

Moment of the Poynting vector is not spin. V. 2

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Abstract

A common opinion that a moment of the Poynting vector is spin is a common delusion due to a serious defect of the general field theory. We present a circularly polarized Laguerre-Gaussian beam without an azimuth phase structure as an evidence of this. It is shown that the moment of the Poynting vector in this beam is an orbital angular momentum, not spin. Meanwhile, this beam, as well as a circularly polarized plane wave, clearly carries spin. But the standard electrodynamics does not sight it. Spin tensor of the standard electrodynamics is zero. So, this delusion causes many conflicts, vagueness, and paradoxes concerning electromagnetic angular momentum. The only way to resolve the problems is to use an electrodynamics spin tensor, which we introduced into the electrodynamics and demonstrate in this paper.

Key words: light beam, electrodynamics torque, classical spin

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1. Introduction

All physicists, who consider polarized light beam, start from an expression for the magnetic potential \mathbf{A}

$$\mathbf{A} = \sqrt{2} \exp(ikz - i\omega t)(\alpha\mathbf{x} + \beta\mathbf{y})u(x, y, z), \quad (1.1)$$

(see, e.g., Loudon [1] (2.1), Allen et al. [2] (2.9), Zambrini et al. [3] (2)). $i(\alpha\beta^* - \alpha^*\beta) = \sigma$ is a degree of a circularity of the polarization, and

$$|\alpha|^2 + |\beta|^2 = 1 \quad (1.2)$$

(see, e.g., [3] p. 1046). Eqn (1.1) leads to

$$\mathbf{E} = \omega\sqrt{2} \exp(ikz - i\omega t)[i\alpha\mathbf{x} + i\beta\mathbf{y} - \frac{1}{k}\mathbf{z}(\alpha\partial_x + \beta\partial_y)]u, \quad (1.3)$$

$$\mathbf{B} = \sqrt{2} \exp(ikz - i\omega t)[-ik\beta\mathbf{x} + ik\alpha\mathbf{y} + \mathbf{z}(\beta\partial_x - \alpha\partial_y)]u. \quad (1.4)$$

(see, e.g., [1] (2.17), [2] (2.10), (2.11)). It is cozily to set, accordingly to (1.2), $\alpha = -i/\sqrt{2}$, $\beta = 1/\sqrt{2}$, $\sigma = 1$ for a right-circularly polarized beam. Then we arrive to

$$\mathbf{E} = \omega \exp(ikz - i\omega t)[\mathbf{x} + i\mathbf{y} + \frac{1}{k}\mathbf{z}(i\partial_x - \partial_y)]u, \quad (1.5)$$

$$\mathbf{B} = \exp(ikz - i\omega t)[-ik\mathbf{x} + ky + \mathbf{z}(\partial_x + i\partial_y)]u. \quad (1.6)$$

These coincide with the Jackson's expressions [4] (p. 350),

$$\mathbf{E} = \exp(ikz - i\omega t)[\mathbf{x} + i\mathbf{y} + \frac{1}{k}\mathbf{z}(i\partial_x - \partial_y)]E_0, \quad \mathbf{B} = -i\frac{k}{\omega}\mathbf{E}, \quad (1.7)$$

if $E_0 = \omega u$.

We set the speed of light $c = 1$. Thus z -component of the time-average Poynting vector, i.e. $\langle T^{tz} \rangle$ - component of the energy-momentum tensor, is equal to the time-average z -component of the momentum density $p^z = \langle T^{zj} \rangle$ and is equal (for arbitrary α , β and u) to

$$p^z = \Re(E_x B_y^* - E_y B_x^*) / 2 = k\omega |u|^2. \quad (1.8)$$

I must note here that the cited authors [1 – 3] mistakenly do not write the factor $\sqrt{2}$ in the formulae (1.1), (1.3), (1.4) [5].

The rest of the Poynting vector or momentum density components, $p^r = \langle T^{rr} \rangle = \langle T^{rr} \rangle$, $p^\phi = \langle T^{r\phi} \rangle = \langle T^{r\phi} \rangle$, depends on a form of the function $u(x, y, z)$. The paraxial approximation means that a Gaussian beam is considered [6].

$$u = u_G = \frac{\sqrt{2/\pi}}{w(z)} \exp\left(-\frac{r^2}{w^2} + \frac{ir^2 z}{w^2 z_R} - i \tan^{-1} \frac{z}{z_R}\right), \quad w^2 = w_0^2 \left(1 + \frac{z^2}{z_R^2}\right), \quad kw_0^2 = 2z_R. \quad (1.9)$$

However, usually generalizations of the Gaussian beam are considered.

For the Hermite-Gaussian beams we have

$$u = u_{mn}(x, y, z) = u_G \sqrt{\pi/2} A_{mn} H_m(x\sqrt{2}/w) H_n(y\sqrt{2}/w) \exp[-i(m+n) \tan^{-1}(z/z_R)], \quad (1.10)$$

where $A_{mn} = \sqrt{2}/(2^{m+n} m! n! \pi)^{1/2}$ is the normalization constant; $H_m(\xi)$ stands for the Hermite polynomial of order m in ξ . For example,

$$H_0 = 1, \quad H_1(\xi) = 2\xi, \quad H_{j+1} = 2\xi H_j - 2j H_{j-1}.$$

For the Laguerre-Gaussian beams we have

$$u = u_p^l(r, \phi, z) = u_G \sqrt{\pi/2} C_p^l L_p^l(2r^2/w^2) (r\sqrt{2}/w)^{|l|} \exp[-i(2p+|l|) \tan^{-1}(z/z_R) + il\phi], \quad (1.11)$$

where C_p^l is the normalization constant; $L_p^l(\xi)$ is a generalized Laguerre polynomial, for example,

$$L_0^0 = 1, \quad L_1^l(\xi) = l+1-\xi, \quad L_2^0(\xi) = 2+\xi^2-4\xi. \quad (1.12)$$

Here l, p are mode indices: l is the azimuthal mode index, and p is the number of dark rings.

Note, $u_{00} = u_0^0 = u_G$.

The radial component of the Poynting vector p^r , relates to the spread of the beam as it propagates, is independent explicitly on the mode indices l, p and on the degree of the circularity σ in the case of the Laguerre-Gaussian beams:

$$p^r = \frac{2\omega r z}{w^2 z_R} |u|^2, \quad (1.13)$$

however, the azimuthal component of the Poynting vector, p^ϕ , consists of two terms:

$$p^\phi = \frac{\omega l}{r} |u|^2 - \frac{\omega \sigma}{2} \partial_r |u|^2. \quad (1.14)$$

The first term is proportional to the azimuthal index l and is independent on σ . This term holds for a linearly polarized beam, but vanishes if the beam has no azimuthal phase variation. Moment of this term is identified as the orbital angular momentum. A volume V of the beam contains

$$L_z = \int_V r p^\phi dV = \int_V \omega l |u|^2 dV \quad (1.15)$$

as a z -component of the orbital angular momentum.

The second term of (1.14) is independent on l . This term is known very well. In the case of the simple Jackson's beam (1.7), or a Gaussian beam (1.9), i.e. the Laguerre-Gaussian beam LG_0^0 , it is the Poynting vector located on the surface of a circularly polarized beam.

Ohanian wrote [7]:

“In a wave of finite transverse extent, the \mathbf{E} and \mathbf{B} fields have a component parallel to the wave vector (the field lines are closed loops), and the energy flow has components perpendicular to the wave vector... The circulating energy flow in the wave implies the existence of angular momentum, whose direction is along the direction of propagation.” (See Fig. 1)

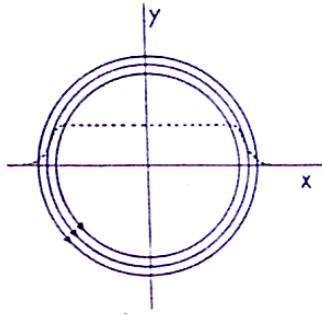


Fig. 1. This pattern of circular flow lines represents the time-average energy flow, or the momentum density, in a circularly polarized electromagnetic wave packet. On a given wave front, say $z = 0$, the fields are assumed to be constant within a circular area and to decrease to zero outside of this area (the dashed line gives the field amplitude as a function of radius). The energy flow has been calculated from an approximate solution of Maxwell's equations. The picture only shows the flow in the transverse directions. The flow in the longitudinal direction is much larger; the net flow is helical.

Simmonds and Guttman [8] wrote: “The electric and magnetic field can have a nonzero z -component only within the skin region of the wave. Having z -component within this region implies the possibility of a nonzero z -component of angular momentum within this region. Since the wave is identically zero outside the skin and constant inside the skin region, the skin region is the only one in which the z -component of angular momentum does not vanish.” (See Fig. 9.3).

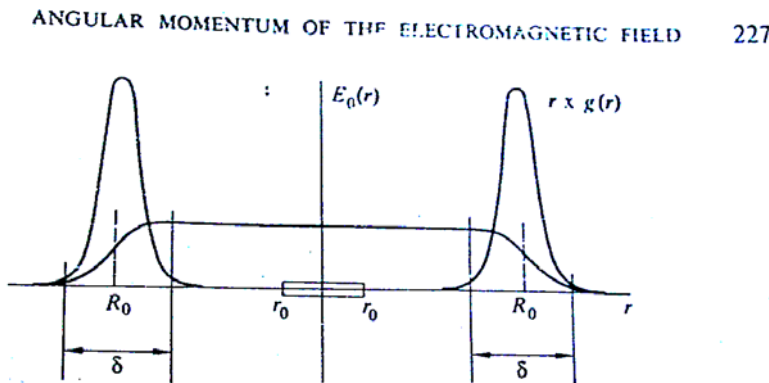
Heitler [9] wrote:

“A plane wave traveling in the z -direction and with infinite extension in the xy -directions can have no angular momentum about the z -

axis, because $\mathbf{E} \times \mathbf{B}$ is in the z -direction and $[\mathbf{r} \times (\mathbf{E} \times \mathbf{B})]_z = 0$. However, this is no longer the case for a wave with finite extension in the xy -plane. Consider a cylindrical wave with its axis in the z -direction and traveling in this direction. At the wall of the cylinder, $r = R$ say, we let the amplitude drop to zero. It can be shown that the wall of such a wave packet gives a finite contribution to J_z .”

Unfortunately, moment of the second term of the Poynting vector (1.14), according to the common opinion, unlike the moment of the first term, is called a spin of the beam. Ohanian [7] wrote: “This angular momentum is the spin of the wave.” Allen and Padgett [10] wrote: “The second term relates to the spin contribution”. Zambrini and Barnett [3] wrote: “The spin angular momentum density depends on the radial gradient of the intensity”.

Fig. 9.3. The electric field amplitude and the angular momentum density across a cylindrical beam.



I raised objections against this delusion [11 - 15]. Now I consider the second term for the Laguerre-Gaussian beam LG_1^0 ($l = 0, p = 1$). LG_1^0 beam has two virtues: this beam contains only the second term; the region where $\partial_r |u|^2 < 0$ is placed inside the beam, and a region where $\partial_r |u|^2 > 0$ exists. These virtues help to prove more convincingly that moment of the Poynting vector is an orbital angular momentum.

2. Consideration of the Laguerre-Gaussian beam

It is a matter of common opinion that the moment of the Poynting vector, i.e. the expression

$$\mathbf{j} = \mathbf{r} \times \langle \mathbf{E} \times \mathbf{B} \rangle, \quad (2.1)$$

is a time average density of the total angular momentum, i.e. it contains both the spin and orbital angular momentum density of electromagnetic field (we set $\epsilon_0 = 1$). We intend to refute this opinion by the use of a concrete example.

A circularly polarized Laguerre-Gaussian beam without an azimuth phase structure of a type $\text{LG}_{p=1}^{l=0}$ is specified by the formulae

$$\mathbf{E} = \exp[i(z-t)][\mathbf{x} + i\mathbf{y} + \mathbf{z}(i\partial_x - \partial_y)]u(r, z), \quad \mathbf{B} = -i\mathbf{E}, \quad (2.2)$$

$$u(r, z) = \frac{C}{w(z)} \left(1 - \frac{2r^2}{w^2}\right) \exp\left[-\frac{r^2}{w^2}(1 - i2z) - i\text{tan}^{-1}2z\right], \quad w^2 = (1 + 4z^2). \quad (2.3)$$

For short we set here: $k = \omega = 1$, the Rayleigh range $z_R = 1/2$, $w(0) = w_0 = 1$. According to (1.14), an angular momentum density (which is considered as a moment of the azimuthal component of the Poynting vector p^ϕ) is

$$j_z(r, z) = rp^\phi = -r\partial_r |u|^2 / 2, \quad (2.4)$$

We intend to show that, contrary to the common opinion, z -component of the expression (2.1), which has the form rp^ϕ (2.4), is an orbital angular momentum rather than spin. Indeed, consider the beam (2.2), (2.3) at the beam waist $z = 0$. We have

$$u(r) = C(1 - 2r^2) \exp(-r^2). \quad (2.5)$$

Calculate p^ϕ by substituting (2.5) into (2.4),

$$p^\phi = -\partial_r |u|^2 / 2 = -C^2 \partial_r [(1 - 2r^2)^2 \exp(-2r^2)] / 2 = 2C^2 (2r^2 - 1)(2r^2 - 3)r \exp(-2r^2). \quad (2.6)$$

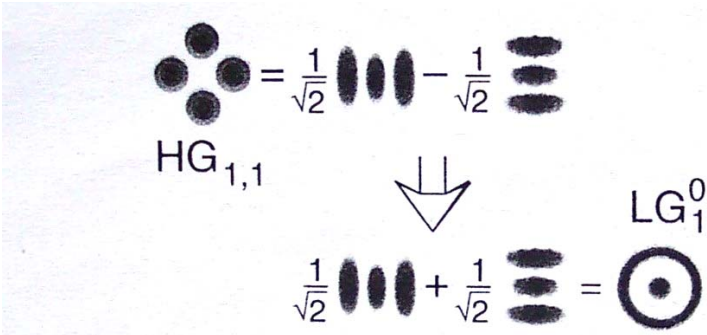


Fig. 2. The transformation of HG modes into LG modes.

This LG_1^0 mode contains one dark ring of radius $r = 1/\sqrt{2} = r_1$ and one light ring of radius $r = \sqrt{3}/2 = r_2$. You can see a negative image in the Fig. 2 from [2]. Between these rings, according to (2.6) and because of the positive derivative, $\partial_r |u|^2 > 0$, electromagnetic mass-energy orbits the beam axis opposite in a direction determined by the handedness of the circular polarization. We have $p^\phi > 0$ for $r < r_1$ and for $r > r_2 = \sqrt{3}/2$, but $p^\phi < 0$ if $r_1 < r < r_2$. This behavior of

mass-energy is incompatible with the concept of spin.

Zambrini and Barnett [3] calculate the moment of a part of a beam relative to the beam axis,

$$dJ_z = -\int dz \int_{\phi_a}^{\phi_b} d\phi \int_{r_a}^{r_b} dr r^2 \partial_r (u^2) / 2, \quad (2.7)$$

but a translation of the coordinate system changes this moment. This shows clearly that the moment of the Poynting vector is not an intrinsic angular momentum. Thus a moment of the Poynting vector is an orbital angular momentum rather than spin, and eqn. (2.4) must be rewritten as

$$l_z(r, z) = rp^\phi, \quad (2.8)$$

If a target absorbs our beam, the target experiences a density of orbital forces $\langle T^{\phi z} \rangle$. This component of the Maxwell tensor [4] can be readily calculated:

$$\langle T^{xz} \rangle = -\Re(E_x E_z^* + B_x B_z^*)/2 = -\Re(E_x E_z^*) = \partial_y |u|^2 / 2, \quad \langle T^{yz} \rangle = -\partial_x |u|^2 / 2, \quad \langle T^{\phi z} \rangle = -\partial_r |u|^2 / 2 \quad (2.9)$$

This means that a surface element $da_z = rd\phi dr$ of the target experiences a force

$$dF^\phi = \langle T^{\phi z} \rangle da_z = -\partial_r |u|^2 rd\phi dr / 2. \quad (2.10)$$

This force is positive, $dF^\phi > 0$, if limits of the Zambrini's integrating are inside the central light spot or outside the light ring, i.e. in the domain $r < r_1$ or $r > r_2$, but the force is negative, $dF^\phi < 0$, in the middle domain $r_1 < r < r_2$. Spin cannot become apparent by such a way. Note that the middle domain is twice as big as the central spot.

Imagine that the target is divided into three concentric parts by circles of radiuses $r_1 = 1/\sqrt{2}$ and $r_2 = \sqrt{3/2}$. Then the inner disk ($r < r_1$) and the outer ring ($r > r_2$) should rotate clockwise, but the middle ring ($r_1 < r < r_2$) should rotate anticlockwise. Reasoning along similar lines we must conclude that small particles, which are trapped off axis near the circle of maximum intensity at r_2 should not orbit the beam axis. But when diffusing towards $r < r_2$ they should orbit anticlockwise, while when diffusing towards $r > r_2$ they should orbit clockwise. This proves that the force (2.10) bears no relation to a spin.

Now calculate the ratio of the energy flux to the angular momentum flux for our beam, i.e. the ratio of power to torque. This ratio is $\omega = 1$ for spin. Eqns (2.2) permit to calculate z -component of the Poynting vector of the beam:

$$\langle \mathbf{E} \times \mathbf{B} \rangle_z = \Re(\tilde{E}_x \bar{B}_y - \tilde{E}_y \bar{B}_x) / 2 = \Re(\tilde{E}_x i \bar{E}_y) = |u|^2. \quad (2.11)$$

Consider power of the beam inside of a circle of radius r , $P(r)$. Then consider torque, which acts inside of the circle, $\tau(r)$.

$$P(r) = \int_0^r u^2 2\pi r dr, \quad \tau(r) = \int r dF^\phi = -\int_0^r r^2 \pi \partial_r |u|^2 dr = P(r) - r^2 \pi |u|^2(r) \quad (2.12)$$

When integrated over the complete cross section of the beam, i.e. from zero to infinity, we find $P/\tau = 1 = \omega$ because $u(\infty) = 0$. However, eqns (2.12) show a lack of torque in the central part of the target. A part of the target near the axis almost does not receive the angular momentum although the energy flux is maximal there. Using two terms of the Taylor expansion of $u(r)$, $u(r) \approx C(1 - 3r^2)$, yields $P(r) \approx r^2 \pi |u|^2$, and $\tau(r) \approx 0$. In other words, the central part receives energy, but does not receive the angular momentum. Attention was paid on this fact in [11]. This proves once more that the angular momentum (2.1) is an orbital angular momentum rather than spin.

At the same time, if we *isolate* the small central part of the target and illuminate it by the beam or by a plane circularly polarized wave, the part will receive j_z (2.1) per unit time that equals to P/ω because $-j_z$ per unit time will be carried away by *the edge of the part's shade*. It was explained in [8]. However, neither plane wave nor the central part of our beam carry the angular momentum of a type (2.1) by themselves, i.e. without an interaction with a small part of a target. Thus this angular momentum (2.1) is not spin.

3. Spin tensor

Meanwhile, spin obviously is present in a circularly polarized light. But it is out of all relation to eqn. (2.1). The standard electrodynamics has not caught sight of spin. Standard electrodynamics' spin tensor is zero because of a serious defect of classical field theory. This causes many conflicts, vagueness, and paradoxes concerning electromagnetic angular momentum. I quote here some evidences of this phenomenon.

“Experimental observations appear to be in conflict with theoretical considerations” [3].

“If the above expression (2.1) was in fact the correct angular momentum flux density, then the angular momentum of a circularly polarized plane would be zero. Since the correct classical angular momentum density must agree with the classical limit of the quantum angular momentum density, this must be incorrect.” [16].

“The angular momentum of a classical electromagnetic plane wave of arbitrary extent is predicted to be exactly zero. However, finite sections of circularly polarized plane waves are found experimentally to carry angular momentum.” [17]

“A circularly polarized plane wave has a linear momentum density only in the z-direction. When this is crossed with r to give the angular momentum density, there is no contribution in the z-direction. Thus, such a beam has no angular momentum to transfer to a waveplate. Yet, Beth was able to make such a transfer – a paradox.” [10]

Note that aforementioned trapped particles, which can orbit in both directions, rotate about their own axes in the direction determined by the handedness of the circular polarization. The only way to resolve the problems is to use the electrodynamics spin tensor [11 – 15]:

$$Y_{\alpha\beta\gamma} = A_{[\alpha}\partial_{|\gamma|}A_{\beta]} + \Pi_{[\alpha}\partial_{|\gamma|}\Pi_{\beta]}, \quad 2\partial_{[\alpha}A_{\beta]} = F_{\alpha\beta}, \quad 2\partial_{[\alpha}\Pi_{\beta]} = -F_{\alpha\beta}^* \quad (3.1)$$

where A_α, Π_α are magnetic and electric vector potentials, $F_{\alpha\beta}, F_{\alpha\beta}^* = e_{\alpha\beta\mu\nu}F^{\mu\nu}$ are the field strength tensor and the dual field strength tensor. Thus, eqn. (2.1) must be replaced by

$$j_z = \rho p^\phi + Y^{\rho\phi r} \quad (3.2)$$

Eq. (3.1) explains [12] the Beth’s experiment [18]. Another use of the spin tensor (13) is presented at web sites www.mai.ru/projects/mai_works, www.sciprint.org.

The result (3.1), (3.2) was submitted to “JETP” on Jan. 27, 1999. It was rejected more than 350 times by scientific journals. For example (I show an approximate number of the rejections in parentheses): JETP Lett. (8), JETP (13), TMP (10), UFN (9), RPJ (70), AJP (14), EJP (4), EPL (5), IJTP (2), PRA (6), PRD (4), PRE (2), PRL (6), APP (5), FP (6), PLA (9), OC (5), JPA (7), JPB (1), JMP (6), JOPA (4), JOSAB (2), JMO (2), CJP (1), OL (2), NJP (2), MREJ (3), arXiv (70).

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