NONCOMMUTATIVE BRANES IN CLIFFORD-SPACE BACKGROUNDS AND MOYAL-YANG STAR PRODUCTS WITH UV-IR CUTOFFS

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Abstract

A novel Moyal-Yang star product deformation of generalized p-brane actions in Clifford-space target backgrounds involving *multivectors* (polyvectors, antisymmetric tensors) valued coordinates is constructed based on the novel Moyal-Yang star product deformations of Generalized-Yang-Mills theories. This novel Moyal-Yang star product requires the use of the Noncommutative Yang's spacetime algebra involving a lower (ultraviolet) scale λ (Planck's scale) and an upper (infrared) scale R simultaneously. The "classical " $\hbar_{eff} = (\hbar\lambda/R) \rightarrow 0$ limit of the Moyal-Yang star products leads naturally to *Noncommutative* p-branes actions in Clifford-space target backgrounds which are associated to the *Noncommutative* "classical" p-brane dynamics described by nontrivial (nonzero) Poisson brackets among the multivector-valued coordinates and multivector-valued momenta of the p-branes moving in Clifford spaces. We derived in previous work, from first principles, why the observed value of the vacuum energy density (cosmological constant) is given by a geometric mean relationship $\rho \sim L_{Planck}^{-2} R^{-2} = L_P^{-4} (L_{Planck}/R)^2 \sim 10^{-122} M_{Planck}^4$ and can be obtained when the infrared scale R is set to be of the order of the present value of the Hubble radius. A *Noncommutative* **QFT** in Clifford spaces (devoid of ultraviolet and infrared divergences) involving both an upper R (infrared) and lower (ultraviolet) scale λ remains to be developed further in order to study in full the physical applications of these *Noncommutative* p-branes living in Clifford space target backgrounds.

1. Introduction

In recent years we have argued that the underlying fundamental physical principle behind string theory, not unlike the principle of equivalence and general covariance in Einstein's general relativity, might well be related to the existence of an invariant minimal length scale (Planck scale) attainable in nature. A scale relativistic theory involving spacetime *resolutions* was developed long ago by Nottale where the Planck scale was postulated as the minimum observer independent invariant resolution [43] in Nature. Since "points" cannot be observed physically with an ultimate resolution, they are fuzzy and smeared out into fuzzy balls of Planck radius of arbitrary dimension. For this reason one must construct a theory that includes all dimensions (and signatures) on the equal footing. Because the notion of dimension is a topological invariant, and the concept of a fixed dimension is lost due to the fuzzy nature of points, dimensions are resolution-dependent, one must also include a theory with *all* topologies as well. It is our belief that this may lead to the proper formulation of string and M theory.

In [15] we applied this Extended Scale Relativity principle to the quantum mechanics of p-branes which led to the construction of C-space (a dimension *category*) where all p-branes were taken to be on the same footing; i.e. transformations in C-space reshuffled a string history for a five-brane history, a membrane history for a string history, for example. It turned out that Clifford algebras contained the appropriate algebro-geometric features to implement this principle of polydimensional transformations [15, 16, 17, 18]. Clifford algebras have been a very useful tool for a description of geometry and physics [39]. In [17,18] it was proposed that every physical quantity is in fact a *polyvector*, that is, a Clifford number or a Clifford aggregate. Also, spinors are the members of left or right minimal ideals of Clifford algebra, which may provide the framework for a deeper understanding of sypersymmetries, i.e., the transformations relating bosons and fermions. The Fock-Stueckelberg theory of a relativistic particle [17] can be embedded in the Clifford algebra of spacetime [3]. Many important aspects of Clifford algebra are described in [39]

Using these methods the bosonic p-brane propagator, in the quenched mini superspace approximation, was constructed in [33]; the logarithmic corrections to the black hole entropy based on the geometry of Clifford space (in short C-space) were obtained in [35]; the action for a higher derivative gravity with *torsion* was obtained directly from the geometry of C-spaces [34] and how the Conformal agebra of spacetime emerges also from the Clifford algebra was described in [40]; the resolution of the ordering ambiguities of QFT in curved spaces was resolved by [17].

In this new physical theory the arena for physics is no longer the ordinary spacetime, but a more general manifold of Clifford algebra valued objects, polyvectors. Such a manifold has been called a pan-dimensional continuum [18] or C-space [15]. The latter describes on a unified basis the objects of various dimensionality: not only points, but also closed lines, surfaces, volumes,.., called 0-loops (points), 1-loops (closed strings) 2-loops (closed membranes), 3-loops, etc.. It is a sort of a *dimension* category, where the role of functorial maps is played by C-space transformations which reshuffles a p-brane history for a p'-brane history or a mixture of all of them, for example.

The above geometric objects may be considered as to corresponding to the well-known physical objects, namely closed p-branes. Technically those transformations in C-space that reshuffle objects of different dimensions are generalizations of the ordinary Lorentz transformations to C-space. In that sense, the C-space is roughly speaking a sort of generalized Penrose-Twistor space from which the ordinary spacetime is a *derived* concept. In [15] we derived the minimal length uncertainty relations as well as the full blown uncertainty relations due to the contributions of *all* branes of *every* dimensionality, ranging from $p = 0$ all the way to $p = \infty$. For further details of the Extended Relativity Theory in Clifford spaces we refer to the review [16] .

Another current important applications of multivectors, (polyvectors) in Physics is the work on Polyvector Super-Poincare Algebras and its relation to the M-theory superalgebra [55] and the formulations of Higher Spin Theories based on twistor-particle dynamics in Tensorial spaces [54]. The role of enlarged superspace coordinates in the context of super p-branes, Born-Infeld and M-theory has recently been investigated by [56] . Clifford Spaces are more fundamental than these tensorial spaces.

This work is organized as follows. Section **2** provides the physical motivation for this work that is based on two different ways to view the physics of branes. One view is to interpret them as composite antisymmetric tensor field field theories possessing an infinite-dimensional group of volume-preserving diffeomorphisms (of the target space of the scalar primitive field constituents) . The other view is related to the large N limit of $SU(N)$ Yang-Mills theories that is tantamount to the "classical " $\hbar = 2\pi/N \rightarrow 0$ limit of the Moyal deformations of (Generalized) Yang-Mills theories.

In **2.1** we will summarize the construction of p'-brane solutions to the rank $p + 1$ composite antisymmetric tensor field theories [2] developed by Guendelman, Nissimov and Pacheva [1] when the condition $D = p + p' + 2$ is satisfied. These field theories posess an infinite-dimensional group of volume-preserving diffeomorphisms of the target space of the scalar primitive field constituents. In section **2.2** we reviewed the interplay between ordinary brane actions in ordinary target spacetime backgrounds and the Moyal deformation quantization of (Generalized) Yang-Mills theories, in the quenched-reduced approximation. Brane actions from Moyal deformations of $SU(N)$ Yang-Mills Theories are obtained. In particular, the connection between the large N limit of $SU(N)$ Yang-Mills (in the quenched-reduced approximation) and p-branes is displayed. New p-branes actions in terms of a new measure of integration via the introduction of auxiliary scalar fields are also studied that are also amenable to Moyal deformations. Section **2** provides the background material necessary for the remaining sections with contain the new results.

In **3.1,3.2** the basic features of the Extended Relativity Theory in Clifford-spaces (C-spaces) are briefly outlined that allowed us [16] to construct for the first time (to our knowledge) a *unified* action encompassing the dynamics of *all* closed p-branes of *different* dimensionality in Clifford spaces : namely, the generalized *master* brane action in Clifford-space target backgrounds is constructed involving *multivector* valued coordinates (antisymmeric tensorial coordinates, representing the holographic areas, volumes, hypervolumes degrees of freedom associated with the projections of the several p-brane world-volumes onto their embedding spacetimes). The latter unified master action of all closed p-branes (p-loops) is what we called the C-space Brane action.

Section **3.3** is *new* where we extend the discussion of composite antisymmetric tensor field theories of volume-preserving diffs in ordinary spaces to Clifford-spaces and provide the Clifford-valued field theory action associated with a Clifford-valued field, that is the generalization of a massless scalar field action to Clifford-spaces. In particular we study the implications of a brane-field theory duality in Clifford spaces. The material in **3.2, 3.3** will be revisited in **6** when we furnish the Noncommutative C-space branes actions based on Moyal-Yang star products in C-spaces.

Section **4** is essential to be able to define later on in section **6** the *novel* Moyal-Yang star products deformations (with ultraviolet and infrared cutoffs) of the Master-Brane actions in Clifford-space target backgrounds . In section **4** we derive the relationship among the Yang's $4D$ Noncommutative space-time algebra [21] (in terms of ordinary $8D$ phase space coordinates), the *holographic area coordinates* algebra of the C-space associated with a $6D$ Clifford algebra, and the *Euclideanized* AdS_5 spaces. The role of AdS_5 was instrumental in explaining the origins of an extra (infrared) scale R in conjunction to the (ultraviolet) Planck scale λ characteristic of C-spaces. Tanaka [23] gave the physical and mathematical derivation of the *discrete* spectra for the spatial coordinates and spatial momenta that yields a *minimum* length-scale λ (ultraviolet cutoff in energy) and a minimum momentum $p = \hbar/R$ (maximal length R, infrared cutoff) .

We will show why one of the most salient features of the results of section **4** is that it agrees with our previous findings of [25] where a *geometric mean* relationship was found from first principles among the vacuum energy density (cosmological constant) ρ_{vacuum} , the Planck area λ^2 and the AdS_4 throat size squared R^2 given by $(\rho_v)^{-1} = (\lambda)^2(R^2)$. By setting the infrared scale R equal to the Hubble radius horizon R_H and λ equal to the Planck scale one reproduces precisely the *observed* value of the vacuum energy density ! [25] : $\rho \sim L_{Planck}^{-2} R_H^{-2} = L_P^{-4}(L_{Planck}/R_H)^2 \sim 10^{-122} M_{Planck}^4$. Cosmological bounds on the vacuum energy density and the universe's entropy based on an upper and lower scale are being investigated nowadays within the framework of the the 'T Hooft-Susskind-Maldacena holographic hypothesis. A different approach to the cosmological constant based on exploiting the pseudo-Euclidean signatures of C-spaces and Jackiw's definition of the vacuum state has been discussed by Pavsic [17] .

In **5** we construct the *generalized* Yang's algebra in Clifford spaces involving multivector-valued coordinates and momenta of arbitrary grade and which yields generalized uncertainty relations consistent with a volume (hypervolume) *quantization* in units of the Planck scale. The *generalized* Yang's algebra in Clifford spaces is essential to define a Moyal-Yang star product and bracket whose deformation parameter is comprised now of 3 fundamental physical parameters $\hbar_{eff} = (\hbar\lambda/R)$. Finally, having constructed the generalized Yang's algebra in Clifford spaces, the crux of the last section **6** is to perform a novel Moyal-Yang Quantization of Generalized Yang-Mills theories in Clifford spaces, in the quenched-reduced approximation, that yields the sought-after star-product deformations of the brane actions in Clifford space target backgrounds based on multivector (polyvector, antisymmetric tensor) valued coordinates. It is shown that the *novel* Moyal-Yang star product deformations of brane actions in Clifford space backgrounds are *induced* from the Moyal-Yang star product deformation quantization of Generalized-Yang-Mills theories in Clifford spaces.

The "classical" limit $\hbar_{eff} \to 0$ is defined by taking $\lambda \to 0$, $R \to \infty$ but maintaining $\hbar = c = 1$ in natural units while implementing simultaneously the *double*-scaling limit $\lambda R = L^2 = finite$. Upon taking the $\hbar_{eff} \to 0$ limit, novel *Noncommutative* Brane actions in Clifford space target backgrounds are obtained which are associated with the *Noncommutative* "classical" brane dynamics described by nontrivial Poisson brackets among the multivector-valued coordinates and momenta of the branes in Clifford spaces. The latter Noncommutative Classical Mechanics has a natural correspondence to the Noncommutative Quantum Mechanics in C-spaces (when $\hbar_{eff} \neq 0$) described by the generalized Noncommutative Yang's algebra in Clifford-spaces constructed in section **5** . This is one of the most relevant findings of this work. A Noncommutative **QFT** in Clifford spaces (devoid of ultraviolet and infrared divergences) involving both an upper R (infrared) and lower (ultraviolet) scale λ remains to be developed.

2. Branes as Gauge Theories of Volume Preserving Diffeomorphisms

The purpose of this section is to discuss two different ways to view the physics of branes. One view is to interpret them as composite antisymmetric tensor field field theories possessing an infinite-dimensional group of volume-preserving diffeomorphisms (of the target space of the scalar primitive field constituents). The other view is related to the large N limit of $SU(N)$ Yang-Mills theories that is tantamount to the "classical " $\hbar = 2\pi/N \to 0$ limit of the Moyal deformations of (Generalized) Yang-Mills theories. This section provides with the physical motivations behind this work in the sense that Noncommutative branes in Clifford-space backgrounds are constructed based on a novel Moyal-Yang star product involving and UV (ultraviolet) and IR (infrared) scale cutoffs.

2.1 Branes as composite antisymmetric tensor field theories

In this section we will review the construction of p'-brane solutions to the rank $p+1$ composite antisymmetric tensor field theories [2] developed by Guendelman, Nissimov and Pacheva [1] when the condition $D = p+p'+2$ is satisfied. These field theories posess an infinite-dimensional group of volume-preserving diffeomorphisms of the target space of the scalar primitive field constituents. The role of local gauge symmetry is traded over to an infinite-dimensional *global* Noether symmetry of volume-preserving diffs. The study of the Ward identities for this infinite-dim global Noether symmetry to obtain non-perturbative information in the mini-QED models (the composite form of QED) was analysed in [1] .

The starting Lagrangian is defined [1,2]:

$$L = -\frac{1}{g^2} F^2_{\mu_1\mu_2\ldots\mu_{p+1}}. \quad F = dA = \epsilon_{a_1 a_2 \ldots. a_{p+1}} \partial_{\mu_1}\phi^{a_1} .... \partial_{\mu_{p+1}}\phi^{a_{p+1}}. \tag{2.1}$$

the rank $p+1$ composite field strength is given in terms of $p+1$ scalar fields $\phi^1(x), \phi^2(x) .... \phi^{p+1}(x)$. Notice that the dimensionality of spacetime where the field theory is defined is *greater* than the number of primitive scalars $D > p+1$. An Euler variation w.r.t the ϕ^a fields yields the following field equations, after pre-multiplying by a factor of $\partial_{\mu_{p+2}}\phi^{a_1}$ and using the Bianchi identity $dF = 0$:

$$\partial_{\mu_1}[\frac{\delta L}{\delta(\partial_{\mu_1}\phi^{p+2})}] = 0 \;\Rightarrow\; F_{\mu_{p+2}\mu_2\ldots\mu_{p+1}} \partial_{\mu_1} F^{\mu_1\mu_2\ldots\mu_{p+1}} = 0. \tag{2.2}$$

Notice that despite the Abelian-looking form $F = dA$ the infinite-dimensional (global) symmetry of volume-preserving diffs is *not* Abelian. The theory we are describing is *not* the standard YM type .

We are going to find now p'-brane solutions to eq-(2) , where $D = p + p' + 2$. These brane solutions obeyed the classical analogs of S and T-duality [2] . Ordinary EM duality for branes requires $D = p+p'+4$. The latter condition is more closely related to the EM duality among two Chern-Simons p, p'-branes [44] which are embeddings of a p, p'-dimensional object into $p+2; p'+2$ dimensions. These co-dimension two objects are nothing but the analog of higher-dimensional "Knots" . A special class of (non-Maxwellian) extended- solutions to eqs-(2.2) requires a *dualization* procedure [2]:

$$G =^* F \;\Rightarrow\; G^{\nu_1\nu_2\ldots\nu_{p'+1}}(\tilde{\phi}(x)) = \epsilon^{\mu_1\mu_2\ldots.\mu_{p+1}\nu_1\nu_2\ldots.\nu_{p'+1}} F_{\mu_1\mu_2\ldots.\mu_{p+1}}(\phi(x)) \tag{2.3}$$

After this dualization procedure the eqs-(2.2) are recast in the form:

$$G^{\mu_1\nu_2\ldots\nu_{p'+1}} \partial_{\mu_1} G_{\nu_2\nu_3\ldots.\nu_{p'+2}}(\tilde{\phi}(x)) = 0. \tag{2.4}$$

The dualized equations (2.4) have a different form than eqs-(2.2) due to the position of the indices (the index contraction differs in both cases). Extended p'-brane solutions to eqs-(2.4) exist based on solutions to the Aurilia-Smailagic-Spallucci local gauge field theory reformulation of extended objects given in [4]. These solutions are

$$G^{\nu_1\nu_2\ldots\nu_{p'+1}}(\tilde{\phi}(x))|_{x=X} = T \; \frac{\{X^{\nu_1}, X^{\nu_2}, ..., X^{\nu_{p'+1}}\}}{\sqrt{-\frac{1}{(p'+1)!}[\{X^{\mu_1}, X^{\mu_2}, ..., X^{\mu_{p'+1}}\}][\{X_{\mu_1}, X_{\mu_2}, ..., X_{\mu_{p'+1}}\}]}}. \tag{2.5}$$

where T is the p'-brane tension and the Nambu-Poisson bracket w.r.t the $p'+1$ world-volume variables is defined as the ordinary determinant /Jacobian:

$$\{X^{\nu_1}, X^{\nu_2}, X^{\nu_3}, ..., X^{\nu_{p'+1}}\}_{NPB} = \epsilon^{\sigma^1\sigma^2\sigma^3\ldots.\sigma^{p'+1}} \partial_{\sigma^1} X^{\nu_1} \partial_{\sigma^2} X^{\nu_2} .... \partial_{\sigma^{p'+1}} X^{\nu_{p'+1}}. \tag{2.6}$$

All quantities are *evaluated* on the $p'+1$-dim world-volume support of the p'-brane; i.e. one must *restrict* the dual-scalar solutions $\tilde{\phi}^a(x)$ to those points in the D-dimensional spacetime which have support on the brane given by $x = X(\sigma^1, \sigma^2, ...)$. Solutions to *all* of the D-dim spacetime region can be extended simply by using delta functionals $\delta(x - X(\sigma))$.

Now we are going to re-interpret these findings in terms of the composite-antisymmetric tensor field theories ($CATF$) of area (volume) preserving-diffs [1] that inspired the work of [2]. The rank-two composite antisymmetric tensor field ($CATF$) strength may be written in terms of a symplectic (antisymmetric) two-form ω_{ab} as :

$$F_{\mu\nu}^{(CATF)} = \omega_{ab}\partial_\mu\phi^a\partial_\nu\phi^b. \tag{2.21}$$

The next step is to relate the $F_{\mu\nu}^{(CATF)}$ in terms of string degrees of feedom associated with the large N limit of $SU(N)$ Yang-Mills and Born-Infeld models in the *quenched* reduced approximation derived in [5,6]. Another procedure to achieve this is by invoking the string (brane) analog of wave-particle duality for point particles which has been coined brane-wave duality in the literature by [7]. For point-particles one has standard QM wave-particle correspondence :

$$x^\mu(\tau) \leftrightarrow \phi(x^\mu). \quad \frac{\partial x^\mu}{\partial\tau} \leftrightarrow \partial_\mu\phi(x^\mu). \tag{2.22}$$

Given the relativistic constraint $p_\mu p^\mu + m^2 = 0$, upon quantization one recovers the Klein-Gordon equation

$$(\partial_\mu\partial^\mu + m^2)\phi(x) = 0. \tag{2.23}$$

. Thus, standard quantization is the basis for the particle-field (wave) correspondence (2-22) .

The brane/wave duality [7] is just a generalization of the point particle case :

$$X^\mu(\sigma^a) \leftrightarrow \phi^a(x^\mu). \quad a = 1, 2. \tag{2.24}$$

but *encoded* via the string (brane) kinetic terms given in terms of the Poisson brackets (Nambu-Poisson Brackets) of the string (brane) coordinates $X^\mu(\sigma^a)$ as follows :

$$\{X_\mu, X_\nu\}_{PB} = \omega^{ab}\partial_a X_\mu(\sigma)\partial_b X_\nu(\sigma) \leftrightarrow$$
$$F_{\mu\nu}^{CATF}[\phi] = \omega_{ab}\partial_\mu\phi^a(x)\partial_\nu\phi^b(x). \tag{2.25}$$

The evaluation of the Poisson bracket with respect to the variables σ^1, σ^2 requires to use the inverse ω^{ab} (a 2×2 matrix) of the symplectic non-degenerate two-form ω_{ab} associated with a 2-dim phase space. Symplectic-diffs are area-preserving.

The above equation (2.25) expresses the composite antisymmetric tensor field strength/string correspondence by interpreting the scalar fields ϕ^a as the *generalized* world-sheet variables σ^a associated with the 2-dim world-sheet swept by a string for $a = 1, 2$. Therefore, the string/wave duality expressed by eqs-(2.25) is basically a world-volume/target space duality since $\phi^a(x)$ represent the mappings (inmersions) from spacetime x^μ to a field-space ϕ^a. Whereas the *inverse* maps from the field-space ϕ^a variables to the spacetime x^μ variables is what the maps/embeddings of the string's world-volume into the target spacetime background represent. The latter are the maps $X^\mu(\sigma^a)$. Hence, eq-(2.25) is just the correspondence between a two-dim area measure of the string's world-sheet and the $CATF$ strength $F_{\mu\nu}[\phi]$. It can be generalized to branes as well.

$$F^2_{\mu_1\mu_2...\mu_{p+1}} \leftrightarrow \{X_{\mu_1}, X_{\mu_2}, ..., X_{\mu_{p+1}}\}^2$$
$$F = dA = \epsilon_{a_1a_2....a_{p+1}}\partial_{\mu_1}\phi^{a_1}.........\partial_{\mu_{p+1}}\phi^{a_{p+1}} \leftrightarrow \{X_{\mu_1}, X_{\mu_2}, ..., X_{\mu_{p+1}}\} \tag{2.26}$$

Since this last *correspondence* between branes and $CATF$ theories, based on brane-wave duality, seems too *heuristic* in the next section we will *show* rigorously why the Poisson bracket has a correspondence to the Lie-algebra commutator of the large N limit of $SU(N)$ Yang-Mills theory in the *quenched* reduced approximation :

$$\{X_\mu, X_\nu\}_{PB} \leftrightarrow [A_\mu, A_\nu] \tag{2.27}$$

In particular, we will see why the large N limit of Yang-Mills theory admits strings, membranes and bag excitations [5,6] .

2.2 Branes from Moyal deformations of Yang-Mills Theories

It is essential now to explain the derivation of how Hadronic Bags (branes) [5,6] and Chern-Simons Branes [44] can be obtained from the Large N limit of Yang-Mills and Generalized Yang-Mills theories [12] in Flat Backgrounds. A Moyal deformation quantization was instrumental in the construction of p-brane actions and Chern-Simons branes from the large N limit of $SU(N)$ YM in flat backgrounds. $SU(N)$ reduced-quenched gauge theories have been shown by us to be related to Hadronic Bags and Chern Simons Membranes in the large N limit [5,6,8] . This is reminiscent of the chiral model approaches to Self Dual Gravity based on Self Dual Yang Mills theories [9] .

A different approach than the quenched-reduced approximation relating the large N limit and the Eguchi-Schild action for strings and Matrix Membranes was undertaken by Zachos et al in [14, 50]. Most recently extensions of Moyal-deformed hierarchies of soliton equations and noncommutative KP hierarchy have been studied by Muller-Hoissen and Dimakis [53]. A Moyal deformation quantization of the Nahm equations associated with a $SU(2)$ YM theory yields the *classical* $N \rightarrow \infty$ limit of the $SU(N)$ YM Nahm equations *directly*, without ever having to use $\infty \times \infty$ matrices in the large N matrix models. By simply taking the classical $\hbar = 0$ limit of the Moyal brackets, the ordinary Poisson bracket algebra associated with area-preserving diffs algebra $SU(\infty)$ [10, 11] is automatically recovered.

This Moyal deformation approach also furnishes dynamical membranes as well [5] when one uses the spatial quenching approximation to a *line* (one dimension) , instead of quenching to a point. In this fashion we constructed what is called a QCD membrane. Basically, a Moyal quantization takes the operator $\hat{A}_\mu(x^\mu)$ into $A_\mu(x^\mu; q, p)$ and commutators into Moyal brackets. A dimensional reduction to one temporal dimension (quenching to a line) brings us to functions of the form $A_\mu(t, q, p)$, which precisely corresponds to the membrane coordinates $X_\mu(t, \sigma^1, \sigma^2)$ after identifying the σ^a variables with q, p. The $\hbar = 0$ limit turns the Moyal bracket (after dividing by $i\hbar$) into a Poisson one. Upon the identification of $\hbar = 2\pi/N$, the classical $\hbar = 0$ limit is tantamount to the $N = \infty$ limit and it is in this fashion how the large N $SU(N)$ matrix model bears a direct relation to the physics of membranes. The Moyal quantization explains this in a straightforward fashion without having to use $\infty \times \infty$ matrices !

We will briefly review [5,6,8] how a $4D$ Yang-Mills theory reduced and quenched to a point, and supplemented by a topological *theta* term can be related through a Weyl-Wigner Groenowold Moyal (WWGM) quantization procedure to an *open* domain of the 3-dim disk D^3. The bulk $D^3 \times R^1$ is the interior of a hadronic bag and the (lateral) boundary is the Chern-Simons world volume $S^2 \times R^1$ of a membrane of topology S^2 (a codimension two object). Hence, we have an example where the world-volume of a boundary $S^2 \times R^1$ is the lateral-boundary of the world-volume of an open 3-brane of topology D^3 : $\partial(D^3 \times R^1) = S^2 \times R^1$ (setting asside the points at infinity). The boundary dynamics is *not* trivial despite the fact that there are no transverse bulk dynamics associated with the interior of the bag. This is due to the fact that the 3-brane is spacetime filling : $3 + 1 = 4$ and therefore has no transverse physical degrees of freedom.

The reduced-quenched action to a point in $D = 4$ is:

$$S = -\frac{1}{4}(\frac{2\pi}{a})^4 \frac{1}{g^2_{YM}} Tr(F_{\mu\nu}F^{\mu\nu}).$$

$$F_{\mu\nu} = i[D_\mu, D_\nu]. \tag{2.28}$$

Notice that the reduced-quenched action is defined at a " point " x_o . The quenched approximation is based essentially by replacing the field strengths by their commutator dropping the ordinary derivative terms . For simplicity we have omitted the matrix $SU(N)$ indices in (2-28). The *theta* term is:

$$S_\theta = -\frac{\theta g^2_{YM}}{16\pi^2}(\frac{2\pi}{a})^4 \epsilon^{\mu\nu\rho\sigma} Tr(F_{\mu\nu}F_{\rho\sigma}). \tag{2.29}$$

The WWGM quantization establishes a one-to-one correspondence between a linear operator $D_\mu = \partial_\mu + A_\mu$ acting on the Hilbert space $\mathcal{H}$ of square integrable functions in R^D and a smooth function $\mathcal{A}_\mu(x, y)$ which is the Fourier transform of $\mathcal{A}_\mu(q, p)$. The latter quantity is obtained by evaluating the *trace* of the $D_\mu = \partial_\mu + A_\mu$ operator summing over the diagonal elements with respect to an orthonormal basis in the Hilbert space. Under the WWGM correspondence , in the quenched-reduced approximation, the matrix product $A_\mu.A_\nu$ is mapped into the *noncommutative* Moyal star product of their symbols $\mathcal{A}_\mu * \mathcal{A}_\nu$ and the commutators are mapped into their Moyal brackets:

$$[A_\mu, A_\nu] \Rightarrow \{\mathcal{A}_\mu, \mathcal{A}_\nu\}_{MB} = \mathcal{A}_\mu * \mathcal{A}_\nu - \mathcal{A}_\nu * \mathcal{A}_\mu \tag{2.30}$$

where the Moyal star product is defined as

$$(\ \mathcal{A}_\mu * \mathcal{A}_\nu\ )(\xi) \equiv exp\ [\ (i\hbar/2)\ \omega^{ab}\ \partial_a^{(\xi_1)}\ \partial_b^{(\xi_2)}]\ \mathcal{A}_\mu(\xi_1)\ \mathcal{A}_\nu(\xi_2)|_{\xi_1=\xi_2=\xi}$$

with

$$\xi = (q_i, p_i) = (q_1, q_2, ...q_n, p_1, p_2, ...p_n) \qquad a, b = 1, 2, 3, ...2n. \qquad ()$$

The *trace* operation is replaced by an integration w.r.t the internal phase space variables , $\sigma \equiv q^i, p^i$

$$(\frac{2\pi}{N})^3\ trace \rightarrow \int d^4\sigma. \tag{2.31}$$

The WWGM deformation quantization of the quenched-reduced orginal actions is:

$$S^* = -\frac{1}{4}(\frac{2\pi}{a})^4 \frac{1}{g^2_{YM}} \int d^4\sigma \mathcal{F}_{\mu\nu}(\sigma) * \mathcal{F}^{\mu\nu}(\sigma).$$

$$\mathcal{F}_{\mu\nu} = i\{\mathcal{A}_\mu, \mathcal{A}_\nu\}_{MB}. \tag{2.32}$$

And the corresponding WWGM deformation of the *theta* term:

$$S^*_\theta = -\frac{\theta g^2_{YM}}{16\pi^2}(\frac{2\pi}{a})^4 \epsilon^{\mu\nu\rho\sigma} \int d^4\sigma \mathcal{F}_{\mu\nu}(\sigma) * \mathcal{F}_{\rho\sigma}(\sigma). \tag{2.33}$$

Performing the following gauge fields/coordinate correspondence:

$$\mathcal{A}_\mu(\sigma) \rightarrow (\frac{2\pi}{N})^{1/4} X_\mu(\sigma)$$

$$\mathcal{F}_{\mu\nu}(\sigma) \rightarrow (\frac{2\pi}{N})^{1/2}\{X_\mu(\sigma), X_\nu(\sigma)\}_{MB}. \tag{2.34}$$

and by setting the Moyal deformation parameter "$\hbar$" $= 2\pi/N$ of the WWGM deformed action, to zero, by taking the "classical" $\hbar = 0$ limit, which is tantamount to taking the $N = \infty$ limit, one can see that the quenched-reduced YM action in the large N limit will become the Dolan-Tchrakian action for a 3-brane [12], in the conformal gauge, moving in a *flat* $D = 4$-dim background

$$S = -\frac{1}{4g^2_{YM}}(\frac{2\pi}{a})^4 \int d^4\sigma \{X^\mu, X^\nu\}_{PB}\ \{X^\rho, X^\tau\}_{PB}\ \eta_{\mu\rho}\eta_{\nu\tau} \tag{2.35a}$$

due to the fact that noncommutative Moyal star products reduce to ordinary commutative pointwise products and Moyal brackets collapse to the ordinary Poisson brackets in the $\hbar = 2\pi/N = 0$ limit (large N limit) as follows :

$$lim_{\ \hbar\rightarrow 0}\ \frac{1}{i\hbar}\ \{\mathcal{A}_\mu, \mathcal{A}_\nu\}_{MB} = \{\mathcal{A}_\mu, \mathcal{A}_\nu\}_{PB}. \tag{2.3b}$$

It is essential to include the $i\hbar$ factors in the denominator as shown.

The θ term S^*_θ in the large N limit becomes the action for a Chern-Simons membrane whose world volume is the 3-dim boundary of a 4-dim hadronic *bag* . For further details and more references of the interplay between $SU(N)$ Born-Infeld actions and Nambu-Gotos strings in the large N limit see [5,6] . A quenched-reduction process to a *line* instead of quenching to a *point* allowed us to construct a Matrix model of a QCD Membrane [5] via the Moyal quantization method of $SU(N)$ YM. The resulting action [5] has the same form as the Matrix model of Banks et al [49] . Other types of p-brane actions were also derived from a Moyal quantization of the Generalized Yang-Mills models (GYM) described in [12] . A confinement mechanism via the relationship between strings and the Wilson Loops associated with the large N limit of $SU(N)$ YM, following this Moyal quantization approach in the quenched-reduced approximation, was attained in [48] and a Moyal deformation of gravity in $AdS_n \times S^m$ backgrounds was presented in [48].

Having reviewed how brane actions can be obtained from the large N limit of quenched-reduced $SU(N)$ YM we proceed with the construction of new types of brane actions that are amenable to Moyal star product deformations in a straightforward fashion. The reparametrization invariant action for a point particle moving in a flat target spacetime:

$$S = -\frac{1}{2}\ m \int d\tau e(\tau)[-\frac{\dot{X}^2}{e(\tau)^2} + 1]. \tag{2.36}$$

after introducing the auxiliary einbein field $e = e(\tau)$ and eliminating it from the action via its algebraic equations of motion and plugging its solution back into the original action gives the well known reparametrization invariant action in terms of a (dimensionless) affine parameter τ along the particle's worldline:

$$S = -m \int d\tau \sqrt{|\dot{X}^\mu \dot{X}_\mu|}. \tag{2.37}$$

Guendelman et al generalized such point particle action to the (super) string case [13] by introducing a modified measure of integration, independent of a metric, involving auxiliary scalar fields. Next section we will construct new types of p-brane actions [48] (in flat spacetime backgrounds) that will be amenable to deformations of the Nambu-Poisson brackets, after making the following correspondence in eq-(2.37):

$$(dX/d\tau)^2 \leftrightarrow (\{X^{\mu_1}, X^{\mu_2}, ..., X^{\mu_{p+1}}\}_{NPB})^2.$$

$$m \leftrightarrow T_p.$$

$$\int d\tau e(\tau) \leftrightarrow \int d^{p+1}\sigma \{\phi^1, \phi^2, ..., \phi^{p+1}\}. \tag{2.38}$$

where the measure in the r.h.s of (2.38) is the one introduced by [13]

The new p-brane actions based on a modified-measure are given then by [48]

$$S = -\frac{1}{2} \int\ d^{p+1}\sigma\ \{\phi^1, \phi^2, ..., \phi^{p+1}\}\ [\ -\frac{T_p^2}{(p+1)!} \frac{\{X^{\mu_1}, X^{\mu_2}, X^{\mu_3}, ..., X^{\mu_{p+1}}\}^2}{\{\phi^1, \phi^2, ..., \phi^{p+1}\}^2}\ +\ 1\ ]. \tag{2.39}$$

Notice that the new measure of integration has the same dimensions (units) as the p-brane tension. Such worldvolume reparametrization invariant actions are just a new version of the reparametrization invariant Schild actions for p-branes where the auxiliary determinant field e is replaced by the Jacobian measure involving the auxiliary Guendelman scalars. Eliminating the Jacobian measure (the auxiliary Guendelman scalars) from the action (2.39) via its equation of motion, and inserting its value back into the action (2.39), yields the standard p-brane Dirac-Nambu-Goto actions given by the $p+1$-dim world volume spanned by the p-brane in its motion across the target spacetime background.

Now we turn to the deformations of these new types of p-brane actions by deforming the Nambu-Poisson brackets. When the dimension of the the worldvolume $p+1 = 2n$ is even, one can recur to the star products applied to the even-dimensional phase space formalism and deform these p-brane actions. After 30 years, the construction of Quantum Nambu Mechanics has been recently settled by Curtright and Zachos [14] since it remains unclear whether or not this problem can be solved by the Zariski Abelian deformation method [14] due to the fact that it does not satisfy the *three* crucial properties of antisymmetry, Liebnitz derivation rule and the fundamental identity that should be obeyed by any (quantum) bracket. Zachos and Curthright [14] defined the quantum Nambu brackets (QNB) by the signed sum over all permutations of the star products of functions of the even-dim phase space variables:

$$[A_1, A_2, ..., A_k]_* = \sum (-1)^{\pi(p)} A_{p_1} * A_{p_2} * .... * A_{p_k}. \tag{2.40}$$

it is explicitly antisymmetric, by construction, however the Leibnitz derivation property and the fundamental identity of the QNB are not explicitly manifest. Even-order Quantum Nambu Brackets (QNB) can always be resolved into sums of products of commutators, for instance [14]

$$[A, B, C, D]_* = [A, B]_* * [C, D]_* - [A, C]_* * [B, D]_* - [A, D]_* * [C, B]_* + \\ [C, D]_* * [A, B]_* - [B, D]_* * [A, C]_* + [B, C]_* * [A, D]_*. \quad (2.41)$$

with

$$[A, B]_* \equiv A * B - B * A. \quad (2.42)$$

In the case of a four-dim space of topology $S^2 \times S^2$ (a four-volume can be written as wedge-products of areas) it follows from the $SU(2)$ Moyal-bracket algebra, and the crucial commutator-resolution of the four-bracket given by (2.41), that the Liebnitz and the Fundamental Identity properties are indeed satisfied.

This procedure can be generalized to spaces of topology $S^2 \times S^2 \times .... \times S^2$. Hence, for even-dimensional worldvolumes, $p+1 = 2n$, the deformation of the p-brane actions (2.39), in the preferred volume *unimodular* gauge:

$$\{\phi^1, \phi^2, ..., \phi^{p+1}\}_{NPB} = 1. \quad (2.43)$$

is attained by replacing:

$$(\{X^{\mu_1}, X^{\mu_2}, X^{\mu_3}, ..., X^{\mu_{p+1}}\}_{NPB})^2 \rightarrow \\ [X_{\mu_1}, X_{\mu_2}, X_{\mu_3}, ..., X_{\mu_{p+1}}]_* * [X^{\mu_1}, X^{\mu_2}, X^{\mu_3}, ..., X^{\mu_{p+1}}]_*. \quad (2.44)$$

Where the QNBs of the p-brane's embedding spacetime coordinates $X^\mu(\sigma)$ in (4-10) is given explicitly in terms of the signed-sum over all the permutations of the star products:

$$[X^{\mu_1}, X^{\mu_2}, X^{\mu_3}, ..., X^{\mu_{p+1}}]_* \equiv X^{\mu_1}(\sigma) * X^{\mu_2}(\sigma) * X^{\mu_3}(\sigma) * ..... * X^{\mu_{p+1}}(\sigma) + permutations \quad (2.45)$$

The even $2n$-dim phase space variables required in the evaluation of the star products $q_1, p_1; q_2, p_2; ...q_n, p_n$ can be identified with the $p + 1 = 2n$ worldvolume variables $\sigma^1, \sigma^2, ...\sigma^{2n}$, respectively, of the p-brane.

The reason we chose to fix the unimodular gauge by choosing a preferred volume in (2.39), leaving a residual symmetry of volume-preserving diffs, is to simplify the deformation procedure of the p-brane action because the presence of denominators in the action (2.39) will complicate matters. Therefore, in the *unimodular* gauge (2.43) , the deformation of the p-brane action (2.39) in flat backgrounds in terms of the Quantum Nambu Brackets (QNB), when $p + 1 = 2n$, is

$$S_* = -\frac{1}{2}\int d^{p+1}\sigma[-\frac{T_p^2}{(p+1)!}\frac{1}{(i\hbar)^{p+1}}([X_{\mu_1}, X_{\mu_2}, ..., X_{\mu_{p+1}}]_*)^2 + 1]. \quad \{\phi^1, \phi^2, ..., \phi^{p+1}\}_{*NPB} = 1. \quad (2.45)$$

In the classical limit $\hbar = 0$ it reduces to the classical action because the star products collapse to ordinary pointwise commutative products and the QNBs divided by $(i\hbar)^{p+1}$ collapse to ordinary NPBs since the denominator factors of $(i\hbar)^{p+1}$ in (2.45) are *absorbed* by the QNBs in the classical $\hbar = 0$ limit leading to ordinary NPBs.

The deformed p-brane action when $p+1 = 2n$ = even is invariant under deformations of a subalgebra of the volume-preserving diffs since an even-dim volume can be written as wedge-products of area-forms. We should notice, however, that there are volume-preserving diffs that do not always amount to area-preserving diffs.

3. Field Theory and Branes in Clifford-space Backgrounds

In this section we will introduce the novel physics of branes in Clifford spaces.

3.1 The Extended Relativity Theory in Clifford Spaces

The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory. For a comprehensive review we refer to [16] . A natural generalization of the notion of a space-time interval in Minkwoski space to C-space is :

$$dX^2 = d\Omega^2 + dx_\mu dx^\mu + dx_{\mu\nu} dx^{\mu\nu} + ... \qquad (3.1)$$

The Clifford valued poly-vector:

$$X = X^M E_M = \Omega\ \mathbf{1} + x^\mu \gamma_\mu + x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + ... x^{\mu_1 \mu_2 .... \mu_D} \gamma_{\mu_1} \wedge \gamma_{\mu_2} .... \wedge \gamma_{\mu_D}. \qquad (3.2a)$$

denotes the position of a polyparticle in a manifold, called Clifford space or C-space. The series of terms in (2) terminates at a $finite$ value depending on the dimension D. A Clifford algebra $Cl(r,q)$ with $r+q = D$ has 2^D basis elements. For simplicity, the gammas γ^μ correspond to a Clifford algebra associated with a flat spacetime :

$$1/2 \{ \gamma^\mu, \gamma^\nu \} = \eta^{\mu\nu}. \qquad (3.2b)$$

but in general one could extend this formulation to curved spacetimes with metric $g^{\mu\nu}$. The multi-graded basis elements E_M of the Clifford-valued poly-vectors are

$$E_M \ \equiv\ \mathbf{1}, \quad \gamma^\mu, \quad \gamma^{\mu_1} \wedge \gamma^{\mu_2}, \quad \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3}, \quad \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} \wedge ..... \wedge \gamma^{\mu_D}. \qquad (3.2c)$$

It is convenient to order the collective M indices as $\mu_1 < \mu_2 < \mu_3 < ...... < \mu_D$.

The connection to strings and p-branes can be seen as follows. In the case of a closed string (a 1-loop) embedded in a target flat spacetime background of D-dimensions, one represents the projections of the closed string (1-loop) onto the embedding spacetime coordinate-planes by the variables $x_{\mu\nu}$. These variables represent the respective *areas* enclosed by the projections of the closed string (1-loop) onto the corresponding embedding spacetime planes. Similary, one can embed a closed membrane (a 2-loop) onto a D-dim flat spacetime, where the projections given by the antisymmetric variables $x_{\mu\nu\rho}$ represent the corresponding *volumes* enclosed by the projections of the 2-loop along the hyperplanes of the flat target spacetimr background.

This procedure can be carried to all closed p-branes (p-loops) where the values of p are $p = 0, 1, 2, 3, ...D-2$. The $p = 0$ value represents the center of mass and the coordinates $x^{\mu\nu}, x^{\mu\nu\rho}$.... have been *coined* in the string-brane literature [32] as the *holographic* areas, volumes, ...projections of the nested family of p-loops (closed p-branes) onto the embedding spacetime coordinate planes/hyperplanes.

The classification of Clifford algebras $Cl(r,q)$ in $D = r+q$ dimensions (modulo 8) for different values of the spacetime signature r, q is discussed, for example, in the book of Porteous [19]. All Clifford algebras can be understood in terms of $CL(8)$ and the $CL(k)$ for k less than 8 due to the modulo 8 Periodicity theorem

$$CL(n) = CL(8) \times Cl(n-8)$$

. $Cl(r,q)$ is a matrix algebra for even $n = r+q$ or the sum of two matrix algebras for odd $n = r+q$. Depending on the signature, the matrix algebras may be real, complex, or quaternionic. For furher details we refer to [19] .

If we take the differential dX and compute the scalar product among two polyvectors $< dX^\dagger dX >_{scalar}$ we obtain the C-space extension of the particles proper time in Minkwoski space. The symbol X^+ denotes the $reversion$ operation and involves reversing the order of all the basis γ^μ elements in the expansion of X . It is the analog of the transpose (Hermitian) conjugation. The C-space proper time associated with a polyparticle motion is then :

$$< dX^\dagger dX >_{scalar} = d\Sigma^2 = (d\Omega)^2 + \Lambda^{2D-2} dx_\mu dx^\mu + \Lambda^{2D-4} dx_{\mu\nu} dx^{\mu\nu} + .. \qquad (3.3)$$

Here we have explicitly introduced the Planck scale Λ since a length parameter is needed in order to tie objects of different dimensionality together: 0-loops, 1-loops,..., p-loops. Einstein introduced the speed of light as a universal absolute invariant in order to "unite" space with time (to match units) in the Minkwoski space interval:

$$ds^2 = c^2 dt^2 - dx_i dx^i. \quad (3.4)$$

A similar unification is needed here to "unite" objects of different dimensions, such as x^μ, $x^{\mu\nu}$, etc... The Planck scale then emerges as another universal invariant in constructing an extended scale relativity theory in C-spaces [16].

To continue along the same path, we consider the analog of Lorentz transformations in C-spaces which transform a poly-vector X into another poly-vector X' given by $X' = RXR^{-1}$ with

$$R = e^{\theta^A E_A} = exp\ [(\theta \mathbf{1} + \theta^\mu \gamma_\mu + \theta^{\mu_1\mu_2}\gamma_{\mu_1} \wedge \gamma_{\mu_2}.....)]. \quad (3.5)$$

and

$$R^{-1} = e^{-\theta^A E_A} = exp\ [-(\theta \mathbf{1} + \theta^\nu \gamma_\nu + \theta^{\nu_1\nu_2}\gamma_{\nu_1} \wedge \gamma_{\nu_2}.....)]. \quad (3.6)$$

where the theta parameters in (2.5, 2.6) are the components of the Clifford-value parameter $\Theta = \theta^A E_A$:

$$\theta; \theta^\mu; \theta^{\mu\nu}; .... \quad (3.7)$$

they are the C-space version of the Lorentz rotations/boosts parameters.

Since a Clifford algebra admits a matrix representation, one can write the norm of a poly-vectors in terms of the trace operation as: $||X||^2 = Trace\ X^2$ Hence under C-space Lorentz transformation the norms of poly-vectors behave like follows:

$$Trace\ X'^2 = Trace\ [RX^2R^{-1}] = Trace\ [RR^{-1}X^2] = Trace\ X^2. \quad (3.8)$$

These norms are invariant under C-space Lorentz transformations due to the cyclic property of the trace operation and $RR^{-1} = 1$. Another way of rewriting the inner product of poly-vectors is by means of the reversal operation that reverses the order of the Clifford basis generators : $(\gamma^\mu \wedge \gamma^\nu)^\dagger = \gamma^\nu \wedge \gamma^\mu$, etc... Hence the inner product can be rewritten as the scalar part of the geometric product $< X^\dagger X >_s$. The analog of an orthogonal matrix in Clifford spaces is $R^\dagger = R^{-1}$ such that

$$< X'^\dagger X' >_s = < (R^{-1})^\dagger X^\dagger R^\dagger R X R^{-1} >_s = < R X^\dagger X R^{-1} >_s = < X^\dagger X >_s = invariant. \quad (3.9a)$$

This condition $R^\dagger = R^{-1}$, of course, will *restrict* the type of terms allowed inside the exponential defining the rotor R in eq-(3-5) because the *reversal* of a p-vector obeys

$$(\gamma_{\mu_1} \wedge \gamma_{\mu_2} ..... \wedge \gamma_{\mu_p})^\dagger = \gamma_{\mu_p} \wedge \gamma_{\mu_{p-1}} ..... \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_1} = (-1)^{p(p-1)/2} \gamma_{\mu_1} \wedge \gamma_{\mu_2} ..... \wedge \gamma_{\mu_p} \quad (3.9b)$$

Hence only those terms that change sign (under the reversal operation) are permitted in the exponential defining $R = exp[\theta^A E_A]$.

Another possibility is to *complexify* the C-space polyvector valued coordinates $= Z = Z^A E_A = X^A E_A + iY^A E_A$ and the boosts/rotation parameters θ allowing the unitarity condition $\bar{U}^\dagger = U^{-1}$ to hold in the generalized Clifford unitary transformations $Z' = UZU^\dagger$ associated with the complexified polyvector $Z = Z^A E_A$ such that the interval

$$< d\bar{Z}^\dagger\ dZ >_s = d\bar{\Omega} d\Omega + d\bar{z}^\mu dz_\mu + d\bar{z}^{\mu\nu} dz_{\mu\nu} + d\bar{z}^{\mu\nu\rho} dz_{\mu\nu\rho} + ... \quad (3.9c)$$

remains invariant (upon setting the Planck scale $\Lambda = 1$).

The unitary condition $\bar{U}^\dagger = U^{-1}$ under the *combined* reversal and complex-conjugate operation will constrain the form of the complexified boosts/rotation parameters θ^A appearing in the rotor : $U = exp[\ \theta^A E_A\ ]$. The theta parameters θ^A are either purely real or purely imaginary depending if the reversal ${E_A}^\dagger = \pm E_A$, to ensure that an overall *change* of sign occurs in the terms $\theta^A E_A$ inside the exponential defining U so that $\bar{U}^\dagger = U^{-1}$ holds and the norm $< \bar{Z}^\dagger Z >_s$ remains invariant under the analog of unitary transformations in *complexified* C-spaces. These techniques are not very different from Penrose Twistor spaces. As far as we know a Clifford-Twistor space construction of C-spaces has not been performed so far.

Another alternative is to define the polyrotations by $R = exp\ (\Theta^{AB}[E_A, E_B])$ where the commutator $[E_A, E_B] = F_{ABC}E_C$ is the C-space analog of the $i[\gamma_\mu, \gamma_\nu]$ commutator which is the generator of the Lorentz algebra, and the theta parameters Θ^{AB} are the C-space analogs of the rotation/boots parameters $\theta^{\mu\nu}$. The diverse parameters Θ^{AB} are purely real or purely imaginary depending whether the reversal $[E_A, E_B]^\dagger = \pm[E_A, E_B]$ to ensure that $R^\dagger = R^{-1}$ so that the scalar part $< X^\dagger X >_s$ remains invariant under the transformations $X' = RXR^{-1}$. This last alternative seems to be more physical because a poly-rotation should map the E_A direction into the E_B direction in C-spaces, hence the meaning of the generator $[E_A, E_B]$ which extends the notion of the $[\gamma_\mu, \gamma_\nu]$ Lorentz generator. We refer to the review [16] for further details about the Extended Relativity Theory in Clifford spaces.

3.2 Brane Actions in Clifford-space Target Backgrounds

In the next sections we will develop further the construction of the generalization to C-spaces of string and p-brane actions [16] as embeddings of world-manifolds onto target spacetime backgrounds which involves the embeddings of polyvector-valued world-manifolds (of dimensions 2^d) onto polyvector-valued target spaces (of dimensions 2^D), given by the Clifford-valued maps $X = X(\Sigma)$. These are maps from the Clifford-valued world-manifold, parametrized by the polyvector-valued variables Σ, onto the Clifford-valued target space parametrized by the polyvector-valued coordinates X. Physically one envisions these maps as taking an n-dimensional simplicial cell (n-loop) of the world-manifold onto an m-dimensional simplicial cell (m-loop) of the target C-space manifold ; i.e. maps from n-dim objects onto m-dim objects generalizing the old maps of taking points onto points. One is basically dealing with a dimension-category of objects. The size of the simplicial cells (p-loops), upon quantization of a generalized harmonic oscillator, for example, are given by multiples of the Planck scale, in area, volume, hypervolume units or Clifford-bits.

In compact multi-index notation $X = X^M\Gamma_M$ one denotes for each one of the components of the target space polyvector X:

$$X^M \equiv X^{\mu_1\mu_2....\mu_r},\ \mu_1 < \mu_2 < ... < \mu_r. \qquad 3.10$$

and for the world-manifold polyvector $\Sigma = \Sigma^A E_A$:

$$\Sigma^A \equiv \xi^{a_1a_2....a_s}, a_1 < a_2 < ... < a_s. \qquad 3.11$$

where $\Gamma_M = (\underline{1}, \gamma_\mu, \gamma_{\mu\nu}, ...)$ and $E_A = (\underline{1}, e_a, e_{ab}, ...)$ form the basis of the target manifold and world manifold Clifford algebra, respectively. It is very important to order the indices within each multi-index M and A as shown above. The above Clifford-valued coordinates X^M, Σ^A correspond to antisymmetric tensors of ranks r, s in the target spacetime background and in the world-manifold, respectively.

There are many different ways to construct C-space brane actions which are on-shell equivalent to the analogs of the Dirac-Nambu-Goto action for extended objects and that are given by the world-volume spanned by the branes in their motion through the target spacetime background.

One of these actions is the Polyakov-Howe-Tucker action:

$$I = \frac{T}{2}\int [D\Sigma]\sqrt{|H|}[H^{AB}\partial_A X^M \partial_B X^N G_{MN} + (2 - 2^d)]. \qquad 3.12$$

with the 2^d-dim world-manifold measure:

$$[D\Sigma] = (d\xi)(d\xi^a)(d\xi^{a_1a_2})(d\xi^{a_1a_2a_3})..... \qquad 3.13$$

Upon the algebraic elimination of the auxiliary world-manifold metric H^{AB} from the action (3.12), via the equations of motion, yields for its on-shell solution the pullback of the target C-space metric onto the C-space world-manifold:

$$H_{AB}(on - shell) = G_{AB} = \partial_A X^M \partial_B X^N G_{MN} \qquad 3.14$$

upon inserting back the on-shell solutions (3.14) into (3.12) gives the Dirac-Nambu-Goto action for the C-space branes directly in terms of the C-space determinant, or measure, of the induced C-space world-manifold metric G_{AB}, as a result of the embedding:

$$I = T \int [D\Sigma] \sqrt{Det\ (\partial_A X^M \partial_B X^N G_{MN})}. \quad 3.15$$

However in C-space, the Polyakov-Howe-Tucker action admits an even further generalization that is comprised of two terms $S_1 + S_2$. The first term is:

$$S_1 = \int [D\Sigma]|E|E^A E^B \partial_A X^M \partial_B X^N \Gamma_M \Gamma_N. \quad 3.16$$

Notice that this is a generalized action which is written in terms of the C-space coordinates $X^M(\Sigma)$ and the C-space analog of the target-spacetime vielbein/frame one-forms $e^m = e^m{}_\mu dx^\mu$ given by the Γ^M variables. The auxiliary world-manifold vielbein variables e^a, are given now by the Clifford-valued frame E^A variables.

In the conventional Polyakov-Howe-Tucker action, the auxiliary world-manifold metric h^{ab} associated with the standard p-brane actions is given by the usual scalar product of the frame vectors $e^a.e^b = e^a_\mu e^b_\nu g^{\mu\nu} = h^{ab}$. Hence, the C-space world-manifold metric H^{AB} appearing in (3.12) is given by scalar product $< (E^A)^\dagger E^B >_0 = H^{AB}$, where $(E^A)^\dagger$ denotes the reversal operation of E^A which requires reversing the orderering of the vectors present in the Clifford aggregate E^A.

Notice, however, that the form of the action (3.12) is far more general than the action in In particular, the S_1 itself can be decomposed futher into two additional pieces by rewriting the Clifford product of two basis elements into a symmetric plus an antisymmetric piece, respectively:

$$E^A E^B = \frac{1}{2}\{E^A, E^B\} + \frac{1}{2}[E^A, E^B]. \quad 3.17$$

$$\Gamma_M \Gamma_N = \frac{1}{2}\{\Gamma_M, \Gamma_N\} + \frac{1}{2}[\Gamma_M, \Gamma_N]. \quad 3.18$$

In this fashion, the S_1 component has *two* kinds of terms. The first term containing the symmetric combination is just the analog of the standard non-linear sigma model action, and the second term is a Wess-Zumino-like term, containing the antisymmetric combination [17] .

To extract the non-linear sigma model part of the generalized action above, we may simply take the scalar product of the vielbein-variables as follows:

$$(S_1)_{sigma} = \frac{T}{2} \int [D\Sigma]|E| < (E^A \partial_A X^M \Gamma_M)^\dagger (E^B \partial_B X^N \Gamma_N) >_0 . \quad 3.19$$

where once again we have made use of the reversal operation (the analog of the hermitian adjoint) before contracting multi-indices. In this fashion we recover again the Clifford-scalar valued action given by (3.19).

Actions like the ones presented here in terms of derivatives with respect to quantities with multi-indices can be mapped to actions involving *higher* derivatives, in the same fashion that the C-space scalar curvature, the analog of the Einstein-Hilbert action, could be recast as a higher derivative gravity with torsion (reviewed in sec. 4). Higher derivatives actions are also related to theories of Higher spin fields Vasiliev and W-geometry, W-algebras [20]

The S_2 (scalar) component of the C-space brane action is the usual cosmological constant term given by the C-space determinant $|E| = \det(H^{AB})$ based on the scalar part of the geometric product $< (E^A)^\dagger E^B >_0 = H^{AB}$

$$S_2 = \frac{T}{2} \int [D\Sigma]|E|(2 - 2^d) \quad 3.20$$

where the C-space determinant $|E| = \sqrt{|\det(H^{AB})|}$ of the $2^d \times 2^d$ generalized world-manifold metric H^{AB} is given by:

$$\det(H^{AB}) = \frac{1}{(2^d)!} \epsilon_{A_1 A_2 .... A_{2^d}} \epsilon_{B_1 B_2 .... B_{2^d}} H^{A_1 B_1} H^{A_2 B_2} .... H^{A_{2^d} B_{2^d}}. \quad 3.21$$

The $\epsilon_{A_1 A_2 .... A_{2^d}}$ is the totally antisymmetric tensor density in C-space. In section **6** we will add some concluding remarks pertaining the quantization program of C-space branes.

3.3 Field Theory and Brane-Wave Duality in Clifford Spaces

The main result of this section is to discuss the implications of the correspondence

$$F_{\mu_1\mu_2...\mu_{p+1}} F^{\mu_1\mu_2...\mu_{p+1}} \leftrightarrow \{ X_{\mu_1}, X_{\mu_2}, ...X_{\mu_{p+1}} \}_{NPB} \{ X^{\mu_1}, X^{\mu_2}, ...X^{\mu_{p+1}} \}_{NPB}. \quad (3.22a)$$

between composite antisymmetric tensor fields and the Nambu-Poisson Brackets (NPB) associated with p-brane coordiantes $X^\mu(\sigma^a)$ and its generalization to Clifford-spaces where

$$F = dA = \epsilon_{a_1 a_2 .... a_{p+1}} \partial_{\mu_1}\phi^{a_1} ....\partial_{\mu_{p+1}}\phi^{a_{p+1}}. \quad (3.22b)$$

the rank $p+1$ composite field strength is given in terms of $p+1$ scalar fields $\phi^1(x), \phi^2(x)....\phi^{p+1}(x)$.

$$F = dA = \epsilon_{a_1 a_2 .... a_{p+1}} \partial_{\mu_1}\phi^{a_1} ....\partial_{\mu_{p+1}}\phi^{a_{p+1}} \leftrightarrow \{ X_{\mu_1}, X_{\mu_2}, ...X_{\mu_{p+1}} \}_{NPB} =$$
$$\epsilon^{\sigma_1 \sigma_2 .... \sigma_{p+1}} (\partial_{\sigma^1} X_{\mu_1})(\partial_{\sigma^2} X_{\mu_2})...........(\partial_{\sigma^{p+1}} X_{\mu_{p+1}}). \quad (3.23)$$

Notice that

$$\{ X_{\mu_1}, X_{\mu_2}, ...X_{\mu_{p+1}} \}_{NPB} \neq \frac{\partial X^{\mu_1\mu_2.....\mu_{p+1}}}{\partial \sigma^{a_1 a_2 ......a_{p+1}}}. \quad (3.24)$$

Hence, very strong *constraints* would be imposed in the C-space dynamics of C-space branes if one were to set :

$$\{ X_{\mu_1}, X_{\mu_2}, ...X_{\mu_{p+1}} \}_{NPB} \sim \frac{\partial X^{\mu_1\mu_2.....\mu_{p+1}}}{\partial \sigma^{a_1 a_2 ......a_{p+1}}} \quad (3.25)$$

The C-space target background coordinates admit the expansion :

$$\mathbf{X}(\mathbf{\Sigma}) = X^M(\mathbf{\Sigma}) E_M = \Omega(\mathbf{\Sigma})\ \mathbf{1} + X^\mu(\mathbf{\Sigma})\gamma_\mu + X^{\mu\nu}(\mathbf{\Sigma})\gamma_\mu \wedge \gamma_\nu + X^{\mu\nu\rho}(\mathbf{\Sigma})\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho + ... \quad (3.26)$$

whereas the C-space brane world-volume coordinates admit the expansion :

$$\mathbf{\Sigma} = \Sigma^A E_A = \sigma\ \mathbf{1} + \sigma^a \gamma_a + \sigma^{ab}\gamma_a \wedge \gamma_b + \sigma^{abc}\gamma_a \wedge \gamma_b \wedge \gamma_c ...... \quad (3.27)$$

The kinetic terms of a C-space scalar field action is

$$S = \int [\mathcal{D}X]\ G^{MN}\ (\partial_M \varphi)\ (\partial_N \varphi). \quad (3.28)$$

where φ is the *scalar* component of the Clifford-valued field $\mathbf{\Phi}$ that is a *section* of the Clifford-polyvector-bundle whose structure group is the generalization of the $GL(dim\ \mathbf{F}, R)$ group acting on the fiber $\mathbf{F}$; namely it is the Clifford group acting on the polyvector-valued-fiber and generated by the basis elements E_A . A special case of a Clifford-polyvector-valued bundle is the Clifford-tangent-bundle when the fiber $\mathbf{F}$ has the *same* dimension as the base manifold $\mathbf{M}$. Hence, the multi-graded components of the *section* $\mathbf{\Phi}$ of the Clifford-polyvector-bundle are

$$\mathbf{\Phi}(\mathbf{X}) = \Phi^A E_A = \varphi(\mathbf{X})\ \mathbf{1} + \Phi^a(\mathbf{X})\gamma_a + \Phi^{ab}(\mathbf{X})\gamma_a \wedge \gamma_b + ... \quad (3.29)$$

and the Clifford-gauge-covariant derivative is

$$D_M \Phi^A = \partial_M \Phi^A\ +\ \mathcal{A}^A_{BM}\ \Phi^B$$

where $\mathcal{A}$ is the conection associated with the Clifford-polyvector-bundle. A natural action associated with the kinetic terms of the Clifford-analog of a massless field $\mathbf{\Phi}$ is

$$S = \int [\mathcal{D}X] \ G^{MN} D_M \Phi^A D_N \Phi^B \Upsilon_{AB} = \int [\mathcal{D}X] \ G^{MN} \ (\partial_M \Phi^A + \mathcal{A}^A_{CM} \Phi^C) \ (\partial_N \Phi^B + \mathcal{A}^B_{DN} \Phi^D) \Upsilon_{AB}. \quad (3.30)$$

The action above in the case that $\mathbf{\Phi}$ is a section of the Clifford-Tangent-Bundle can be rewritten as :

$$S = \int [\mathcal{D}X] \ < (\mathbf{D\Phi})^\dagger (\mathbf{D\Phi}) >_{scalar} = \int [\mathcal{D}X] \ < \ (E^M D_M \Phi^A E_A)^\dagger (E^N D_N \Phi^B E_B) \ >_{scalar} . \quad (3.31)$$

where the frame E_A of the Clifford-Tangent-Bundle is covariantly constant $D_M E_A = 0$ and

$$G^{MN} = \frac{1}{2} < \ (E^M)^\dagger E^N + E^N (E^M)^\dagger \ >_{scalar} . \quad \Upsilon_{AB} = \frac{1}{2} < \ (E_A)^\dagger E_B + E_B (E_A)^\dagger \ >_{scalar} . \quad (3.32)$$

The Geometric product among the Clifford basis elements is multi-graded since it contains objects of different grade given

$$(E^M)^\dagger E^N = \{ < (E^M)^\dagger E^N >_{r+s}, \quad < (E^M)^\dagger E^N >_{r+s-2}, \quad ..... < (E^M)^\dagger E^N >_{|r-s|} \}. \quad (3.33)$$

when $r = s$, the scalar part coincides with

$$< (E^M)^\dagger E^N >_{|r-s|} = < E^N (E^M)^\dagger >_{|r-s|} = < (E^M)^\dagger E^N >_0 = < E^N (E^M)^\dagger >_0 . \quad (3.34)$$

$$\partial_M \Phi^A = \{ \ \frac{\partial \Phi^A}{\partial x^\mu}, \quad \frac{\partial \Phi^A}{\partial x^{\mu\nu}}, \quad \frac{\partial \Phi^A}{\partial x^{\mu\nu\rho}}, \ .... \}. \quad (3.35)$$

where :

$$\frac{\partial \Phi^A}{\partial x^\mu} = \{ \ \frac{\partial \varphi}{\partial x^\mu}, \quad \frac{\partial \Phi^a}{\partial x^\mu}, \quad \frac{\partial \Phi^{ab}}{\partial x^\mu}, \quad \frac{\partial \Phi^{abc}}{\partial x^\mu}, \quad .... \}. \quad (3.36)$$

$$\frac{\partial \Phi^A}{\partial x^{\mu\nu}} = \{ \ \frac{\partial \varphi}{\partial x^{\mu\nu}}, \quad \frac{\partial \Phi^a}{\partial x^{\mu\nu}}, \quad \frac{\partial \Phi^{ab}}{\partial x^{\mu\nu}}, \quad \frac{\partial \Phi^{abc}}{\partial x^{\mu\nu}}, \quad .... \}. \quad (3.37)$$

$$\frac{\partial \Phi^A}{\partial x^{\mu\nu\rho}} = \{ \ \frac{\partial \varphi}{\partial x^{\mu\nu\rho}}, \quad \frac{\partial \Phi^a}{\partial x^{\mu\nu\rho}}, \quad \frac{\partial \Phi^{ab}}{\partial x^{\mu\nu\rho}}, \quad \frac{\partial \Phi^{abc}}{\partial x^{\mu\nu\rho}}, \quad .... \}. \quad (3.38)$$

The measure in the target 2^D-dim C-space background is :

$$[\mathcal{D}X] = [d\Omega] \ [\mathbf{\Pi} \ dx^\mu] \ [\mathbf{\Pi} \ dx^{\mu\nu}] \ [\mathbf{\Pi} \ dx^{\mu\nu\rho}].... \quad (3.39)$$

while the measure in the Clifford-valued 2^d-dim world-volume of the C-space brane is :

$$[\mathcal{D}\Sigma] = [d\sigma] \ [\mathbf{\Pi} \ d\sigma^a] \ [\mathbf{\Pi} \ d\sigma^{ab}] \ [\mathbf{\Pi} \ d\sigma^{abc}]... \quad (3.40)$$

Given a 2^d-multiplet of Clifford-scalars

$$\varphi^I \ = \ \varphi^1, \ \varphi^2, \ \varphi^3, \ ...., \ \varphi^{2^d}. \quad (3.41)$$

that defines the CATF in Clifford-spaces, in the same way that one obtained p-brane solutions to the CATF of volume-preserving diffs in section **2.1** one can find C-space brane solutions to the Clifford extensions of the ordinary CATF , thus the Brane/Composite-Antisymmetric-Tensor-Field Theory (CATF) *duality* in C-spaces is realized in terms of C-space brane actions as follows

$$\partial_M \varphi^I \leftrightarrow \partial_{\Sigma^I} X^M. \quad (3.42)$$

so that

$$\mathbf{F}_{M_1 M_2 ......M_{2^d}} \ [\ \varphi^1, \varphi^2, ..., \varphi^{2^d}\ ] \ \equiv \ \epsilon_{I_1 I_2.....I_{2^d}} \ (\partial_{M_1}\varphi^{I_1})(\partial_{M_2}\varphi^{I_2}).....(\partial_{M_{2^d}}\varphi^{I_{2^d}}) \ \leftrightarrow$$

$$\epsilon^{A_1 A_2.....A_{2^d}} \ (\partial_{\Sigma^{A_1}} X^{M_1})(\partial_{\Sigma^{A_2}} X^{M_2}).....(\partial_{\Sigma^{A_{2^d}}} X^{M_{2^d}}) \ \equiv \ \{\ X^{M_1}, X^{M_2}, ..., X^{M_{2^d}} \}_{CNPB}. \qquad (3.43a)$$

The following identity holds expressing the determinant of the induced metric G_{AB} resulting from the embedding of the 2^d-dim world-volume of the C-space Brane into the 2^D-dim target C-space background

$$det\ [\ G_{AB}\ ] = det\ [\ G_{MN}\ \partial_A X^M\ \partial_B X^N\ ] =$$

$$\frac{1}{(2^d)!} \{\ X_{M_1}, X_{M_2}, ..., X_{M_{2^d}} \}_{CNPB}\ \{\ X^{M_1}, X^{M_2}, ..., X^{M_{2^d}} \}_{CNPB}. \qquad (3.43b)$$

where X^M are the multi-graded components of the target 2^D-dim C-space background and Σ^A are the multi-graded components of the Clifford-valued 2^d-dim world-manifold associated with the C-space brane.

The Clifford-analog of the brane-field duality relationship [7] discussed in section **2.1** is

$$\sqrt{[\ \mathbf{F}_{M_1 M_2......M_{2^d}}\ (\ \varphi^1, \varphi^2, ..., \varphi^{2^d})\ ]^2} \leftrightarrow \sqrt{[\ \{\ X^{M_1}, X^{M_2}, ..., X^{M_{2^d}} \}_{CNPB}\ ]^2}. \qquad (3.44)$$

The C-space branes actions can be explicitly written in terms of the Clifford Nambu-Poisson Brackets (CNPB)

$$S = \mathcal{T} \int [\mathcal{D}\Sigma] \sqrt{\frac{1}{(2^d)!} [\ \epsilon^{A_1 A_2.....A_{2^d}}\ (\partial_{\Sigma^{A_1}} X^{M_1})(\partial_{\Sigma^{A_2}} X^{M_2}).....(\partial_{\Sigma^{A_{2^d}}} X^{M_{2^d}})\ ]^2} =$$

$$\mathcal{T} \int [\mathcal{D}\Sigma] \sqrt{\frac{1}{(2^d)!} [\ \{\ X_{M_1}, X_{M_2}, ..., X_{M_{2^d}} \}_{CNPB} \{\ X^{M_1}, X^{M_2}, ..., X^{M_{2^d}} \}_{CNPB}\ ]}. \qquad (3.45)$$

When $d = D$, all the M indices become *saturated* since the C-space brane is C-space $filling$ and there is *one* term only *inside* the square-root, hence the square root simplifies giving the standard Jacobian of the change of variables from X^M to the Σ^A described by the Nambu-Poisson Bracket :

$$S\ (d = D)\ =\ \mathcal{T} \int [\mathcal{D}\Sigma]\ \{\ X_{M_1}, X_{M_2}, ..., X_{M_{2^d}} \}_{CNPB}\ =\ \int [\mathcal{D}X]. \qquad (3.46)$$

where the measures $[\mathcal{D}\Sigma]$ and $[\mathcal{D}X]$ were given in eqs-(3.39, 3.40). When $D > d$, the number of terms inside the square root in the action (3.45) is given by the binomial coefficient $C^{2^D}_{2^d}$ giving the number of possible independent combinations of 2^d elements among a collection of 2^D elements.

In section **2. 2** we have shown why the *heuristic* duality between p-branes and composite-antisymmetric tensor field theories of volume-preserving diffs in section **2.1** could be me made more *precise* in terms of the large N limit of $SU(N)$ Yang-Mills theories via the Moyal deformation procedure, that could be extended to Generalized Yang-Mills theories (GYM). A Moyal deformation of GYM theories in the quenched-reduced approximation has been shown to yield p-brane actions [5] in the case that $p + 1 = 2n = 4k$. When the group is compact like $SO(2n)$ and when $2n = 4k = D$ the GYM Lagrangians in R^D are defined [12]

$$L = trace\ (\ F_{\mu_1\mu_2....\mu_{2n}}\ )^2 = trace\ (\ F^{i_1 i_2....i_{2n}}_{\mu_1\mu_2....\mu_{2n}}\ \Sigma_{i_1 i_2....i_{2n}}\ )^2. \qquad (3.47a)$$

The antisymmetrization of indices is performed with respect to the totally antisymmetrized product

$$\Sigma_{i_1 i_2 i_3 i_4........i_{2n-1} i_{2n}} = \Sigma_{i_1 i_2} \Sigma_{i_3 i_4}......\Sigma_{i_{2n-1} i_{2n}}\ +\ permutations. \qquad (3.47b)$$

on the indices $i_1 i_2.....i_{2n}$ of the products of the $2^{2k-1} \times 2^{2k-1}$ matrices Σ_{ij} corresponding to the chiral representation of $SO(4k)$. This is just the higher dimensional version of the decomposition of $SO(4)$ into $SU(2) \times SU(2)$.

A Moyal deformation quantization procedure of the GYM actions [12] , in the quenched-reduced approximation, in the classical limit $\hbar \rightarrow 0$ yields upon using the gauge-field/ coordinate correspondence [5,6] $A_\mu \leftrightarrow X_\mu$ and the $trace \leftrightarrow \int$ correspondence the following

$$\mathbf{F}_{\mu_1\mu_2....\mu_{2n}} \rightarrow \{ X_{\mu_1}, X_{\mu_2} \} \{ X_{\mu_3}, X_{\mu_4} \} .....\{ X_{\mu_{2n-1}}, X_{\mu_{2n}} \} + permutations. \quad (3.48)$$

and

$$trace \ [\ \mathbf{F}_{\mu_1\mu_2....\mu_{2n}} \ ]^2 \rightarrow \int [d^{4n}\sigma] \ [\ \mathcal{F}_{\mu_1\mu_2....\mu_{2n}}(\sigma^1, \sigma^2, ...\sigma^{2n}) \ ]^2. \quad (3.49)$$

The Dolan-Tchrakian p-branes Lagrangians in flat backgrounds [12] when the $conformal\ gauge$ is chosen [5] have a similar form as eq-(3.49) after using the expression eq-(3.48) when commutators are replaced by brackets. The integration domain is now $p + 1 = 4n$-dimensional $\int \ d^{4n}\sigma$ and the Poisson brackets must be taken with respect to an *enlarged* phase space of variables $q^1, p^1, ...q^{2n}, p^{2n}$ (since the new domain of integration is now $4n$-dimensional) instead of the variables $q^1, p^1, ...q^n, p^n$ ($2n$-dimensional). See the results of section **2.2** for an example.

Eguchi-Schild types of actions are different than Dolan-Tchrakian actions and involve an integration domain that is $2n$-dimensional over the variables $q^1, p^1, ...q^n, p^n$ When the world-volume of the p-brane is *even* dimensional $2n$ one can decompose the Nambu-Poisson-Bracket (NPB) as sums of antisymmetrized products of ordinary Poisson brackets since an even $2n$-volume form can be rewritten as

$$\Omega^{(2n)} = \omega^{(2)} \wedge \omega^{(2)} .... \wedge \omega^{(2)}. \quad (3.50)$$

Hence the NPB can be decomposed as

$$\{ \ X_{\mu_1}, X_{\mu_2}, ..., X_{\mu_{2n}} \ \}_{NPB} =$$

$$\{ X_{\mu_1}, X_{\mu_2} \} \{ X_{\mu_3}, X_{\mu_4} \} .....\{ X_{\mu_{2n-1}}, X_{\mu_{2n}} \} + permutations. \quad (3.51)$$

$$trace \ [\ \mathbf{F}_{\mu_1\mu_2....\mu_{2n}} \ ]^2 \rightarrow \int [d^{2n}\sigma] \ [\ \mathcal{F}_{\mu_1\mu_2....\mu_{2n}}(\sigma^1, \sigma^2, ...\sigma^{2n}) \ ]^2 =$$

$$\int [d^{2n}\sigma] \ [\ \{ \ X_{\mu_1}, X_{\mu_2}, ..., X_{\mu_{2n}} \ \}_{NPB} \ ]^2. \quad (3.52)$$

this last eq-(3.52) is just the p-brane version of the Eguchi-Schild string action (area-squared) that is only area-preserving diffs invariant. One can fully covariantize these volume-preserving diffs invariant p-brane actions (3.52) by recurring to the $auxiliary\ fields$ of [13] as we discussed in eq-(2.39) of section **2.2** when we constructed the *new* fully reparametrization invariant p-brane actions and/or by introducing an auxiliary metric as it is customary done in the covariant form of Eguchi-Schild actions.

In the next sections we will construct the generalized Noncommutative Yang's spacetime algebra extended to C-spaces, which in turn, will allow us to define a Moyal-Yang star product deformations of Generalized Yang-Mills in C-spaces and establish the relationship with branes in C-spaces . In **6** we will perform the Moyal-Yang deformations of C-space brane models described by the actions

$$S = \mathcal{T} \int [\mathcal{D}\Sigma] \ \sqrt{ [\ \{ \ X_{M_1}, X_{M_2}, ..., X_{M_{2d}} \}_{CNPB} \ ]^2 }. \quad (3.53)$$

and the *new* p-brane actions studied in section **2.2** in terms of a family of scalar fields which define new integration measures in eq-(2.39) .

4. The Noncommutative Yang's Spacetime algebra from Clifford algebras

Prior to constructing the Moyal deformations (based on the novel Moyal-Yang star products with an UV-IR cutoff) of the brane actions in Clifford-space target backgrounds discussed in the previous section, it is essential to discuss in detail the interplay between the Noncommutative Yang's Spacetime algebra in $4D$ involving a lower and upper length scale and Clifford algebras.

The main result of this section is to show that there is a *subalgebra* of the C-space operator-valued coordinates which is *isomorphic* to the Noncommutative Yang's spacetime algebra [21] . This, in conjunction to the discrete spectrum of angular momentum, leads to the discrete area-quantization in multiples of Planck areas. Namely, the $4D$ Yang's Noncommutative space-time (YNST) algebra [21] (written in terms of $8D$ phase-space coordinates) is isomorphic to the 15-dimensional *subalgebra* of the C-space operator-valued coordinates associated with the *holographic areas* of C-space. This connection between Yang's algebra and the $6D$ Clifford algebra is possible because the $8D$ phase-space coordinates x^μ, p^μ (associated to a $4D$ spacetime) have a one-to-one correspondence to the $\hat{X}^{\mu 5}; \hat{X}^{\mu 6}$ holographic area-coordinates of the C-space (corresponding to the $6D$ Clifford algebra).

Furhermore, Tanaka [23] has shown that the Yang's algebra [21] (with 15 generators) is related to the $4D$ conformal algebra (15 generators) which in turn is isomorphic to a subalgebra of the $4D$ Clifford algebra because it is known that the 15 generators of the $4D$ conformal algebra $SO(4,2)$ can be explicitly realized in terms of the $4D$ Clifford algebra as [16] :

$$P^\mu = \mathcal{M}^{\mu 5} + \mathcal{M}^{\mu 6} = \gamma^\mu(\mathbf{1} + \gamma^5). \quad K^\mu = \mathcal{M}^{\mu 5} - \mathcal{M}^{\mu 6} = \gamma^\mu(\mathbf{1} - \gamma^5). \quad D = \gamma^5. \quad M^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]. \tag{4.1}$$

where the Clifford algebra generators :

$$\mathbf{1}. \quad \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 = \gamma^5. \tag{4.2}$$

account for the extra *two* directions within the C-space associated with the $4D$ Cliffiord-algebra leaving effectively $4 + 2 = 6$ degrees of freedom that match the degrees of freedom of a $6D$ spacetime [16] . The relevance of [16] is that it was not necessary to work directly in $6D$ to find a realization of the $4D$ conformal algebra $SO(4,2)$. It was possible to attain this by recurring solely to the $4D$ Clifford algebra as shown in eq-(4.1) .

One can also view the $4D$ conformal algebra $SO(4,2)$ realized in terms of a 15-dim *subalgebra* of the $6D$ Clifford algebra. The bivector holographic area-coordinates $X^{\mu\nu}$ couple to the basis generators $\Gamma_\mu \wedge \Gamma_\nu$. The bivector coordinates $X^{\mu 5}$ couple to the basis generators $\Gamma_\mu \wedge \Gamma_5$ where now the Γ^5 is another generator of the $6D$ Clifford algebra and *must not* be confused with the usual γ^5 defined by eq-(4.2) . The bivector coordinates $X^{\mu 6}$ couple to the basis generators $\Gamma_\mu \wedge \Gamma_6$. The bivector coordinate X^{56} couples to the basis generator $\Gamma_5 \wedge \Gamma_6$.

In view of this fact that these bivector holographic area-coordinates in $6D$ *couple* to the bivectors basis elements $\Gamma_\mu \wedge \Gamma_\nu, ...$, and whose algebra is in turn isomorphic to the $4D$ conformal algebra $SO(4,2)$ via the realization in terms of the $6D$ angular momentum generators (and boosts generators) $\mathcal{M}^{\mu\nu} \sim [\Gamma^\mu, \Gamma^\nu]$, $\mathcal{M}^{\mu 5} \sim [\Gamma^\mu, \Gamma^5]$,.... we shall *define* the holographic *area coordinates algebra* in C-space as the dual algebra to the $SO(4,2)$ conformal algebra (realized in terms of the $6D$ angular momentum, boosts, generators in terms of a $6D$ Clifford algebra generators as shown)

Notice that the conformal boosts K^μ and the translations P^μ in eq-(4.1) do commute $[P^\mu, P^\nu] = [K^\mu, K^\nu] = 0$ and for this reason we shall assign the appropriate correspondence $p^\mu \leftrightarrow X^{\mu 6}$ and $x^\mu \leftrightarrow X^{\mu 5}$, up to numerical factors (lengths) to match dimensions, in order to attain *noncommuting* variables x^μ, p^μ .

Therefore, one has two possible routes to relate Yang's algebra with Clifford algebras. One can relate Yang's algebra with the holographic area-coordinates algebra in the C-space associated to a $6D$ Clifford algebra and/or to the subalgebra of a $4D$ Clifford algebra via the realization of the conformal algebra $SO(4,2)$ in terms of the $4D$ Clifford algebra generators $\mathbf{1}, \gamma^5, \gamma^\mu$ as shown in eq-(4.1).

Since the relation between the $4D$ conformal and Yang's algebra and the implications for the AdS/CFT, dS/CFT duality have been discussed before by Tanaka [23], in this work we shall establish the following correspondence between the C-space holographic-area coordinates algebra (associated to the $6D$ Clifford algebra) and the Yang's spacetime algebra via the angular momentum generators in $6D$ as follows :

$$i\hat{M}^{\mu\nu} = i\hbar\Sigma^{\mu\nu} \leftrightarrow i\frac{\hbar}{\lambda^2}\hat{X}^{\mu\nu}. \tag{4.3}$$

$$i\hat{M}^{56} = i\hbar\Sigma^{56} \leftrightarrow i\frac{\hbar}{\lambda^2}\hat{X}^{56}. \tag{4.4}$$

$$i\lambda^2\Sigma^{\mu 5} = i\lambda\hat{x}^\mu \leftrightarrow i\hat{X}^{\mu 5}. \tag{4.5}$$

$$i\lambda^2\Sigma^{\mu 6} = i\lambda^2\frac{R}{\hbar}\hat{p}^\mu \leftrightarrow i\hat{X}^{\mu 6}. \tag{4.6}$$

With Hermitian (bivector) operator- coordinates :

$$(\hat{X}^{\mu\nu})^\dagger = \hat{X}^{\mu\nu}. \quad (\hat{X}^{\mu 5})^\dagger = \hat{X}^{\mu 5}. \quad (\hat{X}^{\mu 6})^\dagger = \hat{X}^{\mu 6}. \quad (\hat{X}^{56})^\dagger = \hat{X}^{56}. \tag{4.7}$$

The algebra generators can be realized as :

$$\hat{X}^{\mu\nu} = i\lambda^2(X^\mu\frac{\partial}{\partial X_\nu} - X^\nu\frac{\partial}{\partial X_\mu}). \tag{4.8a}$$

$$\hat{X}^{\mu 5} = i\lambda^2(X^\mu\frac{\partial}{\partial X_5} - X^5\frac{\partial}{\partial X_\mu}). \tag{4.8b}$$

$$\hat{X}^{\mu 6} = i\lambda^2(X^\mu\frac{\partial}{\partial X_6} - X^6\frac{\partial}{\partial X_\mu}). \tag{4.8c}$$

$$\hat{X}^{56} = i\lambda^2(X^5\frac{\partial}{\partial X_6} - X^6\frac{\partial}{\partial X_5}). \tag{4.8d}$$

where the angular momentum generators are defined as usual :

$$\hat{M}^{\mu\nu} \equiv \hbar\Sigma^{\mu\nu}. \quad \hat{M}^{\mu 5} \equiv \hbar\Sigma^{\mu 5}. \quad \hat{M}^{\mu 6} \equiv \hbar\Sigma^{\mu 6}. \quad \hat{M}^{56} \equiv \hbar\Sigma^{56}. \tag{4.8e}$$

which have a one-to-one correspondence to the Yang Noncommutative space-time (YNST) algebra generators in $4D$. These generators (angular momentum differential operators) act on the coordinates of a $5D$ hyperboloid AdS_5 space defined by :

$$-(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2 - (x^6)^2 = R^2. \tag{4.9a}$$

where R is the *throat* size of the hyperboloid. This introduces an extra and crucial scale in addition to the Planck scale. Notice that $\eta^{55} = +1$. $\eta^{66} = -1$. $5D$ de Sitter space dS_5 has the topology of $S^4 \times R^1$. Whereas AdS_5 space has the topology of $R^4 \times S^1$ and its conformal (projective) boundary at infinity has a topology $S^3 \times S^1$. Whereas the *Euclideanized* Anti de Sitter space AdS_5 can be represented geometrically as two disconnected branches (sheets) of a $5D$ hyperboloid embedded in $6D$. The topology of these two disconnected branches is that of a $5D$ disc and the metric is the Lobachevsky one of constant negative curvature. The conformal group $SO(4,2)$ leaves the $4D$ lightcone at infinity invariant.

Thus, *Euclideanized* AdS_5 is defined by a Wick rotation of the x^6 coordinate giving :

$$-(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2 + (x^6)^2 = R^2. \tag{4.9b}$$

whereas de Sitter space dS_5 with the topology of a pseudo-sphere $S^4 \times R^1$, and *positive* constant scalar curvature is defined by :

$$-(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2 + (x^6)^2 = -R^2. \tag{4.9c}$$

(Notice that Tanaka [23] uses $different$ conventions than ours in his definition of the $5D$ hyperboloids. He has a sign change from R^2 to $-R^2$ because he introduces i factors in iR) .

After this discussion and upon a direct use of the correspondence in eqs-(4.3, 4.4, 4.5, 4.6 ...) yields the exchange algebra between the position and momentum coordinates :

$$[\hat{X}^{\mu 6}, \hat{X}^{56}] = -i\lambda^2\eta^{66}\hat{X}^{\mu 5} \leftrightarrow [\frac{\lambda^2 R}{\hbar}\hat{p}^\mu,\ \lambda^2\Sigma^{56}] = -i\lambda^2\eta^{66}\lambda\hat{x}^\mu. \tag{4.10}$$

from which we can deduce that :

$$[\hat{p}^\mu,\ \Sigma^{56}] = -i\eta^{66}\frac{\hbar}{\lambda R}\hat{x}^\mu. \tag{4.11}$$

and after using the definition $\mathcal{N} = (\lambda/R)\Sigma^{56}$ one has the exchange algebra commutator of p^μ and $\mathcal{N}$ of the Yang's spacetime algebra :

$$[\hat{p}^\mu, \mathcal{N}] = -i\eta^{66}\frac{\hbar}{R^2}\hat{x}^\mu. \tag{4.12}$$

The other commutator is :

$$[\hat{X}^{\mu 5}, \hat{X}^{56}] = -[\hat{X}^{\mu 5}, \hat{X}^{65}] = i\eta^{55}\lambda^2\hat{X}^{\mu 6} \leftrightarrow [\lambda\hat{x}^\mu, \lambda^2\Sigma^{56}] = i\eta^{55}\lambda^2\lambda^2\frac{R}{\hbar}\hat{p}^\mu. \tag{4.13}$$

from which we can deduce that :

$$[\hat{x}^\mu, \Sigma^{56}] = i\eta^{55}\frac{\lambda R}{\hbar}\hat{p}^\mu. \tag{4.14}$$

and after using the definition $\mathcal{N} = (\lambda/R)\Sigma^{56}$ one has the exchange algebra commutator of x^μ and $\mathcal{N}$ of the Yang's spacetime algebra :

$$[\hat{x}^\mu, \mathcal{N}] = i\eta^{55}\frac{\lambda^2}{\hbar}\hat{p}^\mu. \tag{4.15}$$

The other relevant holographic area-coordinates commutators in C-space are :

$$[\hat{X}^{\mu 5}, \hat{X}^{\nu 5}] = -i\eta^{55}\lambda^2\hat{X}^{\mu\nu} \leftrightarrow [\hat{x}^\mu, \hat{x}^\nu] = -i\eta^{55}\lambda^2\Sigma^{\mu\nu}. \tag{4.16}$$

after using the representation of the C-space operator holographic area-coordinates :

$$i\hat{X}^{\mu\nu} \leftrightarrow i\lambda^2\frac{1}{\hbar}\mathcal{M}^{\mu\nu} = i\lambda^2\Sigma^{\mu\nu} \quad i\hat{X}^{56} \leftrightarrow i\lambda^2\Sigma^{56}. \tag{4.17}$$

where we appropriately introduced the Planck scale λ as one should to match units.

From the correspondence :

$$\hat{p}^\mu = \frac{\hbar}{R}\Sigma^{\mu 6} \leftrightarrow \frac{\hbar}{R}\frac{1}{\lambda^2}\hat{X}^{\mu 6}. \tag{4.18}$$

one can obtain nonvanishing momentum commutator :

$$[\hat{X}^{\mu 6}, \hat{X}^{\nu 6}] = -i\eta^{66}\lambda^2\hat{X}^{\mu\nu} \leftrightarrow [\hat{p}^\mu, \hat{p}^\nu] = -i\eta^{66}\frac{\hbar^2}{R^2}\Sigma^{\mu\nu}. \tag{4.19}$$

The signatures for AdS_5 space are $\eta^{55} = +1$; $\eta^{66} = -1$ and for the *Euclideanized* AdS_5 space are $\eta^{55} = +1$ and $\eta^{66} = +1$. Yang's space-time algebra corresponds to the latter case.

Finally, the *modified* Heisenberg algebra can be read from the following C-space commutators :

$$[\hat{X}^{\mu 5}, \hat{X}^{\nu 6}] = i\eta^{\mu\nu}\lambda^2\hat{X}^{56} \leftrightarrow$$

$$[\hat{x}^\mu, \hat{p}^\mu] = i\hbar\eta^{\mu\nu}\frac{\lambda}{R}\Sigma^{56} = i\hbar\eta^{\mu\nu}\mathcal{N}. \tag{4.20}$$

Eqs-(4.12, 4.15, 4.16, 4.19, 4.20) are the defining relations of Yang's Noncommutative $4D$ spacetime algebra involving the $8D$ phase-space variables. These commutators obey the Jacobi identities. There are other commutation relations like $[\mathcal{M}^{\mu\nu}, x^\rho]$, $[\mathcal{M}^{\mu\nu}, p^\rho]$ that we did not write down. These are just the well known rotations (boosts) of the coordinates and momenta.

When $\lambda \to 0$ and $R \to \infty$ one recovers the ordinary *commutative* spacetime algebra. The Snyder algebra [22] is recovered by setting $R \to \infty$ while leaving λ intact. To recover the ordinary Weyl-Heisenberg algebra is more subtle. Tanaka [23] has shown the the *spectrum* of the operator $\mathcal{N} = (\lambda/R)\Sigma^{56}$ is discrete given by $n(\lambda/R)$. This is not suprising since the angular momentum generator $\mathcal{M}^{56}$ associated with the *Euclideanized* AdS_5 space is a rotation in the now compact $x^5 - x^6$ directions. This is not the case in

AdS_5 space since $\eta^{66} = -1$ and this timelike direction is no longer compact. Rotations involving timelike directions are equivalent to noncompact boosts with a continuous spectrum.

In order to recover the standard Weyl-Heisenberg algebra from Yang's Noncommutative spacetime algebra, and the standard uncertainty relations $\Delta x \Delta p \geq \hbar$ with the ordinary $\hbar$ term , rather than the $n\hbar$ term, one needs to take the limit $n \rightarrow \infty$ limit in such a way that the net combination of $n\frac{\lambda}{R} \rightarrow 1$.

This can be attained when one takes the *double* scaling limit of the quantities as follows :

$$\lambda \rightarrow 0. \quad R \rightarrow \infty. \quad \lambda R \rightarrow L^2.$$

$$lim_{n\rightarrow\infty}\ n\frac{\lambda}{R} = n\frac{\lambda^2}{\lambda R} = \frac{n\lambda^2}{L^2} \rightarrow 1. \tag{4.21}$$

From eq-(4.21) one learns then that :

$$n\lambda^2 = \lambda R = L^2. \tag{4.22}$$

The spectrum n corresponds to the quantization of the angular momentum operator in the $x^5 - x^6$ direction (after embedding the $5D$ hyperboloid of throat size R onto $6D$) . Tanaka [23] has shown why there is a *discrete spectra* for the *spatial* coordinates and *spatial* momenta in Yang's spacetime algebra that yields a *minimum* length λ (ultraviolet cutoff in energy) and a minimum momentum $p = \hbar/R$ (maximal length R , infrared cutoff) . The energy and temporal coordinates had a continous spectrum.

The physical interpretation of the double-scaling limit of eq-(4.21, 4.22) is that the the area $L^2 = \lambda R$ becomes now *quantized* in units of the Planck area λ^2 as $L^2 = n\lambda^2$. Thus the quantization of the area (via the double scaling limit) $L^2 = \lambda R = n\lambda^2$ is a result of the *discrete* angular momentum spectrum in the $x^5 - x^6$ directions of the Yang's Noncommutative spacetime algebra when it is realized by (angular momentum) differential operators acting on the *Euclideanized* AdS_5 space (two branches of a $5D$ hyperboloid embedded in $6D$). A general interplay between quantum of areas and quantum of angular momentum, for arbitrary values of spin, in terms of the square root of the Casimir $\mathbf{A} \sim \lambda^2\sqrt{j(j+1)}$, has been obtained a while ago in Loop Quantum Gravity by using spin-networks techniques and highly technical area-operator regularization procedures [24] . The advantage of this work is that we have arrived at similar (not identical) area-quantization conclusions in terms of minimal Planck areas and a discrete angular momentum spectrum n via the double scaling limit based on Clifford algebraic methods (C-space holographic area-coordinates). This is not surprising since the norm-squared of the holographic Area operator has a correspondence with the quadratic Casimir $\Sigma_{AB}\Sigma^{AB}$ of the conformal algebra $SO(4,2)$ ($SO(5,1)$ in the Euclideanized AdS_5 case). This quadratic Casimir must not be confused with the $SU(2)$ Casimir J^2 with eigenvalues $j(j+1)$. Hence, the correspondence given by eqs-(4.1-4.8) gives $\mathbf{A}^2 \leftrightarrow \lambda^4\Sigma_{AB}\Sigma^{AB}$.

In [25] we have shown why AdS_4 gravity with a topological term; i.e. an Einstein-Hilbert action with a cosmological constant plus Gauss-Bonnet terms can be obtained from the vacuum state of a **BF**-Chern-Simons-Higgs theory *without* introducing by *hand* the zero torsion condition imposed in the MacDowell-Mansouri-Chamsedine-West construction. One of the most salient features of [25] was that a *geometric mean* relationship was found among the cosmological constant ρ_{vacuum} , the Planck area λ^2 and the AdS_4 throat size squared R^2 given by $(\rho_v)^{-1} = (\lambda)^2(R^2)$. A similar geometric mean relation is also obeyed by the condition $\lambda R = L^2 (= n\lambda^2)$ in the double scaling limit of Yang's algebra which suggests to identify the cosmological constant as $\rho_{vacuum} = L^{-4}$. Notice that by setting the infrared scale R equal to the Hubble radius horizon R_H and λ equal to the Planck scale one reproduces precisely the *observed* value of the vacuum energy density ! [25] : $\rho \sim L_{Planck}^{-2}R_H^{-2} = L_P^{-4}(L_{Planck}/R_H)^2 \sim 10^{-122}M_{Planck}^4$. This geometric mean condition remains to be investigated further. In particular, we presented the preliminary steps how to construct a Noncommutative Gravity via the Vasiliev-Moyal star products deformations of the $SO(4,2)$ algebra used in the study of higher conformal massless spin theories in AdS spaces by taking the inverse-throat size $1/R$ as a deformation parameter of the $SO(4,2)$ algebra [20] . A new realization of holography and the geometrical intepretation of AdS_{2n} spaces in terms of $SO(2n-1,2)$ instantons was studied in [27] .

Since the expectation value

$$\frac{\lambda^2}{L^2} < n|\Sigma^{56}|n >= \frac{n\lambda^2}{L^2} = 1. \tag{4.23}$$

in the double-scaling limit one recovers the standard Heisenberg uncertainty relations :

$$\Delta x^\mu \Delta p^\mu \geq \frac{1}{2} || < [x^\mu, p^\mu] > || = \hbar. \tag{4.24}$$

and the commutators become in the double-scaling limit:

$$[\hat{p}^\mu,\ \Sigma^{56}] = -i\eta^{66} \frac{\hbar}{L^2} \hat{x}^\mu. \quad [\hat{p}^\mu,\ \mathcal{N}] = 0. \tag{4.25}$$

$$[\hat{x}^\mu,\ \Sigma^{56}] = -i\eta^{55} \frac{L^2}{\hbar} \hat{p}^\mu. \quad [\hat{x}^\mu,\ \mathcal{N}] = 0. \tag{4.26}$$

$$[\hat{x}^\mu,\ \hat{x}^\nu] = [\hat{p}^\mu,\ \hat{p}^\nu] = 0. \quad [\hat{x}^\mu,\ \hat{p}^\mu] = i\hbar\eta^{\mu\nu} \frac{\lambda^2}{L^2} \Sigma^{56} \rightarrow i\hbar\eta^{\mu\nu} \mathbf{1}. \tag{4.27}$$

Rigorously speaking, when $\lambda \rightarrow 0$ the last commutator $[x^\mu, p^\nu] \rightarrow 0$ since the generator Σ^{56} is well defined. It is the large n limit of the spectrum $< n|\Sigma^{56}|n >$ that reproduces the ordinary Heisenberg uncertainty relations.

5 The Generalized Noncommutative Yang's Algebra In Clifford Spaces

Having studied in detail the Yang's Noncommutative spacetime algebra in section **4** we must generalize such algebra to include multivector-valued coordinates and momenta of arbitrary grade. This will be the final step before one can proceed with the novel Moyal-Yang star products deformations of branes in Clifford-space target backgrounds (involving the UV-IR cutoffs) that is the final goal of this work.

In order to generalize Yang's Noncommutative spacetime algebra to the full Clifford space associated with the 4-dim real Clifford algebra $Cl(4, R)$ one needs to enlarge the number of dimensions from $D = 4$ to $D = 12$ as follows. Since in the previous section we have established the isomorphism $x^\mu \leftrightarrow \lambda\Sigma^{\mu 5}$, $p^\mu \leftrightarrow \frac{\hbar}{R}\Sigma^{\mu 6}$, and $(\lambda/R)\Sigma^{56} = \mathcal{N}$, the generalized nonzero $[x, p]$ commutator must be of the form :

$$[\ x^{\mu_1,\mu_2....\mu_n},\ p^{\nu_1\nu_2...\nu_n}\ ] = [\ \lambda^n \Sigma^{\mu_1\mu_2...\mu_n\ i_1 i_2....i_n},\ (\frac{\hbar}{R})^n \Sigma^{\nu_1\nu_2.....\nu_n\ j_1 j_2...j_n}\ ] =$$

$$i\hbar^n \eta^{\mu_1\nu_1\mu_2\nu_2.....\mu_n\nu_n} (\frac{\lambda}{R})^n \Sigma^{\ [i_1 i_2....i_n]\ [j_1 j_2.....j_n]} = i\hbar^n \eta^{\mu_1\nu_1\mu_2\nu_2.....\mu_n\nu_n} \mathcal{N}^{\ [i_1 i_2....i_n]\ [j_1 j_2.....j_n]}. \tag{5.1}$$

where the indices i, j span the *extra* dimensions as follows : the index i spans over the values $i = 5, 7, 9, 11$ only, and the index j spans over the values $j = 6, 8, 10, 12$ only . The i indices are linked to the poly-vector-valued coordinates $x^{\mu_1\mu_2....\mu_n}$ and the j indices are linked to the poly-momentum variables $p^{\nu_1\nu_2...\nu_n}$. Further possibilities occur when *both* indices i, j span over *all* the internal directions $5, 6, 7, ...12$. In that case the commutator (5.2) will contain additional terms of the form : $\eta^{i_1 j_1....i_n j_n} \Sigma^{\mu_1\mu_2....\mu_n\nu_1\nu_2....\nu_n}$. For the time being we shall *restrict* the indices so that $i = 5, 7, 9, 11$ and $j = 6, 8, 10, 12$ only. The index n spans $1, 2, 3, 4$ since $n = 4$ is the maximal grade of the Clifford polyvector associated with the Clifford algebra in $D = 4$.

$\eta^{\mu_1\nu_1\mu_2\nu_2.....\mu_n\nu_n}$ is given by the determinant of the $N \times N$ matrix $\mathbf{G}_{mn}$ whose entries are $\eta^{\mu_m\nu_n}$:

$$\eta^{\mu_1\nu_1\mu_2\nu_2.....\mu_n\nu_n} = det\ \mathbf{G}_{mn} = \frac{1}{N!} \epsilon_{i_1 i_2....i_n} \epsilon_{j_1 j_2....j_n} \eta^{\mu_{i_1}\nu_{j_1}} \eta^{\mu_{i_2}\nu_{j_2}} ....\eta^{\mu_{i_n}\nu_{j_n}}. \tag{5.2}$$

For example :

$$\eta^{\mu_1\nu_1\mu_2\nu_2} = \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2} - \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_1} \quad etc.... \tag{5.3}$$

Similar results apply to the definition of $\eta^{i_1 j_1....i_n j_n}$.

The generator $\Sigma^{i_1 i_2...}$ which generalizes the $\frac{\lambda}{R}\Sigma^{56} = \mathcal{N}$ Yang's generator is *antisymmetric* under the *collective* exchange of indices :

$$\Sigma^{\ [i_1 i_2....i_n]\ [j_1 j_2.....j_n]} = -\Sigma^{\ [j_1 j_2....j_n]\ [i_1 i_2.....i_n]}. \tag{5,4}$$

and is also antisymmetric in the $[i_1, i_2, ...i_n]$ and $[j_1, j_2, ...j_n]$ indices, respectively.

Hence, the generalized (nonzero) commutators $[x, \mathcal{N}], [p, \mathcal{N}]$ read:

$$[\ X^{\mu_1\mu_2...\mu_n\ k_1k_2......k_n},\ \Sigma^{\ [i_1i_2....i_n]\ [j_1j_2.....j_n]}\ ] = i\eta^{i_1k_1i_2k_2....i_nk_n} X^{\mu_1\mu_2...\mu_n\ j_1j_2......j_n}$$

$$-i\eta^{j_1k_1j_2k_2....j_nk_n} X^{\mu_1\mu_2...\mu_n\ i_1i_2......i_n}. \tag{5.5}$$

If the i, k indices span over the $5, 7, 9, 11$ directions only, the second term in the r.h.s will vanish. If the j, k indices span over the $6, 8, 10, 12$ only, the first term in the r.h.s will vanish. If all the indices i, j, k span over all the $5, 6, 7, ...12$ directions then both terms in the r.h.s will be *nonvanishing*. Hence upon using the correspondence

$$x^{\mu_1\mu_2...\mu_n} \leftrightarrow \lambda^n \Sigma^{\mu_1\mu_2....\mu_n\ i_1i_2...i_n}. \quad p^{\mu_1\mu_2...\mu_n} \leftrightarrow (\frac{\hbar}{R})^n \Sigma^{\mu_1\mu_2....\mu_n\ j_1j_2...j_n}.$$

$$(\frac{\lambda}{R})^n \Sigma^{\ [i_1i_2....i_n]\ [j_1j_2.....j_n]} \equiv \mathcal{N}^{\ [i_1i_2....i_n]\ [j_1j_2.....j_n]} \tag{5.6}$$

where $i = 5, 7, 9, 11$ and $j = 6, 8, 10, 12$, the commutators which exchange coordinates for momenta are :

$$[\ x^{\mu_1\mu_2...\mu_n},\ \mathcal{N}^{\ [i_1i_2....i_n]\ [j_1j_2.....j_n]}\ ] = i\eta^{i_1k_1i_2k_2....i_nk_n} \lambda^n (\frac{1}{(\hbar/\lambda)})^n p^{\mu_1\mu_2...\mu_n} \tag{5.7}$$

and

$$[\ p^{\mu_1\mu_2...\mu_n},\ \mathcal{N}^{\ [i_1i_2....i_n]\ [j_1j_2.....j_n]}\ ] = -i\eta^{j_1k_1j_2k_2....j_nk_n} (\frac{\hbar}{R})^n \frac{x^{\mu_1\mu_2...\mu_n}}{R^n}. \tag{5.8}$$

The generalized (nonzero) commutator of two polyvector-valued coordinate operators is :

$$[\ x^{\mu_1\mu_2...\mu_n},\ x^{\nu_1\nu_2.....\nu_n}\ ] = i\lambda^{2n} \Sigma^{\ [\mu_1\mu_2......\mu_n]\ [\nu_1\nu_2.......\nu_n]}. \tag{5.9}$$

where

$$\Sigma^{\ [\mu_1\mu_2......\mu_n]\ [\nu_1\nu_2.......\nu_n]} = -\Sigma^{\ [\nu_1\nu_2......\nu_n]\ [\mu_1\mu_2.......\mu_n]}. \tag{5.10}$$

and is antisymmetric in the $\mu_1, \mu_2...\mu_n$ and $\nu_1\nu_2.....\nu_n$ indices respectively.

The generalized (nonzero) commutator of two polyvector-valued momentum operators is

$$[\ p^{\mu_1\mu_2...\mu_n},\ p^{\nu_1\nu_2.....\nu_n}\ ] = i(\frac{\hbar}{R})^{2n} \Sigma^{\ [\mu_1\mu_2.......\mu_n]\ [\nu_1\nu_2.......\nu_n]}. \tag{5.11}$$

The remaining (nonzero) commutators are:

$$[\ x^{\mu_1\mu_2...\mu_n},\ \Sigma^{\ [\nu_1\nu_2....\nu_n]\ [\rho_1\rho_2....\rho_n]}\ ] = i\eta^{\mu_1\nu_1\mu_2\nu_2......\mu_n\nu_n} x^{\rho_1\rho_2.....\rho_n} - i\eta^{\mu_1\rho_1\mu_2\rho_2......\mu_n\rho_n} x^{\nu_1\nu_2.....\nu_n}. \tag{5.12}$$

$$[\ p^{\mu_1\mu_2...\mu_n},\ \Sigma^{\ [\nu_1\nu_2....\nu_n]\ [\rho_1\rho_2....\rho_n]}\ ] = i\eta^{\mu_1\nu_1\mu_2\nu_2......\mu_n\nu_n} p^{\rho_1\rho_2.....\rho_n} - i\eta^{\mu_1\rho_1\mu_2\rho_2......\mu_n\rho_n} p^{\nu_1\nu_2.....\nu_n}. \tag{5.13}$$

which are just poly-rotations of poly-vectors and finally the generalized Lorentz algebra in C-space reads:

$$[\ \Sigma^{\ [\mu_1\mu_2...\mu_n]\ [\nu_1\nu_2....\nu_n]},\ \Sigma^{\ [\rho_1\rho_2....\rho_n]\ [\tau_1\tau_2....\tau_n]}\ ] = i\eta^{\mu_1\rho_1\mu_2\rho_2......\mu_n\rho_n} \Sigma^{\ [\nu_1\nu_2.....\nu_n]\ [\tau_1\tau_2.....\tau_n]}$$

$$-i\eta^{\mu_1\tau_1\mu_2\tau_2......\mu_n\tau_n} \Sigma^{\ [\nu_1\nu_2.....\nu_n]\ [\rho_1\rho_2.....\rho_n]} - i\eta^{\nu_1\rho_1\nu_2\rho_2......\nu_n\rho_n} \Sigma^{\ [\mu_1\mu_2.....\mu_n]\ [\tau_1\tau_2.....\tau_n]} +$$

$$i\eta^{\nu_1\tau_1\nu_2\tau_2......\nu_n\tau_n} \Sigma^{\ [\mu_1\mu_2.....\mu_n]\ [\rho_1\rho_2.....\rho_n]}. \tag{5.14}$$

These commutators are the natural generalization of the Yang's Noncommutative spacetime algebra in Clifford spaces and obey the Jacobi identities. Since the poly-vector valued coordinates and momenta don't commute we expect to have uncertainty relations of the form :

$$\Delta x^{\mu_1\mu_2\ldots\mu_n}\Delta p^{\mu_1\mu_2\ldots\mu_n} \geq \hbar^n. \quad \Delta x^{\mu_1\mu_2\ldots\mu_n}\Delta x^{\nu_1\nu_2\ldots\nu_n} \geq \lambda^{2n}. \tag{5.15}$$

These generalized uncertainty relations and the n-volume *quantization* in units of the Planck scale will be the subject of future investigation.

6. Moyal-Yang Star Products and Noncommutative Branes in Clifford spaces

In section **2.2** we reviewed the interplay between ordinary brane actions in ordinary target spacetime backgrounds and the Moyal deformation quantization of (Generalized) Yang-Mills theories, in the quenched-reduced approximation. The crux of this last section is to perform a novel Moyal-Yang Quantization of Generalized Yang-Mills theories in Clifford spaces that yields the sought-after star product deformations of the brane actions in Clifford space target backgrounds based on multivector (polyvector, antisymmetric tensor) valued coordinates. In order to achieve this goal one must recur to the results of sections **4,5** describing the (generalized) Yang's Noncommutative algebra involving the multivector-valued coordinates in Clifford spaces. Ordinary branes actions in ordinary target spacetime backgrounds were given in eqs-(3.48, 3.49) and eq-(3.52). The purpose of this section is to construct generalized generalized brane actions in Clifford spaces (C-space branes) and their deformations based on the novel Moyal-Yang star products with UV-IR cutoffs.

Noncommutative Classical Mechanics has a correspondence to Noncommutative Quantum Mechanics. Denoting the Clifford polyvector-valued indices of different grades by the indices $A, B, C....$ spanning over all the multi-graded components of a Clifford polyvector , the inverse of the poly-symplectic form in C-phase-spaces Ω^{AB} is a $2^{d+1} \times 2^{d+1}$ matrix comprised of *blocks* of $2^d \times 2^d$ antisymmetric matrices consisting of the entries given by $\{q^A, q^B\}$ and $\{p^A, p^B\}$ along the main block-diagonal, and blocks of $2^d \times 2^d$ matrices given by the entries $\{p^A, q^B\}$ and $\{q^B, p^A\}$ off the main block-diagonal such that the Noncommutative Poisson Brackets (NCPB) are defined as

$$\{ \mathcal{F}(q^a, p^a) , \mathcal{G}(q^A, p^A) \}_\Omega = (\partial_{\Upsilon^A}\mathcal{F})\, \Omega^{AB}\, (\partial_{\Upsilon^B}\mathcal{G}) = (\partial_{q^A}\mathcal{F})\, \{q^A, q^B\}\, (\partial_{q^B}\mathcal{G}) + (\partial_{p^A}\mathcal{F})\, \{p^A, p^B\}\, (\partial_{p^B}\mathcal{G}) +$$

$$(\partial_{q^A}\mathcal{F})\, \{q^A, p^B\}\, (\partial_{p^B}\mathcal{G}) \; + \; (\partial_{p^A}\mathcal{F})\, \{p^A, q^B\}\, (\partial_{q^B}\mathcal{G}). \tag{6.1}$$

where the entries $\{q^A, q^B\}$, $\{p^A, p^B\}$, $\{p^A, q^B\}$, $\{p^A, q^B\}$ can be read from the generalized Noncommutative Yang's algebra in C-spaces described in the previous section. In ordinary Classical Mechanics (corresponding to ordinary Quantum Mechanics) the $\{q^A, q^B\}$ and $\{p^A, p^B\}$ brackets are *zero*. This is *not* the case in Noncommutative Classical Mechanics (corresponding to Noncommutative Quantum Mechanics) in C-spaces. For example, the classical-quantum mechanical correspondence among the ordinary phase space variables is

$$\{x^\mu, x^\nu\} \leftrightarrow \frac{1}{i\hbar_{eff}}[\hat{X}^\mu, \hat{X}^\nu]. \quad \{p^\mu, p^\nu\} \leftrightarrow \frac{1}{i\hbar_{eff}}[\hat{P}^\mu, \hat{P}^\nu]. \quad \{x^\mu, p^\nu\} \leftrightarrow \frac{1}{i\hbar_{eff}}[\hat{X}^\mu, \hat{P}^\nu]. \tag{6.2}$$

where the $effective$ Planck constant is now comprised of the 3 fundamental parameters, $\hbar, \lambda, R$. Hence the deformation parameter in the Moyal star products induced from the Generalized Yang's Noncommutative algebra in C-spaces is $\hbar_{eff} \equiv (\hbar\lambda/R)$.

In order to recover the Poisson brackets from the generalized Noncommutative Yang algebra commutators in C-spaces provided in section **5** , when taking the "classical" limit $\hbar_{eff} \to 0$, one needs firstly to *divide* by factors of $i(\hbar_{eff})^n$ where n is the grade of the Clifford-valued coordinate and momentum operators. Hence, after factoring out the i factors in the numerator and denominators, the Poisson brackets corresponding to the *Noncommutative* Classical Mechanics in C-spaces are

$$\{ x^{\mu_1\mu_2\ldots\mu_n},\ x^{\nu_1\nu_2\ldots\ldots\nu_n} \} = lim_{\ \hbar_{eff}\to 0}\ \frac{\lambda^{2n}}{(\hbar\lambda/R)^n}\Sigma^{\ [\mu_1\mu_2\ldots\ldots\mu_n]\ [\nu_1\nu_2\ldots\ldots\nu_n]}. \tag{6.3a}$$

$$\{ p^{\mu_1\mu_2\ldots\mu_n},\ p^{\nu_1\nu_2\ldots\ldots\nu_n} \} = lim_{\ \hbar_{eff}\to 0}\ \frac{(\hbar/R)^{2n}}{(\hbar\lambda/R)^n}\Sigma^{\ [\mu_1\mu_2\ldots\ldots\mu_n]\ [\nu_1\nu_2\ldots\ldots\nu_n]}. \tag{6.3b}$$

$$\{ \ x^{\mu_1,\mu_2....\mu_n}, \ p^{\nu_1\nu_2...\nu_n} \ \} = \eta^{\mu_1\nu_1\mu_2\nu_2.....\mu_n\nu_n} \Sigma^{\ [i_1i_2....i_n] \ [j_1j_2.....j_n]}. \qquad (6.3c)$$

In section **4** we explained how the $double - scaling$ limit behaves :

$$lim \ \lambda \rightarrow 0. \quad lim \ R \rightarrow \infty. \quad \lambda R \rightarrow L^2. \qquad (6.4)$$

We will take the double-scaling limit $\hbar_{eff} = (\hbar\lambda/R) = (\hbar\lambda^2/L^2) \rightarrow 0$ keeping $\hbar = c = 1$ in natural units fixed. Upon doing so one will have a nice cancellation in the r.h.s of eqs-(6.3) without singularities leading to the Noncommutative Poisson Brackets (NCPB) associated with a Noncommutative Classical Mechanics. ($\hbar = c = 1$ units) induced from the generalized Yang's algebra in C-spaces described in section **5**

$$\{ \ x^{\mu_1\mu_2...\mu_n}, \ x^{\nu_1\nu_2.....\nu_n} \ \}_{NCPB} = L^{2n} \ \Sigma^{\ [\mu_1\mu_2......\mu_n] \ [\nu_1\nu_2.......\nu_n]}. \qquad (6.5a)$$

$$\{ \ p^{\mu_1\mu_2...\mu_n}, \ p^{\nu_1\nu_2.....\nu_n} \ \}_{NCPB} = \frac{1}{L^{2n}} \ \Sigma^{\ [\mu_1\mu_2.......\mu_n] \ [\nu_1\nu_2.......\nu_n]}. \qquad (6.5b)$$

$$\{ \ x^{\mu_1,\mu_2....\mu_n}, \ p^{\nu_1\nu_2...\nu_n} \ \}_{NCPB} = \eta^{\mu_1\nu_1\mu_2\nu_2.....\mu_n\nu_n} \Sigma^{\ [i_1i_2....i_n] \ [j_1j_2.....j_n]}. \qquad (6.5c)$$

where $L^2 = \lambda R$ in the double-scaling limit.

Notice that the entries of Ω^{AB} have different units (dimensions) depending on the different grades among the components of the polyvectors in C-phase-space. Therefore, it is convenient to re-scale all the quantities by judicious powers of $\hbar$ such that all the terms appearing in the evaluation of the brackets $\{\mathcal{F}, \mathcal{G}\}$ have the same units. Units that we will choose to be $\hbar^{-d}$ assuming $\mathcal{F}, \mathcal{G}$ are dimensionless. Without this re-scaling the brackets contain terms of different units given by powers of $\hbar$ and whose exponents depend on the different grades of a polyvector as indicated by the r.h.s of eqs-(6.3). To conclude : Upon using the natural units of $\hbar = c = 1$ it automatically solves the adjustment problem of units for all the terms appearing in the evaluation of the brackets $\{\mathcal{F}, \mathcal{G}\}$ in eq-(6.1).

Since the deformation parameter is now $\hbar_{eff} = \frac{\hbar\lambda}{R}$ the Moyal-Yang star product based on the generalized Yang's algebra in Clifford spaces is defined

$$(\ \mathcal{F} * \mathcal{G} \ )(\Upsilon) \ \equiv \ exp \ [\ \ (i\hbar_{eff}/2) \ \Omega^{AB} \ \partial_A^{(\Upsilon_1)} \ \partial_B^{(\Upsilon_2)} \ ] \ \mathcal{F}(\Upsilon_1) \ \mathcal{G}(\Upsilon_2)|_{\Upsilon_1=\Upsilon_2=\Upsilon}. \qquad (6.6a)$$

where the derivatives $\partial_A^{(\Upsilon_1)}$ act only on the $\mathcal{F}(\Upsilon_1)$ term and $\partial_B^{(\Upsilon_2)}$ act only on the $\mathcal{G}(\Upsilon_2)$ term.

The Noncommutative Moyal-Yang Bracket is defined :

$$\{\mathcal{F} \ , \ \mathcal{G}\}_{MYB} \equiv \ \mathcal{F} * \mathcal{G} \ - \mathcal{G} * \mathcal{F}. \qquad (6.6b)$$

Following our discussion in section **2.2** we will see that the C-space world-volume coordinates Σ^A associated with C-space-branes will be identified with the C-phase-space variables as follows $\Upsilon = \Sigma = (q^A, p^A)$

$$q^A = q, \ q^a, \ q^{a_1a_2}, \ q^{a_1a_2a_3}, \ ...., q^{a_1a_2......a_d}. \qquad (6.7a)$$

$$p^A = p, \ p^a, \ p^{a_1a_2}, \ p^{a_1a_2a_3}, \ ...., p^{a_1a_2......a_d}. \qquad (6.7b)$$

the total number of variables (q^A, p^A) is $2 \times 2^d = 2^{d+1}$, which matches the degrees of freedom corresponding to a Clifford space in $d + 1$-dim since the $dim_R \ Cl(d + 1, R) = 2^{d+1}$.

The 2^{d+1} C-phase-space real variables associated with the C-space world-volume of the C-space branes can be recast in terms of 2^d complex variables :

$$Z^A = q^A + ip^A = q + ip, \quad q^a + ip^a, \quad q^{a_1a_2} + ip^{a_1a_2}, \quad ....q^{a_1a_2...a_d} + ip^{a_1a_2...a_d}. \qquad (6.8)$$

Plus their complex conjugates $\bar{Z}^A = q^A - ip^A$. In order to match units it is required to re-scale the variables by suitable powers of λ and $(\hbar/R)$ if one wishes to work with dimensionless variables $Z^A, \bar{Z}^A$. For the time being we should be working with (q^A, p^A) instead of $(Z^A, \bar{Z}^A)$ variables .

One of the most important physical consequences of this section is the following : since the world-volume polyvector-valued coordinates $\Sigma = (q^A, p^A)$ associated with the C-space Branes are *noncommuting* because they obey the Generalized Noncommutative Yang's algebra described in section **5**, like those given by eqs-(6.3, 6.5), the C-space target background coordinates $X^M = X^M(q^A, p^A)$ onto which one *embeds* the C-space world-volumes of the C-space branes, will require the use of *Noncommutative* Poisson brackets themselves $\{X^M(q^A, p^A) \ , \ X^N(q^A, p^A)\}_{NCPB} \neq 0$ as we shall see next. We must emphasize that extreme care must be taken *not to confuse* the world volume variables (q^A, p^A) with the Clifford-space target background coordinates X^M, P^M !

The extension of the Generalized-Yang-Mills (GYM) theories [5,6,12] to C-spaces can be obtained as follows. Define the gauge connection of the Clifford-Tangent-Bundle as

$$A_M = A_M^{AB}\ \Sigma_{AB} = A_M^{AB}\ [E_A, E_B]. \quad F_{[MN]} = \partial_{[N} A_{M]}^{AB}\ \Sigma_{AB} + [\ A_M^{AB}\ \Sigma_{AB}\ ,\ A_N^{CD}\ \Sigma_{CD}\ ]. \tag{6.9}$$

where the E_A, E_B basis elements admit a representation in terms of $2^{d/2} \times 2^{d/2}$ matrices. A Moyal-Yang star product deformation quantization procedure (along similar lines to the ones described in detail in **2.2**) in the *quenched*-reduced approximation leads to the correspondence among Hilbert space operators and functions in phase-space

$$\mathbf{F}_{[MN]} \leftrightarrow \mathcal{F}_{*[MN]}\ (q^A, p^A) = \{\ \mathcal{A}_M(q^A, p^A)\ , \mathcal{A}_N(q^A, p^A)\ \}_{*MYB}. \tag{6.10}$$

since here is no X dependence in the r.h.s of (6.10) due to the *quenched*-reduced approximation. The trace operation corresponds to an integration w.r.t the C-space-brane variables. Thus, the gauge-field/ coordinate correspondence $A_M(q^A, p^A) \leftrightarrow X_M(q^A, p^A) = X_M(\Sigma)$ of section **2.2** yields

$$\mathcal{F}_{*[M_1 M_2 M_3 .... M_{2^d+1}]}\ (q^A, p^A) = \mathcal{F}_{*[M_1 M_2]}\ *\ \mathcal{F}_{*[M_3 M_4]}\ *\ ....*\ \mathcal{F}_{*[M_{2^d} M_{2^d+1}]}\ +\ permutations \leftrightarrow$$

$$\{\ X_{M_1}\ ,\ X_{M_2}\ \}_{*MYB}\ *\ \{\ X_{M_2}\ ,\ X_{M_3}\ \}_{*MYB}\ *\ ....\ *\ \{\ X_{M_{2^d}}\ ,\ X_{M_{2^d+1}}\ \}_{*MYB}\ + permutations =$$
$$\{\ X_{M_1}\ , X_{M_2}\ , X_{M_3}, ..., X_{M_{2^d+1}}\ \}_{MYNPB}. \tag{6.11}$$

where the phase-space coordinates are identified with the world-volume C-space brane coordinates $\Upsilon = \Sigma = (q^A, p^A)$ and this allows us to evaluate the Moyal-Yang star product deformations of the Clifford-Nambu-Poisson-Brackets in terms of the Moyal-Yang Brackets $\{X_M(q^A, p^A)\ ,\ X_N(q^A, p^A)\}_{MYB}$. The Moyal-Yang quantization of GYM theories in C-spaces leads to

$$\mathbf{F}_{*M_1 M_2 ...... M_{2^d+1}}\ \leftrightarrow\ \{\ X_{M_1}, X_{M_2}, ..., X_{M_{2^d+1}}\}_{MYNPB}. \tag{6.12}$$

$$trace\ [\ \mathbf{F}_{*M_1 M_2 ...... M_{2^d+1}}\ *\ \mathbf{F}_*^{M_1 M_2 ...... M_{2^d+1}}\ ]\ \leftrightarrow$$

$$\int [\mathcal{D}\Sigma]\ \{\ X_{M_1}, X_{M_2}, ..., X_{M_{2^d+1}}\}_{MYNPB}\ *\ \{\ X_{M_1}, X_{M_2}, ..., X_{M_{2^d+1}}\}_{MYNPB} \tag{6.13}$$

which is the Clifford-space brane analog of the Eguchi-Schild action for strings that is invariant under area-preserving diffs.

In order to implement the full C-space covariance under world-volume reparametrizations of C-space branes, instead of the restricted invariance under volume-preserving diffs , we can recur to the reparametrization invariant new p-brane actions (2.39) studied in section **2.2** via the introduction of the auxiliary scalars in order to define a new integration measure [13]. *Noncommutative* Clifford-space *extensions* of such new p-branes actions like eqs-(2.39) will be given by eqs-(6.22) below.

The Moyal-Yang star deformations of Dolan-Tchrakian types of actions will require to *enlarge* the integration domain of dimension 2^{d+1} to one of *twice* the dimension 2^{d+2} and the Moyal-Yang brackets must be taken w.r.t an enlarged number of variables as well (twice the number). In the meantime, we proceed to evaluate the Moyal-Yang-Nambu-Poisson Brackets (MYNPB) in terms of the Moyal-Yang Brackets (MYB)

$$\{ X_{\mu_1\mu_2....\mu_n}(\Upsilon) \, , X_{\nu_1\nu_2....\nu_n}(\Upsilon) \}_{MYB} = X_{\mu_1\mu_2....\mu_n} \ * \ X_{\nu_1\nu_2....\nu_n} \ - \ X_{\nu_1\nu_2....\nu_n} \ * \ X_{\mu_1\mu_2....\mu_n}. \qquad (6.14)$$

given in terms of the Moyal-Yang star products defined by eqs-(6.6) and that correspond to the Noncommutative Poisson Brackets (NCPB) in the "classical " $\hbar_{eff} \rightarrow 0$ limit

$$\{ X_{\mu_1\mu_2....\mu_n}(\Upsilon) \, , X_{\nu_1\nu_2....\nu_n}(\Upsilon) \}_{NCPB} \ = \ (\partial_{\Upsilon^A} X_{\mu_1\mu_2....\mu_n}) \ \Omega^{AB} \ (\partial_{\Upsilon^B} X_{\nu_1\nu_2....\nu_n}). \qquad (6.15)$$

A Moyal-Yang star-product deformation of the above Nambu-Poisson Brackets yields the Moyal-Yang-Nambu-Poisson Brackets (MYNPB)

$$\mathbf{F}_{* \ [M_1 M_2 ......M_{2^{d+1}}]} \ \leftrightarrow \ \{ X_{M_1}, X_{M_2}, ..., X_{M_{2^{d+1}}} \}_{MYNPB} =$$

$$\{ X_{M_1}, X_{M_2} \}_{*MYB} \ * \ \{ X_{M_3}, X_{M_4} \}_{*MYB} \ * \ ..... * \ \{ X_{M_{2^d}}, X_{M_{2^{d+1}}} \}_{*MYB} \ + \ permutations. \qquad (6.16)$$

where the Moyal-Yang star-product deformations of the Nambu-Poisson-Brackets (MYNPB) can be decomposed as suitable antisymmetrized sums of Moyal-Yang star products of the Moyal-Yang brackets (MYB) among *pairs* of variables. And the latter MYB given by eqs-(6.6) are those induced from the generalized Noncommutative Yang's spacetime algebra in Clifford spaces whose deformation parameter is $\hbar_{eff} = \hbar\lambda/R = \hbar\lambda^2/L^2$ in the double-scaling limit. This is the generalization of the Quantum Nambu Brackets (QNB) described in section **2.2** .

Concluding, the Moyal-Yang star product deformation of the Noncommutative C-space Brane Action S^{branes}_{*MY} is

$$\mathcal{T} \int [D\Sigma] \sqrt{ \frac{1}{(2^{d+1})! \ (i\hbar_{eff})^{2^{d+1}}} | \ \{ X_{M_1}, X_{M_2}, ..., X_{M_{2^{d+1}}} \}_{MYNPB} \ * \ \{ X^{M_1}, X^{M_2}, ..., X^{M_{2^{d+1}}} \}_{MYNPB} \ |}. \qquad (6.17)$$

where the C-space brane tension $\mathcal{T}$ has the unit of $(mass)^{2^{d+1}}$. The terms inside the square root in the integrand is just the Moyal-Yang star deformation of the determinant of the induced metric G_{AB} resulting from the embedding of the C-space Brane into the C-space target background resulting from the *identity*

$$|det \ (G_{AB})| = |det \ [\ G_{MN} \ \partial_A X^M \ \partial_B X^N \ ]| =$$

$$\frac{1}{(2^{d+1})!} \ | \ \{ X_{M_1}, X_{M_2}, ..., X_{M_{2^{d+1}}} \}_{NPB} \ \{ X^{M_1}, X^{M_2}, ..., X^{M_{2^{d+1}}} \}_{NPB} \ |. \qquad (6.18)$$

In the limit $\hbar_{eff} = h\lambda/R \rightarrow 0$, keeping $\hbar = c = 1$ in the $double - scaling$ limit $\lambda R \rightarrow L^2$, the MYB coalesce to the Noncommutative Poisson Brackets (NCPB) as follows :

$$\{\mathcal{F} \ , \ \mathcal{G}\}_{NCPB} = lim \ _{\hbar_{eff}\rightarrow 0} \ \frac{1}{i\hbar_{eff}} \{\mathcal{F} \ , \ \mathcal{G}\}_{MYB} = lim \ _{\hbar_{eff}\rightarrow 0} \ \frac{1}{i\hbar_{eff}} (\mathcal{F} * \mathcal{G} \ - \mathcal{G} * \mathcal{F})$$

these NCPB are associated with the Noncommutative Classical Mechanics and were defined in eqs-(6.1, 6.3, 6.5, 6.15). Hence, the Noncommutative C-space Brane Action *induced* from the generalized Yang spacetime algebra in Clifford spaces (constructed in section **5**) can be written in terms of the Noncommutative Nambu Poisson Brackets (NCNPB) as

$$S^{branes}_{NC} \ = \ \mathcal{T} \ \int \ [D\Sigma] \ \sqrt{ \frac{1}{(2^{d+1})!} \ | \ \{ X_{M_1}, X_{M_2}, ..., X_{M_{2^{d+1}}} \}_{NCNPB} \ \{ X^{M_1}, X^{M_2}, ..., X^{M_{2^{d+1}}} \}_{NCNPB} \ |}. \qquad (6.19)$$

And the Noncommutative C-space Brane version of the new p-brane actions given by eqs-(2.39) is

$$S_{NC} = -\frac{1}{2} \int \ [\mathcal{D}\Sigma] \ \mathcal{J}[\varphi] \ [\ - \frac{\mathcal{T}^2}{(2^{d+1})!} \mathcal{J}^{-2}[\varphi] \ \{ \ X_{M_1}, X_{M_2}, ..., X_{M_{2^{d+1}}} \ \}^2_{NCPB} \ \ +1 \ \ ]. \qquad (6.21)$$

where the integration measure is given in terms of the 2^{d+1} auxiliary scalar fields

$$\mathcal{J} = \{\varphi^1, \varphi^2, ..., \varphi^{2^{d+1}}\}_{NCPB}. \tag{6.22}$$

A Moyal-Yang star product deformations of the action (6.21) is more subtle since it requires to write the deformation of the product $\mathcal{J}^{-2}\{........\}^2$ as $(1/2)\mathcal{J}_*^{-2} * \{........\}_*^2 + (1/2)\{.........\}_*^2 * \mathcal{J}_*^{-2}$ where the inverse $\mathcal{J}_*^{-2}$ is defined in terms of the star-product Taylor series expansion of the inverse function.

To sum up : the Moyal-Yang C-space Brane actions (6.17) *induced* from the Moyal-Yang star product deformation quantization of Generalized-Yang-Mills theories in C-spaces, upon taking the $\hbar_{eff} \rightarrow 0$ limit, lead naturally to *Noncommutative* C-space Brane actions (6.19) associated to the *Noncommutative* Classical Mechanics described by the brackets in eqs-(6.1, 6.3, 6.5, 6.15). The latter Noncommutative Classical Mechanics has a natural correspondence to the Noncommutative Quantum Mechanics in C-spaces (when $\hbar_{eff} \neq 0$) described by the generalized Noncommutative Yang's algebra in Clifford-spaces constructed in section **5** . This is one of the most relevant findings of this work.

A Noncommutative **QFT** in Clifford spaces involving both an upper R (infrared) and lower (ultraviolet) scale λ remains to be developed. In particular, the full fledged Quantization of C-space Branes and the plausible role of L^2 to the cosmological constant value $\rho \sim L^{-4}$. In [25] we have rigorously derived the geometric mean relationship among the 3 scales $L^2 = \lambda R$. Notice that by setting the infrared scale R equal to the Hubble radius horizon R_H and λ equal to the Planck scale one reproduces precisely the *observed* value of the vacuum energy density ! [25] :

$$\rho \sim L^{-4} = L_{Planck}^{-2} R_H^{-2} = L_P^{-4}(L_{Planck}/R_H)^2 \sim 10^{-122} M_{Planck}^4. \tag{6.23}$$

.

The importance of recurring to an Extended Relativity Theory in Clifford-spaces is that it allows us to work with many branes of *different* dimensions simultaneously. For reviews of Noncommutative Field theories, Yang-Mills and Matrix Models we refer to [36, 38, 46, 51, 52] . Some recent remarks about symplectic methods and Noncommutativity were made in [30, 31, 37] . Noncommutative Riemann-Finsler Geometries within the context of Clifford algebras have been studied by [45] . Applications of Clifford and exceptional algebras in the geometrical calculation of particle masses and coupling constants from first principles have been analyzed by [47] and most recently by us [42]. One final comment we must add is that the diverse Moyal-Yang star deformed actions are not equivalent to the Moyal-Yang deformed Nambu-Goto actions since the equations of motion for the auxiliary fields are not longer algebraic but contain an infinite number of derivatives resulting from the Moyal-Yang star products.

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References

1- E. Guendelman, E. Nissimov, S. Pacheva, Phys. Lett **B 360** (1995) 57. " Volume-preserving diffs versus local gauge symmetry " [arXiv : hep-th/9505128]. H. Aratyn, E. Nissimov, S. Pacheva, Phys. Lett **B 255** (1991) 359.

2 - C. Castro, Int. Jour. Mod. Phys. **A 13** (8) (1998) 1263-1292

3-C. Castro, " Moyal deformations of Gravity via $SU(\infty)$ Gauge Theories, Branes and Topological Chern-Simons-Matrix Models " to appear in General Relativity and Gravitation (2004) .

4- A. Aurilia, A. Smailagic , E. Spallucci, Phys. Rev. **D 47** (1993) 2536. A. Aurilia, E. Spallucci: Class. Quant. Gravity **10** (1993) 1217.

5- C. Castro, "Branes from Moyal deformations of Generalized Yang-Mills Theories" [arXiv: hep-th/9908115]. S. Ansoldi, C. Castro, E. Spallucci, Phys. Lett **B 504** (2001) 174 . Class. Quan. Gravity **18** (2001) L17-L23. Class. Quan. Gravity **18** (2001) 2865 .

6- S. Ansoldi, C. Castro, E. Guendelman, Spallucci, Class. Quant. Gravity vol. 19 (2002) L 135.

7-D. Fairlie, "Brane-Wave Duality" D. Fairlie, "Moyal Brackets in M theory " [arXiv : hep-th/9707190] . Mod. Phys. Letts **A 13** (1998) 263

8-C. Castro, Jour. Math. Phys. **34** (1993) 681. Phys Letts **B 288** (1992) 291.

9- H. Garcia-Compean, J. Plebanski, M. Przanowski, "Geometry associated with SDYM and chiral approaches to Self Dual Gravity" [arXiv : hep-th/9702046] .

10-J. Hoppe, " Quantum Theory of a Relativistic Surface" Ph.D Thesis MIT (1982).

11- Q.H. Park, Int. Jour. Mod. Phys **A 7** (1992) 1415.

12- D. H. Tchrakian, Jour. Math. Phys **21** (1980) 166. B. Dolan, D.H Tchrakian, Phys. Letts **B 198** (4) (1987) 447. B. Dolan, Tchrakian : Phys. Letts **B 202** (2) (1988) 211.

13- E. Guendelman, A. Kaganovich, E. Nissimov and S. Pacheva " New Physics From A Dynamical Volume Element" [hep-th/0411122] . "Novel Aspects in p-Brane Theories: Weyl-Invariant Light-Like Branes" hep-th/0409208. E. Guendelman, Class. Quant. Grav **17** (2002) 361. E. Guendelman, A. Kaganovich, Phys. Rev **D 60** (1999) 065004.

14-T. Curtright and C. Zachos, Phys.Rev. **D68** (2003) 085001. T. Curtright, D. Fairlie and C. Zachos, Phys. Lett. **B 405** (1997) 37-44 . T. Curtright and C. Zachos," Nambu Dynamics, Deformation Quantization and Superintegrability " [math-ph/0211021]. C. Zachos, Phys. Lett **B 570** (2003) 82-88. G. Dito, M. Flato, D. Sternheimer, L. Takhtajan, " The deformation quantization of Nambu-Poisson mechanics " [hep-th/9602016] H. Awata, M. Li, G. Minic, T. Yoneya, " Quantizing the Nambu Poisson brackets " [hep-th/9906248]. G. Minic, " M theory and Deformation Quantization " [hep-th/9909022]

15-C. Castro, "The Programs of the Extended Relativity in C-spaces, towards the physical foundations of string theory" NATO advanced Workshop on *The Nature of Time, Geometry and the Physics of Perception* (2003) 175-185, Kluwer Academic Press. Chaos, Solitons and Fractals **11** (2000) 1663. Foundations of Physics **30** (2000) 1301. Chaos, Solitons and Fractals **11** (2000) 1721. Chaos, Solitons and Fractals **12** (2001) 1585. "The search for the origins of M theory ...." [hep-th/9809102].

16-C. Castro, M. Pavsic, " The Extended Relativity Theory in Clifford Spaces" submitted to journal of General Relativity and Gravitation (2004). CDS-CERN-EXT-2004-002 preprint.

17-M.Pavsic: "The Landscape of Theoretical Physics": A Global View Kluwer, Dordrecht 1993. "Clifford algebra based polydimensional Relativity and Relativistic Dynamics"Talk presented at the IARD Conference in Tel Aviv, June 2000. Foundations of Physics **31** (2001) 1185. hep-th/0011216. Phys. Let **A 242** (1998) 187. Nuovo Cimento **A 110** (1997) 369. Foundations of Physics **33** 1277 (2003). "Kaluza Klein Theory without Extra Dimensions : Curved Clifford Space" arXiv.org: hep-th/0412255. " Clifford Spaces as a Generalization of Spacetime : Prospects for QFT of Point Particles and Strings" hep-th/0501222.

18-W.Pezzagia, "Physical Applications of a Generalized Geometric Calculus" [gr-qc/9710027]. Dimensionally Democratic Calculus and Principles of Polydimensional Physics [gr-qc/9912025].

19 -I. R. Porteous Clifford algebras and Classical Groups Cambridge Univ. Press, 1995.

20-A.B. Zamoldchikov, Teor. Fiz **65** (1985) 347. C. Pope, L. Romans, X. Shen, Nuc. Phys, **B 339** (1990) 191. C. Pope, L. Romans, X. Shen, Phys. Letts **B 236** (1990) 173. C. Pope, L. Romans, X. Shen, Phys. Letts **B 242** (1990) 401. E. Sezgin, " Area-preserving Diffs, w_∞ algebras , w_∞ gravity " [hep-th/9202080]. E. Sezgin : " Aspects of W_∞ Symmetry " [arXiv : hep-th/9112025] . P. Bouwknegt, K. Schouetens, Phys. Reports **223** (1993) 183-276.

21- C.N Yang, Phys. Rev **72** (1947) 874. Proceedings of the International Conference on Elementary Particles, (1965) Kyoto, pp. 322-323.

22- S. Snyder: Phys. Rev. **71** (1947) 38. Phys. Rev. **71** (1947) 78.

23- S.Tanaka, Nuovo Cimento **114 B** (1999) 49. S. Tanaka, " Noncommutative Field Theory on Yang's Space-Time Algebra, Covariant Moyal Star products and Matrix Model " [hep-th/0406166] . " Space-Time Quantization and Nonlocal Field Theory ..." [hep-th/0002001] . "Yang's Quantized Space-Time Algebra and Holographic Hypothesis " [hep-th/0303105] .

24- A. Ashtekar, C. Rovelli and L. Smolin, Phys. Rev. Lett **69** (1992) 237.

25- C. Castro, Mod. Phys. Letters **A 17** (2002) 2095-2103.

26-M. Vasiliev, " Higher Spin Gauge Theories, Star Product and AdS spaces " [hep-th/9910096]; M. Vasiliev , S. Prokushkin, "$3D$ Higher-Spin Gauge Theories with Matter". [hep-th/9812242, hep-th/9806236] .

27- C. Castro, Europhysics Letters **61** (4) (2003) 480-484. Class. Quant. Gravity **20** (16) (2003) 3577-3592.

28- B. Fedosov, " Deformation quantization and Index Theory " Akademie Verlag, Berlin (1996). B. Fedosov, " Jour. Diff. Geometry **40** (1994) 213.

29-C. Castro, Jour. Geometry and Physics **33**. (2000) 173. Jour. Chaos, Solitons and Fractals **7** (5) (1996) 711.

30- I. Martin, A. Restuccia, " Symplectic connections, Noncommutative Yang-Mills Theory and Supermembranes " [hep-th/0108046].

31- E. M Abreu, C. Neves and W. Oliveira, Noncommutativity from the symplectic point of view" [hep-th/0411108].

32- S. Ansoldi, A. Aurilia, E. Spallucci: Chaos, Solitons and Fractals **10** (2-3) (1999).

33- S. Ansoldi, A. Aurilia, C. Castro, E. Spallucci: Phys. Rev. **D 64** 026003 (2001) hep-th/0105027.

34-C. Castro, M. Pavsic, Phys. Letters **B 559** (2003) 74.

35-C. Castro: Jour. of Entropy **3** (2001) 12-26. C.Castro, A. Granik, Foundations of Physics vol **33** no.3 (2003) 445.

36-R. J. Szabo, Phys. Rept. 378 (2003) 207-299. E. Langmann, R. J. Szabo , K. Zarembo, JHEP 0401 (2004) 017

37-M. Cederwall, "Thoughts on Membranes, Matrices and Non-Commutativity" [hep-th/0410110].

38-M. Douglas, N. Nekrasov, Rev. Mod. Phys **73** (2001) 977-1029.

39-J. Chisholm : J. Phys **A 35** (2002) 7359 J. Chisholm and R. Farwell , "Spin Gauge Theories, Principles and Predictions" in *Clifford Algebras and their Application sin Mathematical Physics* eds. F. Brack, R. Delanghe and H. Serras (Dordrecht, Kluwer 1993) p. 367. G. Trayling and W. Baylis : J. Phys. **A 34** (2001) 3309. J. Chisholm and R. Farwell, J. Phys. **A 32** (1999) 2805. J. Chisholm and R. Farwell, Foundations of Physics **25** (1999) 1511. A.Lasenby, C. Doran and S. Gull, Phil. Trans. Roy Soc **356** 487. A.Lasenby, C. Doran and S. Gull , Jour. Math. Phys **39** (1998) 3303.

40-C. Castro, M. Pavsic, Int. J. Theor. Phys. 42 (2003) 1693-1705

41– M. Born, Proc. Royal Society **A 165** (1938) 291. M. Born, Rev. Mod. Physics **21** (1949) 463. S. Low, Jour. Phys. **A 35** (2002) 5711. H. Brandt, Chaos, Solitons and Fractals **10** (2-3) (1999) 267. Contemporary Mathematics **196** (1996) 273.

42-C-Castro, "The Extended Relativity Theory in Born-Clifford Phase Spaces with a Lower and Upper Length Scales and Clifford Group Geometric Unification" CDS-CERN-EXT-2004-128, submitted to Foundations of Physics (2004)

43- L. Nottale*La Relativite dans tous ses Etats* . Hachette Lit. Paris 1999. L. Nottale, *Fractal Spacetime and Microphysics, Towards Scale Relativity* . World Scientific. Singapore, 1992.

R.P Zaikov, Phys. Letts **B 266** (1991) 303. Phys. Letts **B 263** (1991) 209 . R.P Zaikov, Phys. Letts **B 211** (1988) 281. R.P Zaikov : "Chern-Simons p-Branes and p-dimensional Classical W Algebras " [hep-th/9304075].

45-S. Vacaru, "Non-holonomic Clifford-Structures and Noncommutative Riemann-Finsler Geometry " [math.DG/0408121]. S. Vacaru and N Vicol, " Nonlinear Connections and Clifford Structures" [math.DG/0205190]. S. Vacaru, "(Non) Commutative Finsler Geometry from String/M Theory " [hep-th/0211068]. S. Vacaru, A. Nadejda, Int. J. Math. Math. Sci. 23 (2004) 1189-1233.

46-W.Taylor, Rev.Mod.Phys. 73 (2001) 419-462

47-F. Smith, Int. Jour. Theor. Phys. **24** (1985) 155. Int. Jour. Theor. Phys. **25** (1985) 355. "From Sets to Quarks" [arXiv : hep-ph/9708379] and [CERN CDS EXT-2003-087]. G. Gonzalez-Martin, *Physical Geometry* University of Simon Bolivar, publishers, Caracas, June 2000, 265 pages, ISBN : 9800767495. G. Gonzalex-Martin, "The proton/electron Geometric Mass Ratio" [arXiv : physics/0009052] . G. Gonzalez-Martin, "The fine structure constant from Relativistic Groups" [arXiv : physics/0009051]

48-C. Castro Moyal deformations of Gravity via $SU(\infty)$ Gauge Theories , Branes and Topological Chern-Simons Matrix Models **to appear** in General Relativity and Gravitation, **36** no.12 (2004) . C.Castro, " On Wilson Loops, large N limit, Confinement and Composite Antisymmetric Tensor Field Theories " **to appear** in the Int. Jour. Mod. Phys. **A** (2004).

49- T. Banks, W. Fischler, S. Shenker and L. Susskind, Phys. Rev **D 55** (1997) 5112.

50- D. Fairlie, C. Zachos Phys.Lett. **B 224** (1989) 101. D. Fairlie, P. Fletcher and C. Zachos, J.Math.Phys. **31** (1990)1088.

51-K. Morita, " A new gauge-invariant regularization scheme based on Lorentz-invariant Noncommutative Quantum Field Theory " hep-th/0312080.

52- A. P Balachandran, T. Govindarajan, C. Molina and P. Teotonio-Sobrinho, " Unitarity Quantum Physics with Time-Space Noncommutativity " hep-th/0406125.

53- A. Dimakis and F. Muller-Hoissen Dimakis, " Extension of Moyal-deformed hierarchies of soliton equations" nlin.SI/0408023. "An algebraic scheme associated with the noncommutative KP hierarchy and some of its extensions" nlin.SI/0501003.

54-I. Bandos, X. Bekaert, J.A de Azcarraga, D. Sorokin and M. Tsulaia, "Dynamics of Higher Spins and Tensorial Space" hep-th/0501113. F. Toppan, " Hermitian versus holomorphic complex and quaternionic generalized supersymmetries of the M-theory : A classification" hep-th/0406022.

55- D. Alekseevsky, V. Cortes, C. Devchand and A. Van Proeyen " Polyvector Super-Poincare Algebras " hep-th/0311107.

56-J.A de Azcarraga, " Superbranes, D = 11 CJS supergravity and enlarged superspace coordinates/field correspondence" hep-th/0501198.