \ +$$

$$\frac{1}{96} \epsilon_{ijklmnpq} \ \{ \ X^{ij}, \ X^{kl}, \ X^{mn}, \ X^{pq} \ \} \ ]. \qquad (3.11a)$$

where the Nambu-Poisson brackets are defined now with respect to the 28 *bivector* coordinates $\sigma^{ab} = -\sigma^{ba}$ associated with the 8 dim world-volume of the 7-brane

$$\{ \ X^{ij}, \ X_{jk}, \ X^{kl}, \ X_{li} \ \} \ =$$

$$\sum \ \Omega^{a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4} \ \frac{\partial X^{ij}}{\partial \sigma^{a_1 b_1}} \frac{\partial X_{jk}}{\partial \sigma^{a_2 b_2}} \frac{\partial X^{kl}}{\partial \sigma^{a_3 b_3}} \frac{\partial X_{li}}{\partial \sigma^{a_4 b_4}}. \qquad (3.12a)$$

since the indices run $a, b, c..... = 1, 2, 3...8$ there are 28 bivector σ^{ab} degrees of freedom. There are 28 bivector coordinates $X^{ij}(\sigma^{ab})$ and 28 bivector "conjugate momenta" coordinates $X_{ij}(\sigma^{ab}) = P_{ij}(\sigma^{ab})$. The total of $28 + 28 = 56$ "phase space" coordinates match the dimension of the $Fr[O]$ algebra . The four-*biforms* , like $dX^{ij} \wedge dX^{kl} \wedge dX^{mn} \wedge dX^{pq}$, etc.... in (3-11) capture the embeddings of the 8-dim world-volume of the 7-brane onto the C^{28} space (56 real-dim) . Bi-forms and multi-forms as generalizations of ordinary forms have been thoroughly studied in [81].

$\Omega^{a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4}$ is the *bivector*-analog of $\Omega^{abcd} = \omega^{ab}\omega^{cd} - \omega^{cb}\omega^{ad}$ which is defined in terms of the antisymmetric 8×8 matrix ω^{ab}, derived from the inverse of the non-degenerate symplectic form $\omega = \omega_{ab} \ d\sigma^a \wedge d\sigma^b$. Polyvector-valued Phase-spaces have been analyzed by [77]. Generalized actions of the type in (3-11) have been constructed in Clifford-spaces (C-spaces) based on polyvector-valued coordinates (tensorial spaces) in [76], [77]. A generalized supersymmetry in Clifford (polyvector-valued) superspaces has been advanced in [77]. Polyvector-valued super-Poincare algebras and their relationship with M, F theory superlagebras have been studied by [79]. Clifford-space extensions of the Standard Model (generalized Yang-Mills theories, tensorial-gauge theories) have been presented by [80] where new particles are expected to be found at the 10 Tev regime.

• Maps from an 8-dim world-volume onto bivector-valued target spaces $X^{ij}(\sigma^a), X_{ij}(\sigma^a)$:

$$S_8 = \int [\mathcal{D}\sigma^a] \ [\ \{ \ X^{ij}, \ X_{jk}, \ X^{kl}, \ X_{li} \ \} \ - \ \frac{1}{4}\{ \ X^{ij}, \ X_{ij}, \ X^{kl}, \ X_{kl} \ \} \ + \\ \frac{1}{96}\epsilon^{ijklmnpq} \ \{ \ X_{ij}, \ X_{kl}, \ X_{mn}, \ X_{pq} \ \} \ + \\ \frac{1}{96}\epsilon_{ijklmnpq} \ \{ \ X^{ij}, \ X^{kl}, \ X^{mn}, \ X^{pq} \ \} \ ]. \tag{3.11b}$$

where the Nambu-Poisson brackets are defined now with respect to the 8 coordinates σ^a associated with the 8 dim world-volume of the 7-brane

$$\{ \ X^{ij}, \ X_{jk}, \ X^{kl}, \ X_{li} \ \} \ = \\ \sum \ \Omega^{a_1 a_2 a_3 a_4} \ \frac{\partial X^{ij}}{\partial \sigma^{a_1}} \frac{\partial X_{jk}}{\partial \sigma^{a_2}} \frac{\partial X^{kl}}{\partial \sigma^{a_3}} \frac{\partial X_{li}}{\partial \sigma^{a_4}}. \tag{3.12b}$$

therefore, the 8-dim action (3-11b) describes the global dynamics of a 7-brane, embedded in 28-complex dim (56 real dim) corresponding to a complexified bosonic F theory. Identical results can be are attained when the coordinates X^{ij}, X_{ij} belong to the $Fr[C \times O]$ algebra of 4×28 real-dimensions. In this case one has the quaternionic version of the bosonic F theory. The connection between F theory and Jordan algebras of degree four $J_4[H]$ have been described by Smith [53].

• Maps from 8-dim world-volume onto an 8-dim target space $X^i(\sigma^a)$

The action (3-11a) is the *bivector partner action* of the following 8-dim action that is associated with an 8-dim non-linear sigma model described by the maps $X^i(\sigma^a)$ from an 8-dim world volume described by the coordinates $\sigma^a, \ a = 1, 2, 3, ...8$ to an 8-dim target space background described by the coordinates $X^i, \ i = 1, 2, 3, ...8$:

$$S_8 = \int d^8\sigma \ [\ \{X^i, X^j\}\{X_j, X_k\}\{X^k, X^l\}\{X_l, X_i\} \ - \\ \frac{1}{4}\{X^i, X^j\}\{X_i, X_j\}\{X^k, X^l\}\{X_k, X_l\} \ + \\ \frac{1}{96}\epsilon^{ijklmnpq}\{X_i, X_j\}\{X_k, X_l\}\{X_m, X_n\}\{X_p, X_q\} \ + \\ \frac{1}{96}\epsilon_{ijklmnpq}\{X^i, X^j\}\{X^k, X^l\}\{X^m, X^n\}\{X^p, X^q\} \ ]. \tag{3.13}$$

where $\{X, Y\}$ are the ordinary Poisson brackets

$$\{X, Y\} = \omega^{ab} \ \partial_a X \ \partial_b Y \tag{3.14}$$

defined in terms of the antisymmetric 8×8 matrix ω^{ab} that is derived from the inverse of the non-degenerate symplectic form $\omega = \omega_{ab} \ d\sigma^a \wedge d\sigma^b$

The last two terms of (3-13) resemble the theta terms in section **1** , but not the first two terms which require a "metric" to raise and lower indices. The latter terms resemble (up to numerical factors) those terms present in the class of Dolan-Tchrakian [27] conformally invariant p-brane actions (when $p+1 = 2n=$ even) after the conformal gauge has been chosen. The bulk terms of the action have no local dynamics since there are *no* transverse degrees of freedom $8-8=0$. However, there are still non-trivial *boundary* dynamics if $\partial\Omega^{(8)} = \Sigma^{(7)}$.

Since the Nambu-Poisson brackets can be written as

$$\{X^i, X^j, X^k, ..., X^q\}_{NPB} = \{X^i, X^j\}\{X^k, X^l\}\{X^m, X^n\}\{X^p, X^q\} \ \pm$$

$$\textit{signed permutations}. \tag{3.15}$$

This fact allows us to integrate the last two terms of the action (3-13) giving $S_8 = S_{bulk} + S_{boundary}$ where

$$S(boundary) \ \sim$$

$$\int [d^7\Sigma]_a \ \epsilon^{ijklmnpq} \ \epsilon^{abcdefgh} X_i \ \partial_b X_j \ \partial_c X_k \ \partial_d X_l \ \partial_e X_m \ \partial_f X_n \ \partial_g X_p \ \partial_h X_q. \tag{3.16}$$

The boundary term belongs to the same class of Chern-Simons brane actions (higher-dimensional "knots") described by Zaikov [1]. with an underlying W_∞ symmetry structure in the case of a Chern-Simons string whose world volume is the two-dimensional boundary of a three-dim region.

Concluding, the action (3-11a), which is the *bivector partner* of the action (3-13), in conjunction with the ordinary action (3-13) and the F theory action in eq-(3-11b), can *all* be embedded into more fundamental and generalized p-brane actions in Clifford-spaces [76], [77], defined as maps of polyvector-valued world-volumes onto polyvector-valued target spaces. Namely these actions are based on Clifford-valued hyper-complex maps and describe the unified dynamics of many different p-branes of different dimensionalities on equal footing.

Having discussed the action (3-13) and its bivector-partner (3-11a) action, and the bosonic (complexified) F theory action (3-11b), we finalize this section by explaining how one could built generalized exceptional non-linear sigma models on curved backgrounds. Gunaydin et al [42] proceeded to built a quasi-conformal nonlinear realization of $E_{8(8)}$ based on the 5-grading decomposition w.r.t the subgroup $E_{8(8)} \times \mathcal{D}$. Namely, one may exhibit a nonlinear realization of $E_{8(8)}$ on the $1+56=57$-dim real vector space with coordinates living in the $Fr[O] \oplus R$ algebra and given by $\mathcal{X} = (X^{ij}, X_{ij}, x)$. The $\mathcal{X}$ forms the $\mathbf{56} \oplus \mathbf{1}$ representation of $E_{7(7)}$. It can be shown that there is a *quartic* invariant (a "light-cone" in 57 dim) under the action of the $E_{8(8)}$ group given by :

$$\mathcal{N}_4 = I_4(X^{ij}, X_{ij}) \ - x^2. \tag{3.17}$$

The displacement in the 57-dim generalized spacetime is defined by the coordinates :

$$\delta(X,Y) = (X^{ij} - Y^{ij},\ X_{ij} - Y_{ij},\ x - y \ + \ < X, Y > \ ) = (Z^{ij}, Z_{ij}, z) \tag{3.18}$$

the "light-cone" in 57-dim invariant under $E_{8(8)}$ is defined by

$$\mathcal{N}_4 \ [\delta(X,Y)] = I_4(Z^{ij}, Z_{ij}) \ - z^2 = 0. \tag{3.19}$$

This geometrical expression is remarkable similar to the light-cone in Clifford-spaces ! [76], [77].

The construction of a quartic $E_{8(8)}$-invariant in $D = 57$ dim, allows also to built generalized non-linear sigma models [27], [25] on curved 56-dim backgrounds defined by the constraint in a 57-dim space

$$\mathcal{N}_4 \ [\delta(X,Y)] = I_4(Z^{ij}, Z_{ij}) \ - z^2 = R^2. \tag{3.20}$$

this constraint is just the analog of the definition of hyperboloids $\mathcal{H}^n$ as noncompact hypersurfaces embedded in flat R^{n+1} pseudo-Ecuclidean spaces; like de Sitter and Anti de Sitter spaces dS_n, AdS_n. The poly-disc $\mathcal{D}_n = SO(n,2)/SO(n) \times SO(2)$ with n complex-dim ($2n$ real dimensions) is the curved phase space corresponding to the dynamics of particles moving in AdS_n. The Shilov-boundary of the poly-disc is n real-dim and has the same topology as the compactified Minkowski spacetime $S^{n-1} \times RP^1$. The Geometry and Topology of these symmetric spaces (and their Shilov boundaries) has been instrumental in the derivation of the observed values of the coupling constants of the Electromagnetic, Weak, Strong and Gravitational forces, measured at four very specific scales corresponding to the Bohr radius, the Z boson mass, the pion π mass and the Planck mass, respectively [53], [20].

To finalize this section, we should add that a 7-graded decomposition of the E_8 algebra has been provided by Larsson that reflects the underlying Clifford algebraic structures of $Cl(8), Cl(6)$ behind E_8 [53] and which will permit us to embed *all* of these models discussed in this work within the C-space-branes construction of [76].

4 Concluding Remarks : Future Projects

The seminal work by [54] on representation theory and the Exceptional Projective Geometry of points, lines, planes and symplecta; the Magic-square construction of Freudenthal-Tits [58]; the triality and **4**-ality of $SO(4,4)$; del Pezzo surfaces, Severi varieties and knots; sextonions and the missing exceptional $E_{7_{1/2}}$, etc ... contains the rigorous mathematical foundations to explore deeper the results of this work.

A thorough construction of the exceptional gauge symmetries and dynamics of bound states comprised of $D0-D8$ branes using exceptional Jordan algebras has been attained by [53] within the context of a Clifford-group geometric unification of forces that was based on a $SO(8)-SO(10)-E_6-E_7-E_8$ model. A more recent discussion on the geometry of Exceptional Matrix Models and $D0, D8$-branes has been presented by [41].

Beyond the topological aspects of Chern-Simons foliations described here, it is warranted to investigate the local dynamics of higher-dim non-linear sigma models defined on exceptional group manifolds and coset spaces; i.e to study the propagation of p-branes in *generalized* spacetimes described by Jordan algebras and consistent with the conformal (quasi-conformal) group symmetries of Freudenthal-Kantor triple systems [42], [51]. Unified actions for all p-branes, for all values of p, have been displayed in Clifford-spaces [76], [77]. Index theorems in Clifford Modules and Nonholonomic Clifford Structures in Noncommutative Riemann–Finsler Geometry have been analyzed in detail by [69] .

Despite that some authors [53] interpret the triality symetry of $SO(8)$ as some manifestation of "supersymmetry", the supersymmetric extensions deserves further investigation. For instance, the topological G_2 String has been studied in great detail by [75] . Since the number of super-Jordan algebras is extremely vast it makes this project a very difficult and arduous one. In particular, Non-associative $N=8$ superconformal algebras have been investigated by [74] . There is always the issue of local and global anomalies that will select certain theories over others. The fact that $D=27,28,56,112,224$ dimensions are essential ingredients of this work suggests that there may be many more consistent and anomaly free string theories in higher dimensions left to be explored and whose compactifications should yield a very large variety of symmetry groups.

A generalization of a determinant for matrix elements of non-associative Jordan algebra has been provided by Freudenthal $detX = \frac{1}{3}(X,X,X)$ in terms of the cubic-form . Despite the non-associativity of octonions that precludes the ordinary definition of a determinant, another interesting possibility to explore is to write the *cubic* matrix X^{ABC} of $3\times3\times3=27$ entries that matches precisely the number of 27 independent components of the Jordan 3×3 hermitian matrices belonging to $J_3[O]$ algebra, and whose hyper-determinant is :

$$Det\ \mathbf{X} \sim \epsilon^{A_1A_2A_3}\ \epsilon^{B_1B_2B_3}\ \epsilon^{C_1C_2C_3}\ X_{A_1B_1C_1}\ X_{A_2B_2C_2}\ X_{A_3B_3C_3}. \tag{4.1}$$

one could then construct a generalization of the Dirac-Nambu-Goto membrane action :

$$S = \int d^3\sigma\ [\ |Det\ \mathbf{H}|\ ]^{1/3}. \tag{4.2}$$

where the hyper-metric $\mathbf{H}$ represented by the $3\times3\times3$ hyper-matrix (cubic matrix) H_{abc} is defined as the pullback of $H_{\mu_1\mu_2\mu_3}$

$$H_{abc} = H_{\mu_1\mu_2\mu_3}\ \partial_a X^{\mu_1}\ \partial_b X^{\mu_1}\ \partial_c X^{\mu_1}. \tag{4.3}$$

and the Finslerian-like space-time interval is of the form :

$$(ds)^2 = [\ H_{\mu_1\mu_2\mu_3}\ dx^{\mu_1} dx^{\mu_2} dx^{\mu_3}\ ]^{2/3}. \tag{4.4}$$

Finslerian-like geometries are related to W_N geometries . The Exceptional (magical) Jordan algebras $J_3[R, C, H, O]$ were instrumental in W_3, W_N gravity [63]. This Jordan algebras/ W_N geometry interplay has to be explored further within the context of W_∞ strings. We were able to show upon using a BRST analysis [25] that a nilpotent BRST charge operator $\mathcal{Q}^2 = 0$, associated with the non-critical W_∞ superstring, can be constructed by *adjoining* a $q = N + 1$ unitary superconformal model of the super W_N algebra, to a critical W_∞ superstring spectrum, in the $N = \infty$ limit. Therefore, we have an anomaly-free *noncritical* W_∞ superstring in $D = 11$. Similar BRST analysis followed for the bosonic *noncritical* W_∞ string , and we found that $D = 27$ was the required critical dimension of the target spacetime of the non-critical W_∞ string. Since $D = 11, 27$ are the dimensions, respectively, of the (alledgedly) anomaly-free (super) membranes, as was shown by Marquard, Scholl and Kaiser, this W_∞ (super) strings/(super) membranes connections warrants a further investigation Higher-spin theories have reached frontier research status nowadays [7].

Analog of Heterotic string compactifications on 16 dim lattices can also be achieved since Lattices based on *discrete* Jordan algebras have been studied by Gursey [44]. In the case of pure gravity in (curved) M^{27} spaces, one could as well perform a Kaluza-Klein compactification of $M^{27} \rightarrow M^{11} \times N^{16}$ along the internal spaces $(R \times O)P^2, (C \times O)P^2, \ldots..$ and generate the F_4, E_6, E_7, E_8 *local* gauge symmetries resulting from the *isometries* of the respective projective planes. For example, the isometries of $(O \times O)P^2$ is E_8 so one will end up in this case with an Einstein-Yang-Mills-Higgs theory in M^{11} based on the local E_8 gauge symmetry group. Grand Unified Theories based on E_8 go back to Gursey [44], [11], [12] .

Contrary to the standard lore that is not possible to obtain the $SU(3) \times SU(2) \times U(1)$ gauge field structure from a Kaluza-Klein framework in $D = 8$, Batakis [10] uncovered an extra $SU(2) \times U(1)$ gauge field structure to the $SU(3)$ gauge field stucture from a Kaluza-Klein mechanism in $M^4 \times CP^2$, provided a nontrivial *torsion* in the total space is incorporated. Such torsion creates a new and nontrivial possibility for the accomodation of a fully unified theory in $D = 8$ not envisioned before in the physics literature. Clifford spaces have torsion [76]. The important results of Smith [53] based on Clifford algebraic structures in $D = 8$ have also uncovered a CP^2 internal space. CP^2 internal spaces are also essential ingredients in the work of [49].

Gursey was among the first to realize the importance of Exceptional Groups and Division Algebras in Physics [62]; see the monographs by [61], [55], [56]. In this work we have seen the relevance of complex, quaternionic and octonionic spaces within the context of Chern-Simons global dynamics of membranes and 7-branes. Catto [59] has pointed out the interplay among real-analyticity, complex-analyticity and quaternionic-analyticity with point-particle world lines, string worldsheets and membrane world-volumes, respectively. In particular, how four-dim conformal field theories can also be formulated on Kulkarni four-folds leading to a formalism similar to that of 2 dim conformal field theory on Riemann surfaces [60] where the notion of Fueter quaternionic analyticity (versus complex analyticity) plays an essential role. A quaternionic formulation of the $D = 4$ conformal group and its association with twistors, based on quaternionic analyticity, has also been investigated by [59].

The notion (and applications) of Octonionic analyticity and the corresponding construction of Octonionic twistors remains yet to be developed and analyzed, to our knowledge. Non-associative Octonionic gravity and Octonionic Yang-Mills theories have been constructed by [71] and [46]. More recently an octonionic Ashtekar formulation of gravity was presented by [73]. Octonionic geometries have been studied to develop Octonionic Hilbert spaces [62] that describe the colored quark states and provide a geometrical interpretation of confinement; i.e due to the non-Desargues property of Octonionic Geometry it makes them non-embeddeble in higher-dim spaces so these Octonionic Hilbert spaces are finite dimensional and comprised of color-singlets. For algebraic theories of confinement and Z_3 graded extensions/generalizations of ordinary supersymmetry (Z_2 graded Lie algebras) coined Hypersymmetry, and based on *ternary* algebras see [50]. A lot of work remains to be done. We hope that the present work has been useful in the advancement of the prior work [37], [40], [43], [57], [62]

Acknowledgements

We are indebted to M. Bowers for invaluable assistance in preparing the manuscript. Special thanks to T. Smith and M. Pitkanen for numerous discussions of their work.

References

[1] R.P Zaikov, Phys. Letts **B 266** (1991) 303. Phys. Letts **B 263** (1991) 209 . R.P Zaikov, Phys. Letts **B 211** (1988) 281. R.P Zaikov : "Chern-Simons p-Branes and p-dimensional Classical W Algebras " [arXiv : hep-th/9304075].

[2] J. Hoppe, " Quantum Theory of a Relativistic Surface" Ph.D Thesis MIT (1982).

[3] A.B. Zamoldchikov, Teor. Fiz **65** (1985) 347.

[4] C. Pope, L. Romans, X. Shen, Nuc. Phys, **B 339** (1990) 191. C. Pope, L. Romans, X. Shen, Phys. Letts **B 236** (1990) 173. C. Pope, L. Romans, X. Shen, Phys. Letts **B 242** (1990) 401.

[5] E. Sezgin, " Area-preserving Diffs, w_∞ algebras , w_∞ gravity " [arXiv : hep-th/9202080]. E. Sezgin : " Aspects of W_∞ Symmetry " [arXiv : hep-th/9112025] .

[6] P. Bouwknegt, K. Schouetens, Phys. Reports **223** (1993) 183-276.

[7] M. Vasiliev, " Higher Spin Gauge Theories, Star Product and AdS spaces " [arXiv : hep-th/9910096] . M. Vasiliev , S. Prokushkin, "$3D$ Higher-Spin Gauge Theories with Matter". [arXiv : hep-th/9812242, hep-th/9806236] .

[8] S. Ansoldi, C. Castro, E. Spallucci, Phys. Lett **B 504** (2001) 174. S. Ansoldi, C. Castro, E. Spallucci, Class. Quant. Grav. **18** (2001) L23. C. Castro, " Branes from Moyal Deformation quantization of Generalized Yang-Mills " [arXiv : hep-th/9908115]. S. Ansoldi, C. Castro, E. Spallucci, Class. Quan. Gravity **18** (2001) L17-L23.

[9] S. Ansoldi, C. Castro, E. Guendelmann and E. Spallucci, Class. Quant. Gravity **19** (2002) L 135.

[10] N. Batakis, Class. Quant. Gravity **3** (1986) L 99.

[11] , N. Baaklini, Phys. Lett **B 91** (1980) 376.

[12] I. Bars and M. Gunaydin, Phys. Rev. Lett **45** , no. 11 (1980) 859.

[13] S. Korden, " W algebras from the AdS/CFT correspondence " [arXiv : hep-th/0011292].

[14] M. Fukuma, S. Matsuura, T. Sakai, " Higher Derivative Gravity and the AdS/CFT correspondence " [arXiv : hep-th/0103187].

[15] E Sezgin, P. Sundell, "Higher Spin $N = 8$ Supergravity in AdS_4" [arXiv : hep-th/9805125; hep-th/9903020] .

[16] I. Bakas. B. Khesin. E. Kiritsis, Comm. Math . Phy **151** (1993) 233. I. Bakas, E. Kiritsis, " Comon Trends in Mathematics and Physics " Eds. T. Eguchi, ...Prog. Theor. Physics Suppl. (1990)

[17] E. Bergshoeff, P. Howe, C. Pope, L. Romans, E. Sezgin , X. Shen and K. Stelle , " Quantization Deforms w_∞ to W_∞ Gravity " .

[18] C.Pope, "$SL(\infty, R)$ Symmetry of W_∞ Gravity" [arXiv : hep-th/9201023].

[19] I. Oda, Chaos, Solitons and Fractals **10** , no. 2-3 (1999) 483.

[20] C. Castro, Progress in Physics **2** (2005) 30.

[21] C. Castro, Jour. Math. Phys. **34** (1993) 681.

[22] C. Castro, Phys Lets **B 288** (1992) 291.

[23] C. Castro, Jour. Geometry and Physics **33** (2000) 173.

[24] C. Castro : Jour. Chaos, Solitons and Fractals **7** (5) (1996) 711.

[25] C. Castro, Class. Quant. Gravity **20** (16) (2003) 3577-3592. Europhysics Letters **61** (4) (2003) 480-484.

[26] C. Castro, Mod. Phys. **A 17** (2002) 2095-2103.

[27] B. Dolan, D.H Tchrakian, Phys. Lett **B 198** (1987) 447. Phys. Lett **B 202** (1988) 211.

[28] Q.H. Park : Int. Jour. Mod. Phys **A 7** (1992) 1415.

[29] H. Garcia-Compean, J. Plebanski, M. Przanowski, "Geometry associated with SDYM and chiral approaches to Self Dual Gravity" [arXiv : hep-th/9702046] .

[30] M. Calixto : "Higher $U(2,2)$ spin fields and higher-dim W gravities : quantum AdS space and radiation phenomea " [arXiv : hep-th/0102111].

[31] A. Jevicki and B. Sakita, Nuc. Phys. **B 165** (1980) 511. J. Avan and A. Jevicki, Phys. Letts **B 266** (1991) 35.

[32] A. Jevicki, " Matrix Models, Open Strings and Quantization of Membranes " [arXiv : hep-th/9607187].

[33] F. Yu and Y.S Wu , " Nonlinear W_∞ current algrbra in the $SL(2,R)/U(1)$ Coset model " University of Utah preprint-91-19. I.Bakas and E.Kiritsis , Int. J. Mod. Phys **A7** (1992) 55-81.

[34] E Nissimov, S. Pacheva , I. Vaysburd, "W_∞ Gravity, a Geometric Approach ". [arXiv : hep-th/9207048].

[35] G. Chapline, K. Yamagishi, Class. Quan. Gravity **8** (1991) 427.

[36] T. Branson, R. P. Lano and V.G.J Rodgers, " Yang-Mills, Gravity and String Symmetries " [arXiv : hep-th/9610023].

[37] T. Banks, W. Fischler, S. Shenker and L. Susskind, Phys. Rev **D 55** (1997) 5112. R. Dijkgraaf, E. Verlinde and H. Verlinde, " Matrix String Theory" hep-th/9703030.

[38] Y. Ohwashi, " E_6 Matrix Model " hep-th/0110106

[39] L. Smolin, " The exceptional Jordan Algebra and the Matrix String" hep-th/0104050

[40] G. Horowtitz, L. Susskind, " J. Math. Phys **42** (2001) 3152.

[41] M. Rios, " The Geometry of Jordan Matrix Models" hep-th/0503015.

[42] M. Gunaydin and O. Pavlyk, " Generalized spacetimes defined by cubic forms and the minimal unitary realzations of their quasi-conformal groups" hep-th/0506010. M. Gunaydin, K. Koepsell, and H. Nicolai, Comm. Math. Phys **221** (2001) 57. Adv. Teor. Math. Phys **5** (2002) 923.

[43] M. Gunaydin, G. Sierra and P. Townsend, Nuc. Phys. **B 242** (1984) 244. Phys. Letts **B 133** (1983) 72.

[44] F. Gursey, Mod. Phys. Letts **A 3** (1988) 115. F. Gursey, Proc. Conf. on Group theorertical methods in Physics 1978.

[45] G. Sierra, Class. Quan. Grav. **4** (1987) 227. M. Cederwall, Phys. Letts **B 210** (1988) 169. E. Corrigan and T. Hollowood, Comm. Math. Phys. **122** (1989) 393.

[46] R. Foot and G. Joshi, Phys. Rev. **D 36** (1987) 1169. Phys. Letts **B 199** (1987) 203. Int. Jour. Theor. Phys **28** (1988) 1449.

[47] P. Jordan, J von Neumann and E. Wigner, Ann. Math **35** (1934) 2964. K. MacCrimmon, "A Taste of Jordan Algebras" H. Freudenthal, Nederl. Akad. Wetensch. Proc. Ser **57 A** (1954) 218. J. Tits, Nederl. Akad. Wetensch. Proc. Ser **65 A** (1962) 530. T. Springer, Nederl. Akad. Wetensch. Proc. Ser **65 A** (1962) 259.

[48] R. Brown, J. Reine Angew Math **236** (1969) 79. K. Meyberg, Nederl. Akad. Wetensch. Proc. Ser **71 A** (1968). J. Faulkner, Trans. Am. Math. Soc. **155** (1971) 397.

[49] M. Pikannen, "Topological Geometry Dynamics" http://www.physics.helsinki.fi/ matpitka/tgd.html

[50] R. Kerner, " Ternary Algebriac Structures and their applications in Physics" math-ph/0011023.

[51] J. Palmkvist, " A realization of the Lie Algebra associated to a Kantor Triple Systems " math.RA/0504544.

[52] J. Schray and C. Manogue, "Octonionic representations of Clifford algebras and triality " hep-th/9407179.

[53] F.D. Smith Jr, "E_6, Strings, Branes and the Standrad Model" [CERN CDS EXT-2004-031]. Int. J. Theor. Phys **24** , 155 (1985); Int. J. Theor. Phys **25** , 355 (1985) . "From Sets to Quarks" [arXiv : hep-ph/9708379] . "The $D_4 - D_5 - E_6 - E_7 - E_8$ Model" [CERN CDS EXT-2003-087] .

[54] J. Landsberg and L. Manivel, " Representation Theory and Projective Geometry" math.AG/0203260. "The sextenions and $E_{71/2}$" math.AG/0107032. B. Westbury, " Sextenions and the Magic Square"

[55] J. Adams, " Lectures on Exceptional Lie Groups" Chicago Lectures in Mathematics, Univ. of Chicago Press 1996.

[56] C. H Tze and F. Gursey, "On the role of Divison, Jordan and Related Algebras in Particle Physics" World Scientific 1996. S. Okubo, Introduction to Octonion and other Nonassociative Algebras in Physics" Cambridge Univ. Press R. Schafer, " An introduction to Nonassociative Algebras" Academic Press, New York 1966.

[57] P. Ramond, Exceptional Groups and Physics, hep-th/0301050.

[58] J. Baez, " Bull. Amer. Math. Soc (N. S) **39** no. 2 (2002) 145.

[59] S. Catto, " Exceptional Projective Geometries and Internal Symmetries" hep-th/0302079. "Self-Dual Fields and Quaternion Analyticity" hep-th/0302139.

[60] R. Zucchini, J. Geom. Phys **27** (1998) 113. R. Kulkarni, J. Diff. Geom **13** (1978) 109/

[61] G. Dixon, " Division Algebras, Octonions, Quaternions, Complex Numbers, and the Algebraic Design of Physics" (Kluwer, Dordrecht, 1994). J. Math. Phys **45** , no 10 (2004) 3678.

[62] , M. Gunaydin, J. Math. Phys **17** , no. 10 (1976) 1875.

[63] L. Romans, Nuc. Phys. **B 352** (1991) 829. C. Hull, Nuc. Phys. **B 353** (1991) 707.

[64] C. Hull and P. Townsend, Nuc. Phys **B 438** (1995) 109. E. Witten, Nuc. Phys **B 443** (1985) 85 .

[65] E. Witten, Comm. Math. Phys **92** (1986) 455. Comm. Math. Phys **117** (1988) 353. Comm. Math. Phys **121** (1989) 35. Nuc. Phys. **B 311** (1988) 46

[66] P. Horava, E. Witten, " Heterotic and type I String Dynamics from eleven Dimensions " hep-th/9510209.

[67] I. Zois , "Multiple "parallel" D-branes seen as leaves of foliations and Duminy's theorem" hep-th/0011228. " On search for the M-Theory Lagrangian" hep-th/9703067. "Foliations with nonvanishing GV-class and gauge invariance" hep-th/0303158. "Gelfand-Fuchs cohomology and M-Theory as a Topological Quantum Field Theory" hep-th/9906045.

[68] J. Polchinski, Phys. Rev. Lett **75** (1995) 4724. " String Theory" vol I, II , (Cambridge Univ. Press 1998). C. Johnson, " D-branes " (Cambridge Univ. Press 2003) R. Szabo, " An introduction to String Theory and D-brane dynamics" hep-th/0207142.

[69] S. Vacaru, J.F Gonzalez-Hernandez, "Nonlinear Connections on Gerbes, Clifford Modules, and the Index Theorems" math.DG/0506437. S. Vacaru, "Clifford Algebroids and Nonholonomic Einstein–Dirac Structures" hep-th/0501217. S. Vacaru, "Nonholonomic Clifford Structures and Noncommutative Riemann–Finsler Geometry" math.DG/0408121.

[70] C. Vafa, Nucl.Phys. B469 (1996) 403-418.

[71] S. Marques, C. Oliveira, Phys. Rev. **D 36** (1987) 1716.

[72] A. Waldron, G. Joshi, " Gauging the Octonionic Algebra" hep-th/9211123. A. Ritza, G. Joshi, " A Non-associative Deformation of Yang-Mills Gauge Theory" hep-th/9508147.

[73] A. Nieto, Class. Quan. Grav **22** (2005) 947. " Ashtekar Formalims and Two Times " hep-th/0506253.

[74] Z. Hasiewicz, F. Defever and W. Troost, J. Math. Phys **32** , no. 9 (1991) 2285.

[75] J. de Boer, A. Naqvi and A. Shomer "The Topological G_2 String" hep-th/0506211.

[76] C. Castro, M. Pavsic, Progress in Physics **1**, 31 (2005). Phys. Lett **B 559**, 74 (2003). Int. J. Theor. Phys **42**, 1693-1705 (2003)

[77] C. Castro, "The Extended Relativity Theory in Born-Clifford Phase Spaces with a Lower and Upper Length Scales and Clifford Group Geometric Unification", to appear in *Foundations of Physics* **35** , no. 6 (2005). Progress in Physics **1** (2005) 20. C. Castro, "Polyvector Super Poincare Algebras, M, F theory algebras and Generalized Supersymmetry in Clifford Spaces" submitted to the Int. Journal of Mod. Phys. **A** , February 2005.

[78] C. Castro : "Moyal deformations of Gravity via $SU(\infty)$ Gauge Theories , Branes and Topological Chern-Simons Matrix Models" General Relativity and Gravitation, **36** no.12 (2004) 2605-2634

[79] D. Alekseevsky, V. Cortes, C. Devchand and A. Van Proeyen " Polyvector Super-Poincare Algebras " hep-th/0311107. I. Rudychev, E. Sezgin, " Superparticles, p-form Coordinates and the BPS Condition" hep-th/9711128. I. Bars, C. Kounnas, "A new supersymmetry" hep-th/9612119. I. Bandos, J. Lukierski, "Generalized Superconformal Symmetries and Supertwistor Dynamics" hep-th/9912264.

[80] C.Castro, "On Generalized Yang-Mills Theories and Extensions of the Standard Model in Clifford (Tensorial) Spaces", submitted to Annals of Physics.

[81] P. de Madeiros, "Massive gauge-invariant field theories on spaces of constant curvature " hep-th/0311254.