

SMARANDACHE STEPPED FUNCTIONS

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Abstract

The discovery of mathematical complements, assembled under the name of the eccentric mathematics, gave the opportunity for a series of applications, amongst which, in this article, are presented the impulse, step, and unitary ramp functions. The difference, in comparison with the same classic functions, from the distributions theory, is that those eccentric are periodical with a 2π period. By combining these between them, new mathematical functions have been defined; united under the name **Smarandache stepped functions**.

1. Introduction

Romanian mathematician Octavian Stănăşilă sustains that the physics became a science when the calculus (mathematical analysis) has been discovered. In turn, the physics' development imposed the calculus' development.

The theoretical physics, and especially, the quantum mechanics, optics, wave propagation, different electromagnetism phenomena, and the solving of certain limit problems, imposed the introduction of new notions, which are not confined anymore to classical calculus (mathematical analysis), and whose justification could not be made within this frame [6]. This does not mean that it will not come a moment, in mathematics, when this thing can be done. It consists in the discovery of some mathematical complements, included in eccentric mathematics (**EM**) [8], [9], [10], [11], [12] etc., which extend at infinitum all current mathematical forms and objects, ensuring a vast extension of classical/ordinary mathematics, which will be named centric mathematics (**CM**). The reunion of this two mathematics forms what is called the supermathematics (**SM**).

2. The representation of derivatives of some functions

The fact that not every continue function is derivable, having as consequence the inexistence of velocity of a material point, in every moment of its movement, which, evidently, does not correspond to the reality, constitutes a sever difficulty in the **CM**, which affects the unity and the generalization of the results, which is not the case in the **EM**.

For example, let's consider the first nowhere-derivable function, presented by Weierstrass [7, p. 105, Chapter 11 “About Peano curves and their non-differentiability”]:

$$(1) \quad W(t) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi t), \quad 0 < a < 1 \text{ and } b = 1, 3, 5, \dots, (2n - 1), \text{ an odd integer, such that}$$
$$a, b > 1 + \frac{3\pi}{2} = 5,712$$

A modification of Weierstrass' example will be obtained by the substitution in equation (1) of $\cos \pi t$ with linear Euler spline $E(t)$, which interpolates the argument $\cos \pi t$ in all integer values of t , and we obtain the graph [7] from figure 1. Next to it, it was presented the eccentric supermathematics functions family, of eccentric variable $\theta \equiv t$, named $bex t$, and which is a component/term of the eccentric amplitude function $(aex\theta)$, defined by the relation

$$\begin{aligned}
 \alpha(\theta) &= aex\theta = \theta \\
 \beta(\theta) &= \theta \\
 (2) \quad bex\theta &= 0 \\
 &= \arcsin[s \cdot \sin(\theta - \varepsilon)]
 \end{aligned}$$

where θ is the eccentric variable or the angle that a positive semi straight line, revolving around the excenter $S(s, \varepsilon)$ – or solar point, (Kepler affirmed that planets rotate around the Sun on circular orbits, but the Sun is not in the center of the orbits) – it makes it with Ox axis [8], [9], [10], and α is the centric variable or the circular arc, of the unity circle, from the origin of the arc $A(1,0)$ to a current point on circle

$W(1, \alpha) \equiv W(r = rex\theta, \theta)$. The unitary eccentricity is $s = \frac{e}{R}$, or the distance between S and O , and excenter S or E are ejected from the center O on the ε direction.

For $\theta \Rightarrow \pi t$ and a phase difference $\varepsilon = -\frac{\pi}{2}$ will obtain the function or, more precisely, the functions family.

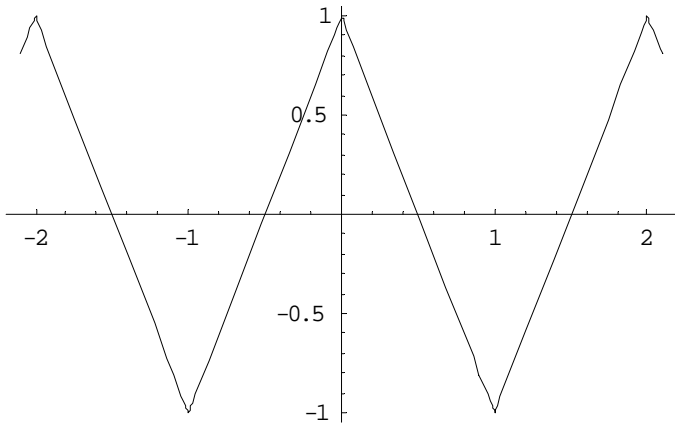


Fig. 1 Modified Weierstrass' function

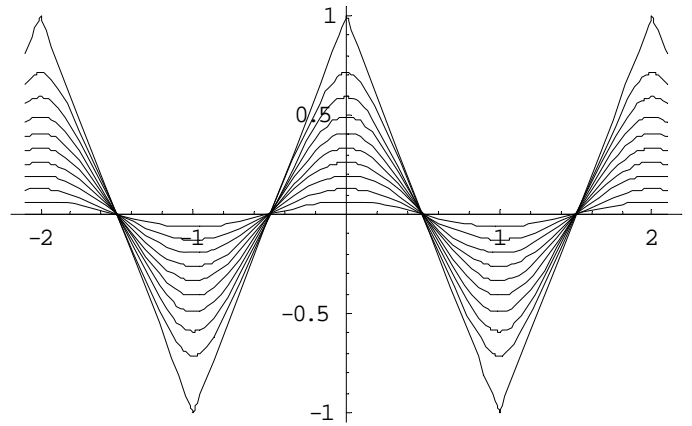


Fig.2. The Eccentric SM function bex t

$$(3) \quad bex t = \arcsin \left[s \sin \left(\pi \cdot t + \frac{\pi}{2} \right) \right],$$

whose graphs, of the numeric eccentricity $s \in [0,1]$, with the step 0.1, are presented in figure 2.

It can be observed, without difficulty, that for $s = 0 \rightarrow aex t = 0$ and for $s = 1$, the maximum limit (in graphs) of s , we obtain the graph of a function in “symmetric triangle teeth” (Fig. 1).

Because, the derivative of the function $aex t$ is the eccentric derivative function $dex\theta$:

$$(4) \quad \frac{d(aex t)}{dt} = \frac{d\alpha}{d\theta} = dex \theta = 1 - \frac{s \cdot \cos(\theta - \varepsilon)}{\sqrt{1 - s^2 \sin^2(\theta - \varepsilon)}},$$

it results that the second term from the relation (4) is exactly the derivative of the function $bex \theta$, that is:

$$(5) \quad \frac{d(bex \theta)}{d\theta} = \frac{s \cdot \cos(\theta - \varepsilon)}{\sqrt{1 - s^2 \sin^2(\theta - \varepsilon)}} = s \cdot coq \left(\pi \cdot t + \frac{\pi}{2} \right) = -s(\sin \pi \cdot t),$$

which is the product between the numerical eccentricity s and the quadrilob cosine function $coq\theta$ [12], with a phase difference $\varepsilon = -\frac{\pi}{2}$, therefore it results $-s \cdot siq\theta$, whose graphs family are presented in the figure 4, for $s \in [0,1]$, with the step 0.1 and in the figure 3, for $s = 1$.

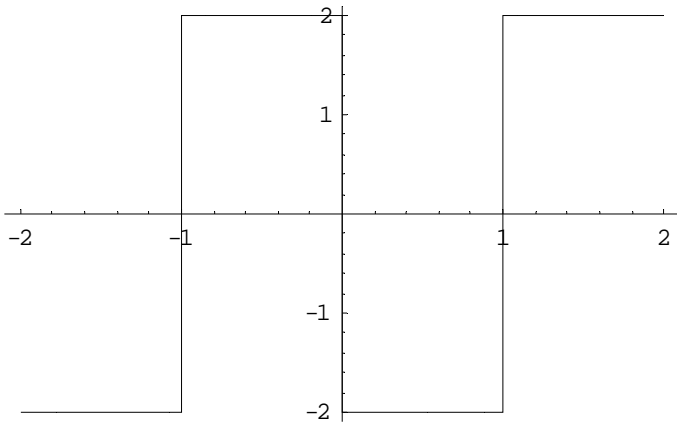


FIGURE 3. Modified derivative of function Weierstrass

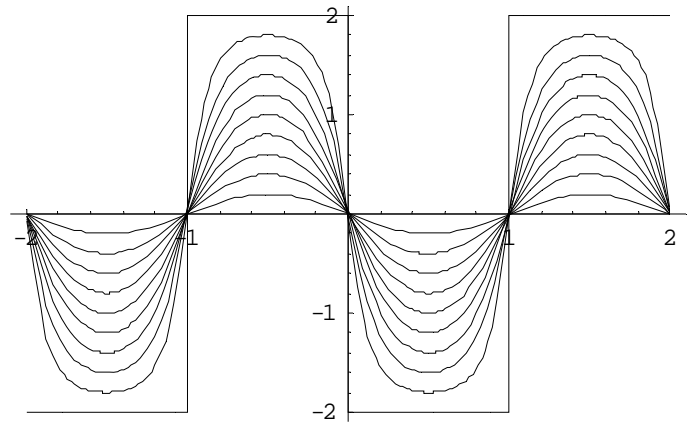


FIGURE 4. The derivatives of the function $bex \theta$

The quadrilob sinus function ($siq\theta$), for the numerical eccentricity $s=1$, represents, in the signals theory, the response of a relay to a sinusoidal signal; this function is also called **square sinus** [13, p. 31], which is exactly the eccentric sinus trigonometric function, with the numerical eccentricity $s=1$, defined on a square, non-rotated with $\frac{\pi}{4}$, as in the case of Alaci quadratic functions [12].

Corroborating the functions and their derivatives, it can be observed that they correspond between them. Thus, modified Weierstrass function, from figure 1, viewed as a $bex t$ function of numerical eccentricity $s=1$, becomes complete derivable.

3. About distributions

In 1926 P. A. M. Dirac introduced, in the quantum mechanics, the delta “function” (δ), which is over all null, with the exception of a point (in origin taking ∞ value), defined as follows:

$$(6) \quad \delta(x) =: \begin{cases} 0, & t \neq 0 \\ +\infty, & t = 0 \end{cases}$$

and whose integral is:

$$(7) \quad \int_{-\infty}^{\infty} \delta(x) dx = 1$$

The same value of the integral is also for the unitary impulse function $\Delta(x, \lambda)$ defined by

$$(8) \quad \Delta(x, \lambda) = \begin{cases} 0, & x < -\frac{\lambda}{2} \\ \frac{1}{\lambda}, & -\frac{\lambda}{2} \leq x \leq \frac{\lambda}{2} \\ 0, & x > \frac{\lambda}{2} \end{cases}$$

It can be observed that for $\lambda \rightarrow 0$ we obtain the Dirac function $\delta(x)$. It must be mentioned that a rigorous definition of Dirac’s impulse can be given within distributions theory [6] or of generalized functions, a chapter of functional analysis.

The unitary impulse can be viewed also as the derivative of the (ideal) unitary step function, or as of Heaviside function $\Gamma(x)$, defined as:

$$(9) \quad \Gamma(x) =: \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

admitting, in this way, the derivability of any continue function on sections.

The unitary ramp function is defined as:

$$(10) \quad R(x) = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$$

and its derivative is the ideal unitary step function (Heaviside).

4. Periodical unitary step, impulse and ramp functions expressed as eccentric circular supermathematics functions (EC-SMF) and with eccentric quadrilob supermathematics functions (QL-SMF)

In figures 5 and 6 are presented the graphs of the eccentric cosine functions ($cex t$) and eccentric quadrilob cosine ($coq t$) respectively [12] for super-unitary numerical eccentricities s .

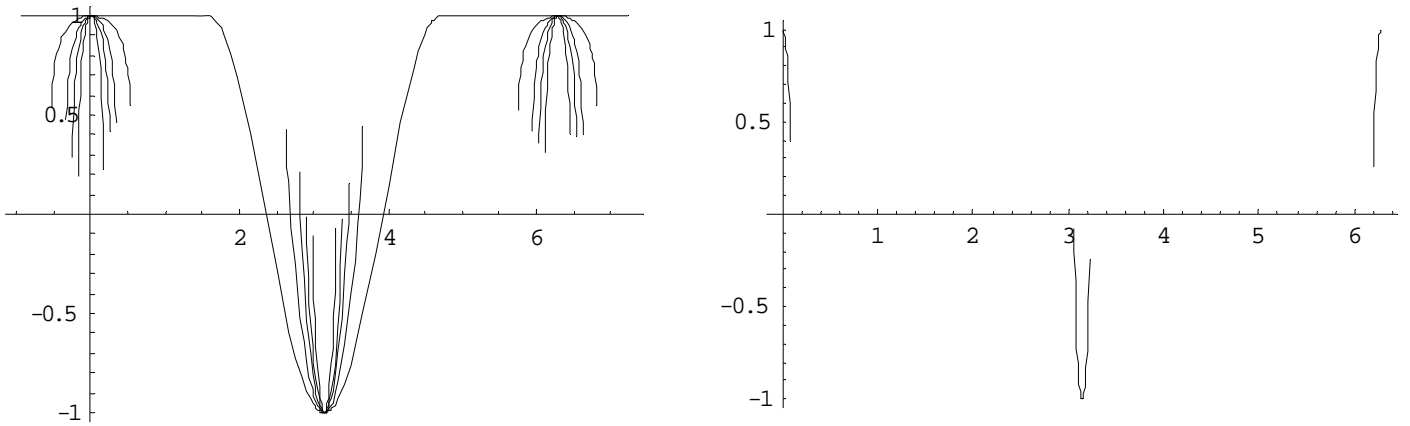


FIGURE 5,A. The function $cex t$, for $s = 1, 2, 3, 4$ and 6 FIGURE 5,B. The function $cex t$, for $s = 4\pi = 12,566$

It can be observed, in the same time with the increase of the numerical eccentricity value s , the functions existence domain becomes restricted to the interval where a line, revolving from the excenter $S(s, \varepsilon)$, external to unity circle, intersects the unity circle. This interval I is periodical, with the period of 2π and it is defined by relation (9).

$$(11) \quad I = t_{final} - t_{initial} = 2\gamma = 2 \arcsin\left(\frac{1}{s}\right),$$

for which the function

$$(12) \quad del\theta = \sqrt{1 - s^2 \sin^2(\theta - \varepsilon)} = 0,$$

where,

$$(13) \quad t_{initial} = \pi + \varepsilon - \gamma = \pi \quad \text{and} \quad t_{final} = \pi + \varepsilon + \gamma = \pi,$$

the eccentric variable: $t \equiv \theta(\text{mod } 2\pi)$ such that, for the excenter S going, on the x axis ($\varepsilon = 0$), to infinite ($s \rightarrow \infty$), the domain I goes to zero ($I \rightarrow 0$). From (12) it results that at $t = \pi$ and $s \rightarrow \infty$, the function $cex_1 t$ is an impulse signal of amplitude -1 , that periodically repeats with a 2π period, and the second determination - with index 2 - of the function, $cex_2 t = 1$ for $t = 0 + 2k\pi$, ($k = 0, 1, 2, \dots$), therefore also at $t = 2\pi$, for $s \rightarrow \infty$, as it results also from the figures 5,A and 5,B.

We will call, these functions “**periodical impulse functions cext of unity amplitude** with $s \rightarrow \infty$ ”. For $\varepsilon = \frac{\pi}{2}$, analogously, for $s \rightarrow \infty$, we obtain “**periodical impulse functions sext of unity amplitude**”.

Because

$$(14) \quad cex_{1,2}^2 \theta + sex_{1,2}^2 \theta = 1, \text{ where } cex_{1,2} t = \pm 1 \rightarrow sex_{1,2} t = 0 \text{ and vice versa.}$$

Therefore, at $t = \pi$, the function $sex_{1,2} t = 0$ with the period 2π and the function

$$I(t, s) = \frac{1}{sex_{1,2} t} \Rightarrow \infty, \text{ obtaining periodical unitary impulse functions, of an infinite amplitude, similar to}$$

Dirac's function, the difference being that it is periodic with a 2π period.

Also the quadrilob cosine function [12]

$$(15) \quad coq_1 t = \frac{\cos \theta}{\sqrt{1 - s^2 \sin^2(\theta - \varepsilon)}} \quad (\theta \equiv t), \text{ for } s \rightarrow \infty \text{ has at } t = \pi \pm 2k\pi, \quad (k = 0, 1, 2, \dots) \text{ the denominator}$$

del $t = 0$ and $\cos 0 = 1$, such that the amplitude goes to infinite and, this way, will obtain, again, a periodical unitary impulse function (Figure 6).

A periodical rectangular function of unity amplitude (Figure 6a) is given by the supermathematics eccentric quadrilob function:

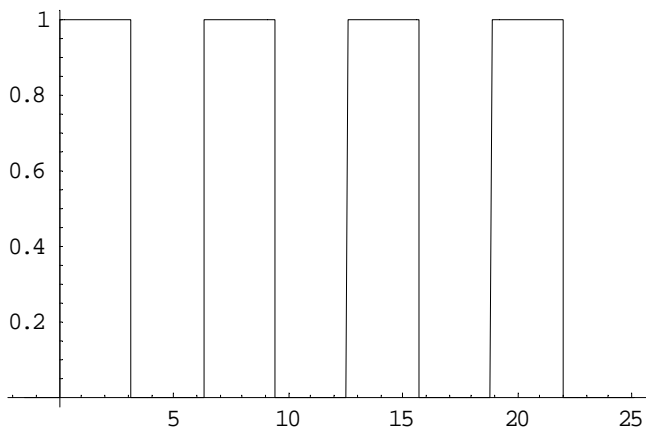


Figure 6,a. The function $0,5.dext$, by numerical eccentricity $s = 1$, as periodical rectangular function or periodical unity stepped function

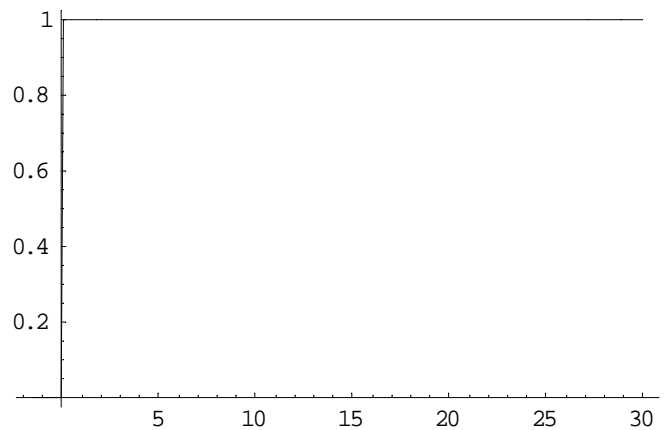


Figure 6,b. The function $0,5.dex(t/10)$ as unity step function

$0,5(1 - siq t)$, with a phase difference π

$$(16) \quad D(t, s) = \begin{cases} \frac{1}{2} \left[1 - \frac{\sin(t + \pi)}{\sqrt{1 - s^2 \cos^2(t + \pi)}} \right], & t \geq 0 \\ 0, & t < 0 \end{cases}$$

which can be named periodical unitary step function, if the numerical eccentricity $s = 1$.

If $t \rightarrow \frac{t}{10}$, (Figure 6b), the first step extends from π to 10π . It results that for $t \rightarrow \frac{t}{\infty} \rightarrow 0$ it will be obtained a unitary step function on all axis $t > 0$.

An analogous function can be obtained also with the eccentric derivative function $dex t$ of $s = 1$ (Figure 7a) and with $t \rightarrow \frac{t}{10}$ and $\varepsilon = -\pi$ (Figure 7b).

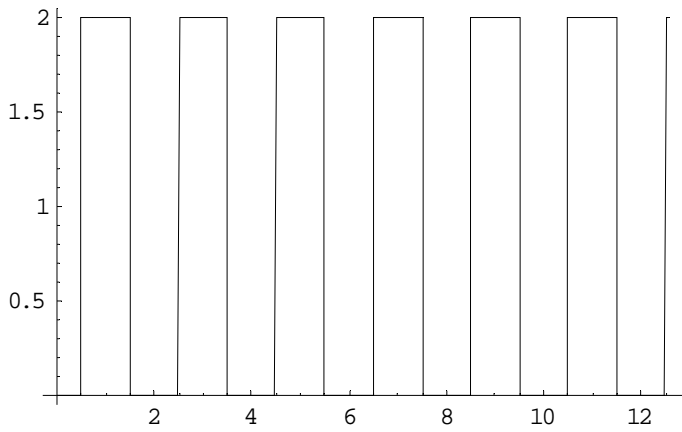


Figure 7a The function $\text{dex } t$ as periodical step function

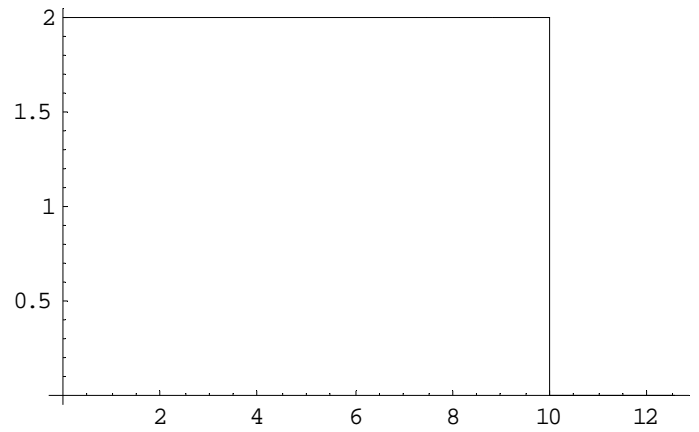


Figure 7b

An ideal unitary ramp function can be obtained as a straight line passing through origin, of an angular coefficient m equal with unity ($m=1$).

$$(17) \quad y = \begin{cases} mx, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

A real unitary ramp function, that will admit certain aberrations from linearity, can also be obtained as a twisted [13] which passes through the origin $O(0,0)$.

A twisted family, obtained for $s \neq 0$ in the interval $s \in [-1,1]$, are presented in figure 8a, where, for $s = 0$, will obtain a ramp for $t \in [0, \infty]$.

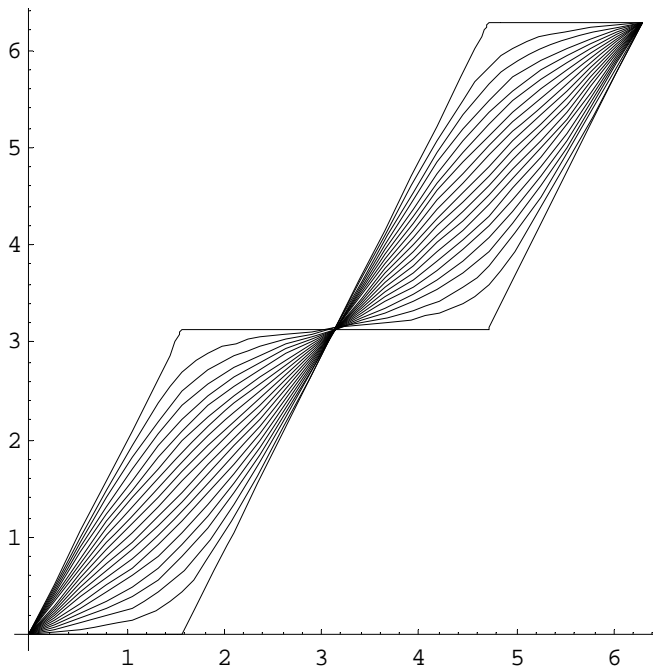


Figure 8a. The function $\text{aex } t$ with $s = -1, \dots, +1$

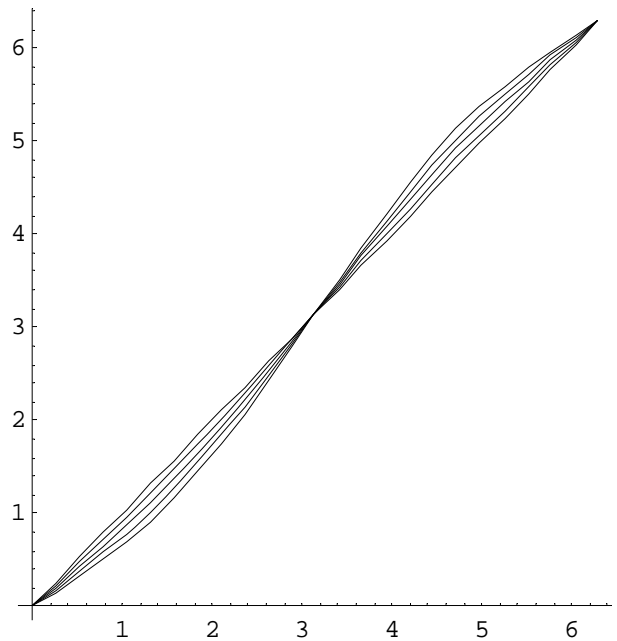


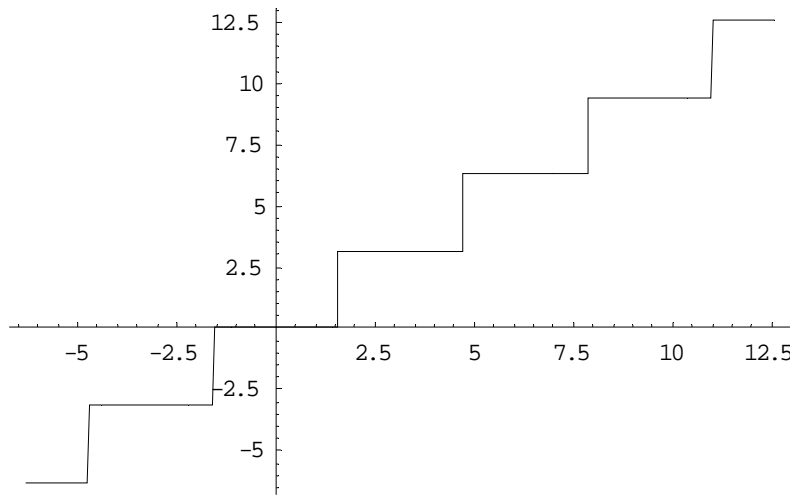
Figure 8b. The function $\text{aex } t$ as real unitary ramp functions. For $s = 0$ we will obtain the ideal ramp

Unitary ramp functions can be obtained by the substitution of the constant $m = \tan \alpha$ with the variable $m = \text{tex} \theta$ for a unitary eccentricity $s = 0,1$.

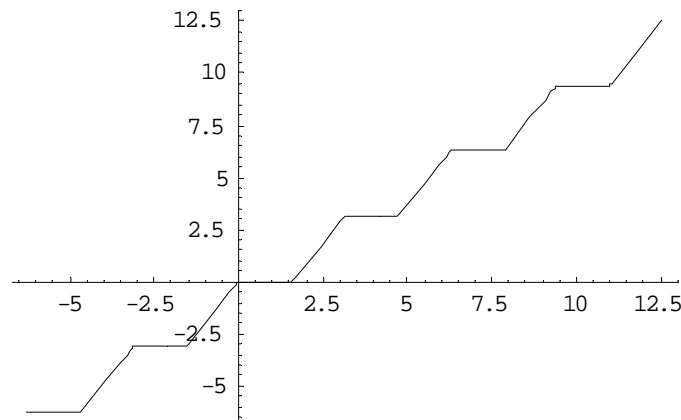
5. Smarandache stepped functions

Combining the eccentric ramp functions, of numerical eccentricity $s=1$, with eccentrically rectangular functions will result the stepped functions, called Smarandache stepped functions, in honor of the Romanian mathematician Florentin Smarandache. Some of these functions, along with their relations of definition, are presented in the following graphs.

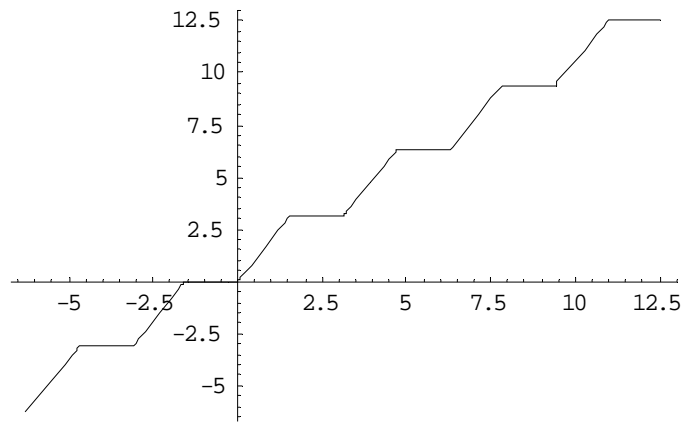
$$\text{Parametric} [\text{Plot}[\{t, t - \text{ArcSin}[\text{Sin}[t]] \text{Cos}[t] / \text{Sqrt}[1 - \text{Sin}[t]^2]\}, \{t, -2\text{Pi}, 4\text{Pi}\}]]$$



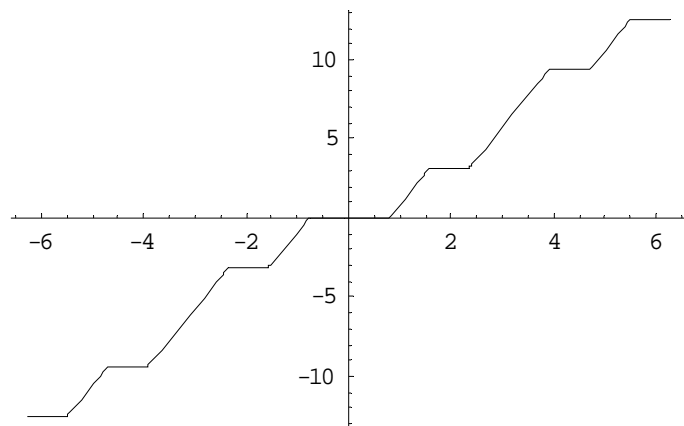
$$\text{Parametric} [\text{Plot}[\{t, t - \text{ArcSin}[\text{Sin}[t]] \text{Sin}[t] / \text{qrt}[1 - \text{Cos}[t]^2]\}, \{t, -2\text{Pi}, 4\text{Pi}\}]]$$



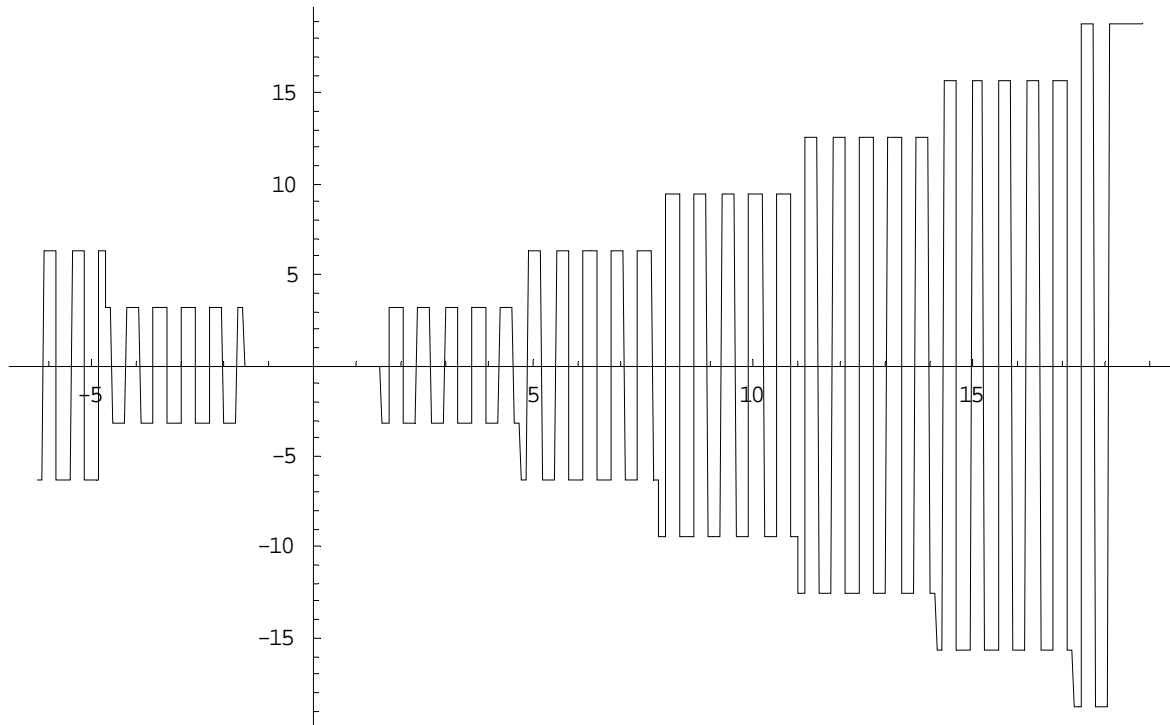
$$\text{Parametric} [\text{Plot}[\{t, t + \text{ArcSin}[\text{Sin}[t]] (\text{Sin}[t] / \text{Sqrt}[1 - \text{Cos}[t]^2])\}, \{t, -2\text{Pi}, 4\text{Pi}\}]]$$



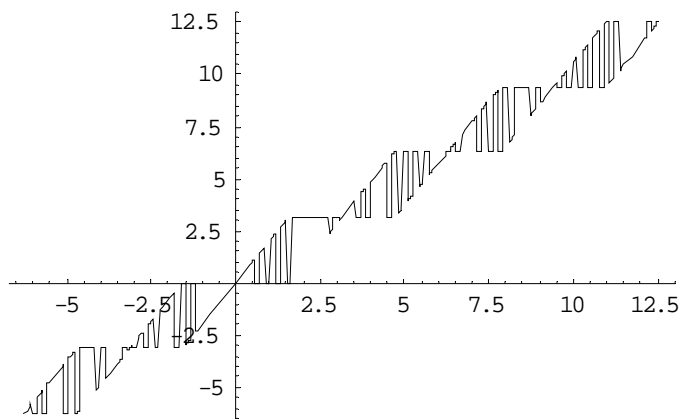
$Parametric[Plot[\{t, 2t - ArcSin[Sin[2t]] \ Cos[t] / Sqrt[1 - Sin[t]^2]\}, \{t, -2Pi, 2Pi\}]$



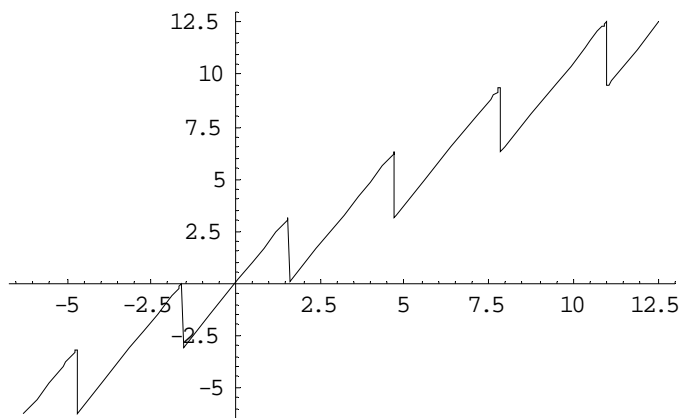
$Parametric[Plot[\{t, (t - ArcSin[Sin[t]] \ (Cos[t] / Sqrt[1 - Sin[t]^2])) \ (Cos[10t] / Sqrt[1 - Sin[10t]^2])\}, \{t, -2Pi, 6Pi\}]$



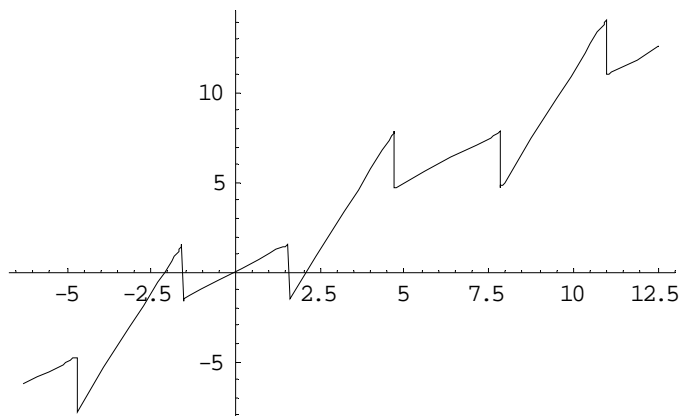
$Parametric[Plot[\{t, t - ArcSin[Sin[t]] (Cos[20t] / Sqrt[1 - Sin[20t]^2])\}, \{t, -2Pi, 4Pi\}]$



$Parametric[Plot[\{t, t + ArcSin[Sin[t]] (Cos[t] / Sqrt[1 - Sin[t]^2])\}, \{t, -2Pi, 4Pi\}]$



$Parametric[Plot[\{t, t - ArcSin[Sin[t]] (1 - Cos[t] / Sqrt[1 - Sin[t]^2])\}, \{t, -2Pi, 4Pi\}]$



$Parametric[Plot[\{t, t - ArcSin[Sin[t]] Cos[t] / Sqrt[1 - Sin[t]^2]\} (t - ArcSin[Sin[10t]] Cos[10t] / Sqrt[1 - Sin[10t]^2])\}, \{t, -2Pi, 4Pi\}]$

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