Does volume force, or does surface force act on polarized dielectric?

Radi I. Khrapko
Moscow Aviation Institute, 125993, Moscow, Russia
E-mail: khrapko_ri@hotmail.com, khrapko_ri@mai.ru

Abstract
Static electric field in linear uniform neutral dielectric is divergence-free and irrotational. Thus the Maxwell stress tensor is divergence-free as well. So, according to Maxwell, the volume force is ZERO. Only surface forces act on the polarized dielectric.

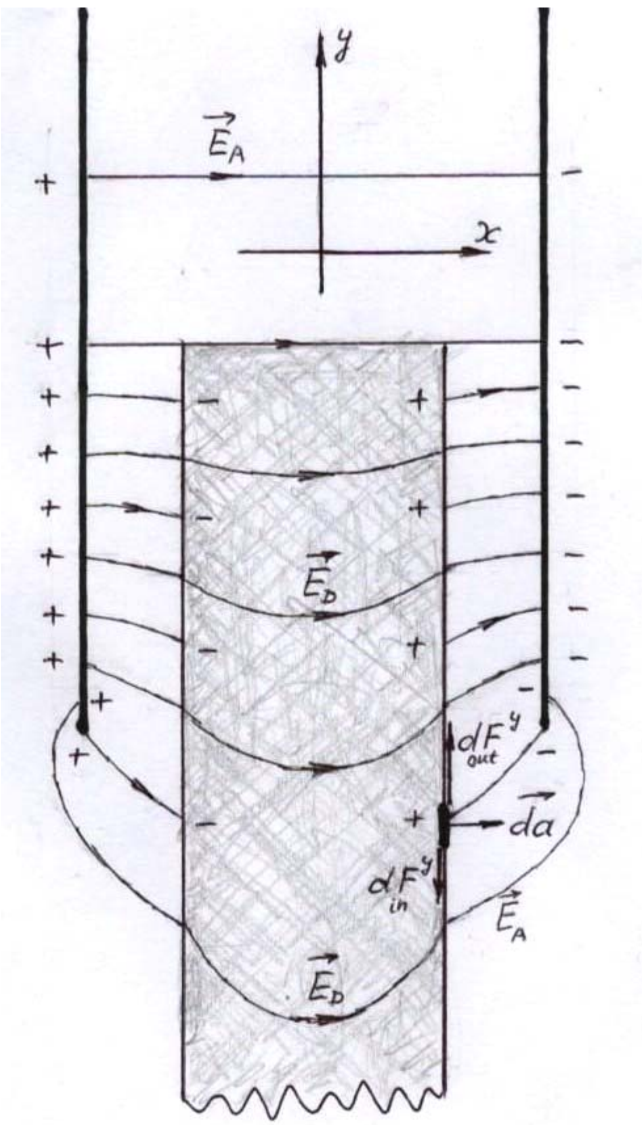

Fig. 1. Dielectric slab is inside of a parallel plate capacitor. Unbalanced y-components of the forces dF are shown acting on the inner and outer sides of a surface element of the slab.

Static electric field E^i in linear uniform neutral dielectric is divergence-free, $\partial_i E^i = 0$, and irrotational, $\partial_{[i} E_{j]} = 0$. Thus the Maxwell stress tensor [1, p. 261].

$$T^{ij} = E^i E^j - \delta^{ij} E^k E_k / 2 \qquad (1)$$

is divergence-free as well, $\partial_j T^{ij} = 0$. So, according to the Maxwell electrodynamics, the volume force acting on the dielectric is zero [1, p.610],

$$f^i = \partial_t g^i = \partial_j T^{ij} = 0 . \qquad (2)$$

But this is not the case on a surface of the dielectric because of bound electric charges.

Let us consider, as an example, a dielectric slab partially inserted between parallel conducting charged plates that is shown in Fig.1. The electric field is weaker in the slab than on the outside the slab. Lines of **E** break at the surface, and E^x_A in air is κ times bigger than E^x_D in the dielectric where κ is the relative permittivity (see also Figure 4.5 from [1]).

The component $dF^y = -T^{yx} da_x$ of the pondermotive force $dF^i = -T^{ij} da_j$ acts on the back side of an oriented element da_x of the side surface of the slab. Here $T^{yx} = E^y E^x$ is a component of the Maxwell stress tensor. Because $E^x_A > E^x_D$, we have $\underset{out}{dF^y} > |\underset{in}{dF^y}|$ in spite of the continuity of E^y at the surface.

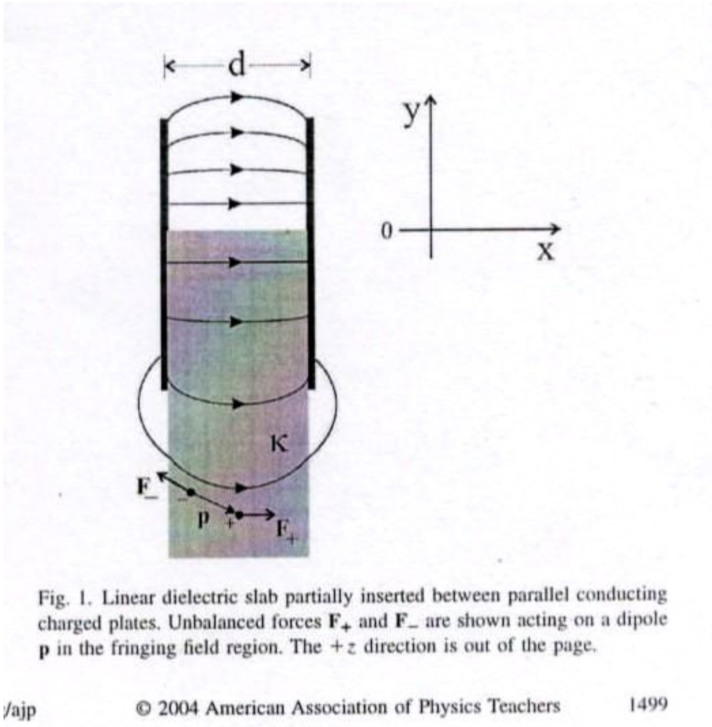

Fig. 2

The question of whether the force is actually acts on the surface of the slab or on the bulk of the slab is not difficult because there is a measurable distinction between the internal mechanical stresses of the slab in these two cases.

A remarkable article [2] is devoted to the force acting on the dielectric slab. But the author argues that the slab experiences a volume force rather than a surface force. He considers electric dipoles **p** within the dielectric and the forces $(\mathbf{p}\cdot\nabla)\mathbf{E}$ acting on the dipoles. I reproduce Fig. 1 from [2] as my Fig. 2. Then the author argue that a force acting on a volume element dV of the dielectric is

$$d\mathbf{F}_D = (\mathbf{P}\cdot\nabla)\mathbf{E}dV \tag{3}$$

where $\mathbf{P} = (\kappa - 1)\mathbf{E}$ is the polarization. (I use V as volume instead of τ because τ is torque and I put $\varepsilon_0 = 1$).

It seems that this statement is incorrect. The point is a boundary of any volume cuts dipoles and thus creates a surface charge which experiences a force as well as dipoles themselves. This force compensates the force acting on dipoles. I think a volume density of force is zero within the dielectric because a volume density of charge within the dielectric is zero. But the expression (3) may be used for integrating over the volume of the whole slab. The boundary of this volume does not cut dipoles, and the volume does not contain discontinuities of **E**. Because any interior volume of integrating gives zero, as I have argued, the surface bound charge of the slab is responsible for the force attracting the slab to the plates.

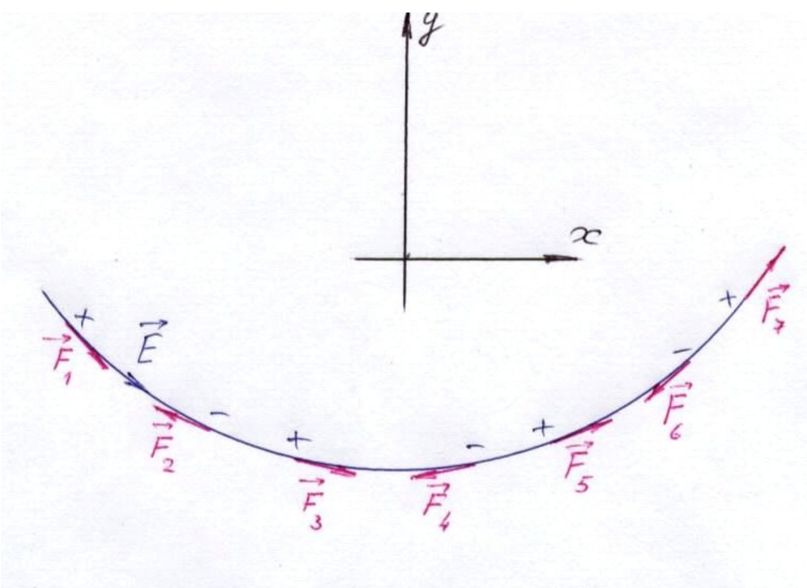

Fig. 3. Three dipoles and a half are shown here.
F1, F2, .. are the forces acting on the charges of the dipoles

We cannot consider a part of the slab as composed of individual aligned dipoles, which feel the field gradient. Indeed, I show dipoles, which are aligned along a field line in Fig. 3. If you add together $\mathbf{F}_2$ and $\mathbf{F}_3$, $\mathbf{F}_4$ and $\mathbf{F}_5$, $\mathbf{F}_6$ and $\mathbf{F}_7$, you obtain a $+y$-directed force. But if you add together $\mathbf{F}_1$

and $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$, $\mathbf{F}_5$ and $\mathbf{F}_6$, you obtain a $-y$-directed force. A conclusion is the following: we cannot consider forces acting on an individual charge or on an individual dipole if the average macroscopic volume bound charge density is zero (as it always is true for neutral linear dielectrics).

Nevertheless, integral of Eq. 3 over the volume of the whole slab can give y-component of the force acting on the slab,

$$F_D^y = (\kappa - 1)\int E^i \partial_i E^y dV = (\kappa - 1)\int (E^x \partial_x E^y + E^y \partial_y E^y + E^z \partial_z E^y) dV \,. \tag{4}$$

As is shown in [2], Eq. (4) can be rewritten as

$$F_D^y = (\kappa - 1)\int E^x \partial_x E^y dV \tag{5}$$

and then, because of $\nabla \times \mathbf{E} = 0$, i.e. $\partial_x E^y = \partial_y E^x$, as

$$F_D^y = (\kappa - 1)\int \partial_y (E^x)^2 dV / 2 = (\kappa - 1)(E_0^x)^2 wd / 2 \tag{6}$$

in accordance with the standard expression for the attractive force, which is obtained by the energy method [3]. (Here E_0^x is x-component of $\mathbf{E}$ in the center of the capacitor and w is a width of the slab along the z-direction).

It is important that the same result (6) may be obtained by integrating of our expression $dF^y = -T^{yi} da_i$ over the surface of the slab. Because

$$E_A^x = \kappa E_D^x, \quad E_A^y = E_D^y, \quad dF^y = \underset{out}{dF}{}^y - \underset{in}{dF}{}^y \,, \tag{7}$$

we have

$$F^y = (\kappa - 1)\oint E_D^y E_D^i \, da_i \;. \tag{8}$$

Using the Green's theorem yields

$$F^y = (\kappa - 1)\int \partial_i (E_D^y E_D^i) dV = (\kappa - 1)\int (\partial_x E_D^y E_D^x + \partial_y E_D^y E_D^y + E_D^y \partial_x E_D^x + E_D^y \partial_y E_D^y) dV \,. \tag{9}$$

Because $\nabla \cdot \mathbf{E} = 0$, i.e. $\partial_x E^x + \partial_y E^y = 0$, Eq. (9) coincides with Eq. (4) except the negligible term $E^z \partial_z E^y$.

Our result means that no force acts on the face of the slab (y = 0 in my Fig.2).

Unfortunately, *AJP* rejected this paper though their author's incorrectness is discussed here

References

[1] J D Jackson, *Classical Electrodynamics* (Wiley, 1999).

[2] Eric R Dietz, "Force on a dielectric slab: Fringing field approach" Am. J. Phys. **72** 1499 (2004)

[3] David J Griffiths, *Introduction to Electrodynamics* (Prentice Hall, Englewood Cliffs, N.Y., 1981) p. 189.