On Dark Energy, Weyl's Geometry, Different Derivations of the Vacuum Energy Density and the Pioneer Anomaly

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Abstract

Two different derivations of the observed vacuum energy density are presented. One is based on a class of proper and novel *generalizations* of the (Anti) de Sitter solutions in terms of a family of radial functions $R(r)$ that provides an explicit formula for the cosmological constant along with a natural explanation of the ultraviolet/infrared (UV/IR) entanglement required to solve this problem. A *nonvanishing* value of the vacuum energy density of the order of $10^{-123} M^4_{Planck}$ is derived in agreement with the experimental observations. A correct lower estimate of the mass of the observable universe related to the Dirac-Eddington-Weyl's large number $N = 10^{80}$ is also obtained. The presence of the radial function $R(r)$ is instrumental to understand *why* the cosmological constant is not *zero* and why it is so *tiny* and provides a natural derivation of the Pioneer's anomaly. Finally, we rigorously prove why the proper use of Weyl's Geometry within the context of Friedman-Lemaitre-Robertson-Walker cosmological models can account for *both* the *origins* and the *value* of the observed vacuum energy density (dark energy). The source of dark energy is just the dilaton-like Jordan-Brans-Dicke scalar field that is required to implement Weyl invariance of the most *simple* of all possible actions. The full theory involving the dynamics of Weyl's gauge field A_μ is very rich and may explain also the anomalous Pioneer acceleration and the temporal variations (over cosmological scales) of the fundamental constants resulting from the expansion of the Universe. This is consistent with Dirac's old idea of the plausible variation of the physical constants but with the advantage that it is *not* necessary to invoke extra dimensions.

[1]Dedicated to the loving memory of Rachael Bowers

1 SCHWARZSCHILD VERSUS HILBERT : WHY A POINT-MASS HAS PROPER AREA

We begin by writing down the class of static spherically symmetric (SSS) vacuum solutions of Einstein's equations studied by Abrams [5] (where there are *NO* mass sources *anywhere*) given by a *infinite* family of solutions parametrized by a family of admissible radial functions $R(r)$

$$(ds)^2 = g_{00}\ (dt)^2\ - g_{RR}\ (dR)^2\ - R^2\ (d\Omega)^2 =$$

$$g_{00}\ (dt)^2\ - g_{RR}\ (\frac{dR}{dr})^2\ (dr)^2\ - R^2\ (d\Omega)^2 = g_{00}\ (dt)^2\ - g_{rr}\ (dr)^2\ - (R(r))^2\ (d\Omega)^2 \quad (1.1)$$

where the solid angle infinitesimal element is

$$(d\Omega)^2 = sin^2(\phi)(d\theta)^2 + (d\phi)^2. \quad (1.2)$$

and

$$g_{00} = (1 - \frac{\alpha}{R(r)})\ ; \quad g_{RR} = \frac{1}{g_{00}} = \frac{1}{1-\alpha/R}.$$

$$g_{rr} = g_{RR}\ (dR/dr)^2 = (1 - \frac{\alpha}{R(r)})^{-1}\ (\frac{dR(r)}{dr})^2. \quad (1.3)$$

where α is an arbitrary constant that *happens* to have dimensions of *mass* when $c = G = 1$ (but there are no masses at all in this vacuum case). When a point mass source is introduced at the location $r = 0$ one must set $\alpha = 2M$ and replace r for $|r|$ [16].

Notice that the static spherically symmetric (SSS) vacuum solutions of Einstein's equations, with and without a cosmological constant, do *not* determine the form of the radial function $R(r)$. In Appendix **A** and **B** we prove why $R(r)$ is an arbitrary function. In particular it can can be chosen to be an *infinite* family of functions like

$$R(r) = r + \alpha; \quad R(r) = [r^3 + \alpha^3]^{1/3}; \quad R(r) = [r^n + \alpha^n]^{1/n}; \quad R(r) = \frac{\alpha}{1 - e^{-\alpha/r}}. \quad (1.4)$$

found by Brillouin [3] , Schwarzschild [2], Crothers [7], and Fiziev-Manev [18] respectively obeying the conditions that

$$R(r=0) = \alpha; \quad and\ when\ \ r >> \alpha\ \Rightarrow\ R(r)\ \rightarrow r \quad (1.5)$$

Numerous authors have corroborated over the years through lengthy but straightforward calculations [5], [6], [7], [8], [9] , [18], that there exist an infinite class of solutions to the vacuum SSS Einstein's equations $\mathcal{R}_{\mu\nu} = \mathcal{R} = 0$ for an *arbitrary* family of radial functions $R(r)$. In particular, for functions of the type displayed above (but the curvature Riemnan tensor $\mathcal{R}^\mu_{\nu\rho\sigma} \neq 0$).

The most *salient* feature of these solutions eqs-(1.1-1.3) for the radial functions in eq-(1.4) is that at $r = 0$ one has $R(r = 0) = \alpha$ such that $g_{00}(r = 0) = 0$. Therefore, there is *no* horizon. For the historical implications of these truly *horizonless* SSS solutions of Einstein's equations see [5] and the book by [6]. The solutions for a mass point that we have been all accustomed to were those given by Hilbert-Droste-Weyl [4] and can be recovered from eqs-(1.1-1.4) by setting $R(r) = r$. The genuine and original Schwarzshild solution in 1916 was based on $R^3 = r^3 + (2M)^3$ (upon equating $\alpha = 2M$) and *not* based in setting $R = r$. The cubic form of the radial function yields upon differentiation $4\pi R^2 dR = 4\pi r^2 dr$, so the spatial measure coincides with the ordinary spatial measure in flat spaces in spherical coordinates. However, this does *not* mean that $R = r$ as the Hilbert solution imposes. The consequences of the use of $R(r) \neq 0$ are enormous as we shall prove next.

There are many physical differences among the Hilbert and Schwarzchild solutions, in particular in the global properties. The Schwarzchild solution is *not* a radial reparametrization of the Hilbert solution as erroneously argued in the physics circles. In particular, because the radial function $R^3 = |r|^3 + (2M)^3$ can *never zero*. The absolute value $|r|$ properly accounts for the field of a point mass source. Thus, the lower bound of R is given by $2M$, and R cannot be zero for a nonvanishing point mass source.

The Fronsdal-Kruskal-Szekeres analytical continuation of the Hilbert solution for $r < 2M$ yields a *spacelike* singularity at $r = 0$ and the roles of t and r are interchanged when one crosses $r = 2M$; so the interior region $r < 2M$ is *no* longer static. The Schwarzchild solution is *static* for $r < 2M$; there are no horizons at $r = 2M$ and there is a *timelike* singularity at $r = 0$, the true location of the point mass source. Notice that when $r >> 2M$ the Schwarzchild solution reduces to the Hilbert solution and one has the correct Newtonian limt.

A large number of people have fallen for the erroneous claims circulating in physics circles that the original and genuine Schwarzschild solution is just a radial reparametrization of the Hilbert solution. This is *not* true at all. The Topology of the solution forbids it. $R(r)$ can never be zero. It has a lower bound given by $2M$. The source of the misconception (which has propagated for 90 years) was due to the fact that in the genuine and original Schwarzschild solution, despite that $r = 0$ has the topology of a *point*, it does *not* have zero area due to the curvature singularity resulting from the presence of the point mass source at that location. It has zero volume but a non-zero area. Therefore, there is *no interior region* to the point $r = 0$ since it "encloses" a *zero* volume. The location $r = 0$ *is* a true *boundary* of spacetime.

When one correctly uses the radial function found by Schwarzschild $R^3 = r^3 + (2M)^3$ one has upon differentiation $4\pi R^2 dR = 4\pi r^2 dr$. Since dR is *not* equal to dr then the proper area $4\pi R^2$ is *not* equal to $4\pi r^2$. The proper area at the point $r = 0$ is given by $4\pi R^2(r = 0) = 4\pi(2M)^2$ which is *not* equal to *zero*. Thus, the point $r = 0$ (location of the point mass source) does have an area $4\pi(2M)^2$ and this misled people into thinking that an event horizon exits of area $4\pi(2M)^2$ at the location $r = 2M$, when there isn't an event horizon at all in that location. The area $4\pi(2M)^2$ is the one associated with the point mass source at $r = 0$ and *not* to an event horizon at $r = 2M$. Since such area $4\pi(2M)^2$ associated with the point $r = 0$ "encloses" a *zero* volume, there is no interior region to the point $r = 0$. To understand how a point can have a geometrical area enclosing a zero volume, due to the singularity at $r = 0$, let us write the 4-dim space-time measure (after integrating over the angles)

$$4\pi R^2 dR dt = 4\pi R^2 \ (|g_{RR}|^{1/2} dR) \ (|g_{tt}|^{1/2} dt) = 4\pi r^2 \ dr \ dt. \tag{1.6}$$

because $g_{RR} g_{tt} = -1$ and $R^2 dR = r^2 dr$ when the radial function is $R^3 = r^3 + (2M)^3$. The proper radius infinitesimal displacement is $|g_{RR}|^{1/2} dR$ and the proper time infinitesimal displacement is $(|g_{tt}|^{1/2} dt)$; hence $4\pi R^2$ can be seen as a *proper* area which must *not* be confused with the expression for the *spatial* area $A(r) = (4\pi r^2)(|g_{RR}|)^{1/2}$ which is derived next.

To derive the expression for the area in the spherically static symmetric solution, we will recur to the relation $dV(r) = A(r) dr$, where the 3-dim spatial infinitesimal volume element dV is given in terms of the square-root of the absolute value of the determinant of the spatial 3-metric. Thus, the volume infinitesimal is

$$dV = [\ |g_{RR} \ g_{\theta\theta} \ g_{\phi\phi}| \ ]^{1/2} \ (dR \ d\theta \ d\phi) =$$
$$[\ | \ (1 - 2M/R)^{-1} \ R^2 \ R^2 sin^2\phi \ | \ ]^{1/2} \ (dR \ d\theta \ d\phi) = (1 - 2M/R)^{-1/2} \ R^2 \ sin\phi \ (dR \ d\theta \ d\phi). \tag{1.7}$$

Integrating w.r.t the angles gives

$$dV = \frac{4\pi R^2 dR}{(1 - 2M/R)^{1/2}}. \tag{1.8}$$

Since the Schwarzschild solution $R^3 = r^3 + (2M)^3$ upon differentiation yields $R^2 dR = r^2 dr$, then the last relation becomes

$$dV = \frac{4\pi R^2 dR}{(1 - 2M/R)^{1/2}} = \frac{4\pi r^2 dr}{(1 - 2M/R(r))^{1/2}} = A(r) dr = dV(r). \tag{1.9}$$

and from which we arrive, by just factoring out the dr piece, at the area :

$$A(r) = \frac{4\pi r^2}{(1 - 2M/R(r))^{1/2}}, \tag{1.10}$$

the value $A(r = 0) = 0/0$ is undetermined since when $R^3 = r^3 + (2M)^3$ we have $R(r = 0) = 2M$. After performing L'Hopital's rule twice, using $dR/dr = r^2/R^2$, and stetting $r = 0$ we still end up with $0/0$. Performing a Taylor expansion for small r yields for the radial function

$$R(r) \ \sim \ 2M \ [\ 1 + \frac{1}{3} \frac{r^3}{(2M)^3} + ... \ ]. \tag{1.11}$$

so that

$$A(r = 0) = \frac{4\pi r^2}{(1 - 2M/R(r))^{1/2}} \ \sim \ (4\pi) \ \sqrt{3} \ (2M)^{3/2} \ r^{1/2} = 0. \tag{1.12}$$

therefore the *spatial* area at $A(r = 0) = 0$ is *not* equal to the *proper* area at $4\pi R^2(r = 0) = 4\pi(2M)^2$. The scaling behaviour of the spatial area $r^{1/2}$ (instead of r^2) suggests that the geometry nearby the location of the point-mass source may be $fractal$. The proper area of a point-mass at $r = 0$ is *not* zero because there is a physical scalar-curvature delta-function (timelike) singularity at $r = 0$ and the component $g_{RR}(R(r = 0)) = g_{RR}(R = 2M) = \infty$.

Notice that due to the fact that $(dR/dr)^2$ goes to zero faster than $1/g_{RR} = 1-2M/R$, this means that $g_{rr}(r = 0) = g_{RR}(dR/dr)^2 \rightarrow 0$ instead of ∞. However, a completely different result for the spatial area and g_{rr} would be obtained had one used a radial function of the form given by [17] with the correct behaviour at $r = 0$ and $r = \infty$ given by $R(r = 0) = 2M$ and $R(r \rightarrow \infty) \sim r$, respectively :

$$R + 2M\ ln\ (\frac{R-2M}{2M}) = 2M\ ln\ [\ sinh\ \frac{r}{2M}\ ] \Rightarrow \frac{dR}{dr} = \frac{1-2M/R}{tanh\ r/2M}. \tag{1.13}$$

such that for very small values of r on has

$$R(r \rightarrow 0) \rightarrow 2M\ [1 + sinh\ \frac{r}{2M}]; \quad \frac{dR}{dr}(r \rightarrow 0) \rightarrow 1 \tag{1.14}$$

Hence, the spatial area for this choice of $R(r)$ at $r = 0$ is the opposite as before :

$$A(r \rightarrow 0) = \frac{4\pi R^2\ (dR/dr)(r = 0)}{\sqrt{1 - 2M/R(r = 0)}} \rightarrow \infty\ !. \tag{1.15}$$

and $g_{rr}(r = 0) = g_{RR}(dR/dr)^2 \rightarrow \infty$. Thus, the spatial area and g_{rr} now diverge at $r = 0$ instead of being zero.

Once again, the fact that one gets different answers for the spatial areas and g_{rr} depending on the choices of the radial functions $R(r)$ is reminiscent of fractal geometry where the length, areas, volumes.....depend on the *resolutions* of the rulers used to measure them. In our case we may say that the different radial functions $R(r)$ play the role of different "rulers". In fact, one can always rewrite $R(r) = (R(r)/r)\ r = \lambda(r)\ r$ where $\lambda(r)$ is the space dependent *scaling* factor. Thus, different choices of the scaling factors $\lambda(r)$ will furnish different answers as expected. For a rigorous study of how spacetime resolutions affect the physics at small scales see the work of [26], [27].

Despite the fact that one can have an infinite number of metrics with arbitrary radial functions $R(r)$ with the desired behaviour at $r = 0$ and $r = \infty$ and different results for the values of the *spatial* area and g_{rr} the relevant invariant physical quantity is the Einstein-Hilbert action. In particular we will show that the Euclidean action after a compactification of the temporal interval yields an invariant quantity which is precisely equal to the "black hole" entropy in Planck area units. The invariant area is the proper area at $r = 0$ given by $4\pi R(r = 0)^2 = 4\pi(2M)^2$. We shall see that the source of entropy is due entirely to the scalar curvature delta function singularity at the location of the point mass source given by $\mathcal{R} = -[2M/R^2(dR/dr)]\delta(r)$ [16] after using the 4-dim measure $4\pi R^2\ |g_{RR}|^{1/2}dR\ |g_{tt}|^{1/2}dt = 4\pi R^2 dRdt$ in the Euclidean Einstein-Hilbert action.

A point mass source located at $r = 0$, with zero volume, has an infinite density (because the volume is zero) such that there is delta function singularity of the scalar curvature [16]

$$\mathcal{R} = -\frac{2M}{R^2(dR/dr)}\ \delta(r) = -\frac{2M}{r^2}\ \delta(r); \quad when \quad R^3 = r^3 + (2M)^3. \tag{1.16}$$

there is a metric singularity at the location $r = 0$ of the point-mass source $g_{RR}(r = 0) = \infty$, and $g_{tt}(r = 0) = 0$ (time freezes, the arrow of time ceases to flow at $r = 0$). The Euclideanized Einstein-Hilbert action associated with the scalar curvature delta function in eq-1.18) as a result of the condition $4\pi R^2 dR = 4\pi r^2 dr$, is

$$S_E = -\frac{1}{16\pi G}\int \left(-\frac{2M}{r^2}\delta(r) \right) (4\pi r^2 \ dr \ dt) \ = \ \frac{4\pi M^2}{L^2_{Planck}} \ = \ \frac{4\pi \ (2M)^2}{4 \ L^2_{Planck}} = \frac{4\pi \ R^2 \ (r=0)}{4 \ L^2_{Planck}} = \frac{Area \ (r=0)}{4 \ L^2_{Planck}}. \tag{1.17}$$

where the Euclidean time coordinate interval $2\pi t_E$ is defined in terms of the Hawking temperature T_H and Boltzman constant k_B as $2\pi t_E = (1/k_B T_H) = 8\pi GM$. It is interesting that the Euclidean action (1.17) is the same as the "black hole" entropy in Planck area units. The source of entropy is due entirely to the scalar curvature delta function singularity at the location of the point mass source. This result (1.17) remains the same (it is an invariant) for other arbitrary and different choices of the radial function $R(r)$ as long as $R(r = 0) = 2M$ by using $\mathcal{R} = -[2M/R^2(dR/dr)]\delta(r)$ and for the 4-dim measure the result $4\pi R^2|g_{RR}|^{1/2}dR \ |g_{tt}|^{1/2}dt = 4\pi R^2 dRdt$. Furthermore, this result that the Euclidean action is equal to the entropy in Planck units can be generalized to higher dimensions upon recurring to Schwarzschild-like metrics in higher dimensions displayed in Appendix **A**.

The volume, when $R^3 = r^3 + (2M)^3$, is

$$V(r=0) = \int_{r=0}^{r=0} \frac{4\pi r^2}{(1 - 2M/R(r))^{1/2}} \ dr = 0. \tag{1.18}$$

Also we can verify that the integral

$$\mathcal{V} = \int_{r=0}^{r} 4\pi r^2 dr = \frac{4\pi}{3} r^3 = \frac{4\pi}{3}[R^3 - (2M)^3] = \int_{2M}^{R} 4\pi R^2 dR. \tag{1.19}$$

Hence $\mathcal{V}(r = 0) = 0$ consistent with the fact that $R(r = 0) = 2M$. The same result applies for any arbitrary radial function $R(r)$.

Therefore, the most important result is that due to the curvature singularity at $r = 0$ a point mass source at $r = 0$ can have a non-zero proper area $4\pi(2M)^2$ directly related to its mass squared. The volume is indeed zero, but not the proper area, hence we have an infinite (mass/volume) density (a singularity at $r = 0$). We should not forget what is the global topology of the solution. The topology in the R-picture of the genuine Schwarzschild solution is that a of spherical shell extending from $R = 2M$ to $R = infinity$. We truly have a *void* in the R-picture in the region from $R = 0$ to $R = 2M$ because R has a lower bound given by $2M$. R can never be zero. Therefore, it is *meaningless* trying to fill in this *void* via a Fronsdal-Szekeres-Kruskal analytical continuation when the topology does *not* allow us to do this. The spacetime is *not* simply connected. The void region of $0 < R < 2M$ is empty space. There is *no* such a thing as *a spacetime* "living" in that region and into which we can perform a so-called analytical continuation. The Topology of the genuine Schwarzschild solution forbids it. Since the point mass at $r = 0$ has *zero* volume it is *devoid* of *an interior* into which one can analytically continue the solutions.

To sum up : Hilbert's solution is *not* diffeomorphic to the genuine Schwarzschild solution. The proper area associated with the mass-point source at $r = 0$ is *not* zero; the proper area at $r = 0$ is $4\pi(2M)^2$, but the volume $V(r = 0) = 0$. The point mass source at $r = 0$ can have a geometrical area $4\pi(2M)^2$ *enclosing* a *zero volume* due to the spacetime singularity at $r = 0$. This has been the source of the confusion in the past 90 years that misled people into thinking there is an event horizon at $r = 2M$ enclosing the singularity $r = 0$. One cannot analytically continue into the interior of a point, because there is NO spacetime interior beyond $r = 0$, since the volume enclosed by the area of the point mass at $r = 0$ is *zero*, and $r = 0$ is both the timelike singularity as well as the true *boundary* of spacetime.

In [16] we studied the many subtleties behind the introduction of a true point-mass source at $r = 0$ (that couples to the vacuum field) and the physical consequences of the delta function singularity (of the scalar curvature) at the location of the point mass source $r = 0$. Those solutions were obtained from the vacuum SSS solutions simply by replacing r for $|r|$ and α for $2M$. For instance, the Laplacian in spherical coordinates in flat space of $1/|r|$ is equal to $-(1/r^2)\delta(r)$, but the Laplacian of $1/r$ is *zero*. Thus, to account for the presence of a true mass-point source at $r = 0$ one must use solutions depending on the modulus $|r|$ instead of r. The scalar curvature was $\mathcal{R} = -[2M/R^2(dR/dr)]\delta(r)$ [16] . It is interesting that for the Hilbert $R = r$ and Schwarzschild solutions $R^3 = r^3 + (2M)^3$, the scalar curvature is the *same* $\mathcal{R} = -(2M/r^2)\delta(r)$ and also the spatial measures $4\pi R^2 dR = 4\pi r^2 dr$. This deserves further investigation.

A different and detailed treatment of point masses, point charges, delta function sources and the physical implications of the many different choices of the radial functions $R(r)$ in General Relativity has been given by Fiziev [18]. A thorough mathematical analysis on the theory of tensor-valued distributions and delta function singularities in nonlinear theories (like General Relativity) based on Colombeau's theory of nonlinear distributions can be found in [12] since the standard Schwarz theory of linear distributions is not valid. After this historical preamble, lets focus next on the study of the static spherically symmetric (SSS) solutions with a *nonvanishing* cosmological constant based on the introduction of an admissible family of radial functions $R(r)$.

2 ONE RESOLUTION OF THE COSMOLOGICAL CONSTANT PROBLEM AND THE PIONEER ANOMALY

2.1 Generalized de Sitter and Anti de Sitter Metrics

We begin with the generalized de Sitter and Anti de Sitter metrics that will help us understand the nature of the infrared cutoff required to solve the cosmological constant problem. In Appendix **B** we prove why the most general *static* form of the (Anti) de

Sitter-Schwarzschild solutions are given in terms of an arbitrary radial function by

$$g_{00} = (\ 1 - \frac{2M}{R(|r|)} - \lambda\ R(|r|)^2\ ). \quad g_{rr} = -(\ 1 - \frac{2M}{R(|r|)} - \lambda\ R(|r|)^2\ )^{-1}\ (dR(|r|)/dr)^2 \quad (2.1)$$

The angular part is given as usual in terms of the solid angle by $-(R(|r|))^2(d\Omega)^2$. We choose the parameter $\lambda = \Lambda/3$ where Λ is the cosmological constant. The $\lambda < 0$ case corresponds to Anti de Sitter-Schwarzschild solution and $\lambda > 0$ corresponds to the de Sitter-Schwarzschild solution. The physical interpretation of these solutions is that they correspond to "black holes" in curved backgrounds that are not asymptotically flat. For very small values of R one recovers the ordinary Schwarzschild solution. For very large values of R one recovers asymptotically the (Anti) de Sitter backgrounds of constant scalar curvature.

The solutions we are interested correspond to the case $M = 0$ and given in terms of the radial functions $R(r)$ which *are* the most general SSS solutions to Einstein's equations *with* a cosmological constant. These solutions were studied earlier by [7] but unfortunate his analysis was *erroneous* and his conclusions are invalid. We will show below that there are nontrivial solutions with a *nonvanishing* cosmological constant (contrary to the assertions made in [7]) λ when the *correct* family of admisisble radial functions $R(r)$ are introduced.

We must emphasize that the *novel* derivation below of the cosmological constant is *not* based on the vacuum fluctuations models of [20]; [21]; nor on the Scale Relativity Theory [26]; nor on the schemes based on a two-measure theory [32]; the holographic renormalization group program, quintessence, non-critical strings [23] etc.... nor with an ad hoc introduction of the Hubble distance. This derivation is based on an entirely $different$ physical reason than all of the others described so far (to our knowledge) ; i.e . it is based solely on the physical implications of the radial function $R(r)$ of eq-(2.1). It is warranted to study the connection (if any) among our derivation of the cosmological constant with all of the prior calculations, for example, [20], [26], [21].

It is important, of course, to find a physical explanation of the *origins* of the cosmological constant besides computing its observed value. This was attained in [15] without imposing any assumptions whatsoever on the calculations as it is done in the literature by showing why an Einstein-Hilbert action with the correct value of the cosmological constant (plus Gauss-Bonnet terms) can be obtained from the vacuum state of a **BF**-Chern-Simons-Higgs theory based on the (Anti) de Sitter group, after the (Anti) de Sitter symmetry is broken to the Lorentz one.

We will show why its connection with the Hubble constant is *not* ad hoc at all ; on the contrary it explains why the Hubble constant (the Hubble horizon) has to appear in the derivation. It also implements naturally the UV/IR (ultra-violet/infrared) entanglement, (without postulating it ad hoc) necessary to derive the cosmological constant and we provide a lower bound for the mass of the observable universe. To sum up, to our knowledge, the crux of the derivation below does not rely *whatsoever* on *any* of the other prior derivations employed to derive the value of the cosmological constant.

One particular expression for the radial function in the de Sitter-Schwarzschild ($\lambda > 0$) case is

$$\frac{1}{R^2 - (2M)^2} = \frac{1}{r^2} + \lambda. \tag{2.2}$$

since $r^2 = |r|^2$ there is no need to explicitly write the modulus sign in (2.2) and in the discussion below. When $\lambda = 0$ one recovers $R^2 = r^2 + (2M)^2$ that has a similar behaviour at $r = 0$ and $r = \infty$ as the pure Schwarzschild case given by $R^3 = r^3 + (2M)^3$; i.e. $R(r = 0) = 2M$ and $R(r \to \infty) \sim r$ respectively. When $M = 0$ one recovers the pure de Sitter case and the radial function becomes

$$\frac{1}{R^2} = \frac{1}{r^2} + \lambda. \tag{2.3a}$$

In this case, one encounters the *reciprocal* situation (the "dual" picture) of the Schwarzschild solutions : (i) when r tends to zero (instead of $r = \infty$) the radial function behaves $R(r \to 0) \to r$; in particular $R(r = 0) = 0$ and (ii) when $r = \infty$ (instead of $r = 0$) the value of $R(r = \infty) = R_{Horizon} = \sqrt{\frac{1}{\lambda}}$ and one reaches the location of the *horizon* given by the condition $g_{00}[R(r = \infty)] = 0$.

It is very important to emphasize that for other choices of admissible radial functions other than eq-(2.3a) they must obey similar boundary conditions at $r = 0$ and $r = \infty$ as the conditions described above : $R(r = 0) = 0$ and $R(r = \infty) = R_{Horizon} = \sqrt{\frac{1}{\lambda}}$. The family of admissible radial functions obeying the required boundary conditions at $r = 0$ and $r = \infty$ are of the form

$$(\frac{1}{R^2}) = [\ (\frac{1}{r^2})^n + (\lambda)^n \ ]^{1/n}. \quad n > 0. \tag{2.3b}$$

The correct *proper* radius $R_p(r)$ (an invariant quantity under radial reparametrizations) is given by the integral

$$R_p(r) \ = \ \int \frac{dR}{\sqrt{1 - \lambda \ R^2}} = \frac{1}{\sqrt{\lambda}} \ arcsin \ [\ R(r)\sqrt{\lambda} \ ] \ \Rightarrow$$

$$R_p(r = 0) = 0 \ \ since \ R(r = 0) = 0; \ and \ R_p(r = \infty) = \frac{\pi}{2} \frac{1}{\sqrt{\lambda}} = \frac{\pi}{2} \ R_{Horizon}. \tag{2.4}$$

Therefore, the pure de Sitter case has a well behaved proper radius $R_p(r)$. When $M \neq 0$ one has for the de Sitter-Schwarzcshild case

$$g_{00}(r_*) = 0 \ \Rightarrow \ 1 - \frac{2M}{R(r_*)} - \lambda \ R(r_*)^2 = 0 \tag{2.5}$$

a cubic equation whose solutions R_* will restrict the values of the radial function $R_* = R(r_*)$ at $r = r_* \neq \infty$, in terms of the mass parameters M and the cosmological constant $\Lambda = 8\pi G \rho_{vacuum}$. The cubic equation was solved *exactly* in [16].

In the pure de Sitter case the condition

$$g_{00}(r = \infty) = 0 \ \Rightarrow \ 1 - \lambda \ R(r = \infty)^2 = 0 \tag{2.6}$$

has a real valued solution

$$R(r = \infty) = \sqrt{\frac{1}{\lambda}} = R_{Horizon}. \qquad (2.7)$$

and the correct order of magnitude of the observed cosmological constant can be derived from eq-(2.7) by equating $R(r = \infty) = R_{Horizon}$ = Hubble Horizon radius as seen today since the Hubble radius is *constant* in the *late time* pure inflationary de Sitter phase of the evolution of the universe because the Hubble parameter is constant H_o. Eq-(2.1) is the *static* form of the generalized de Sitter (Anti de Sitter) metric associated with a *constant* Hubble parameter. Therefore, by setting the Hubble radius to be of the order of 10^{61} L_{Planck} and by setting $G = L^2_{Planck}$ ($\hbar = c = 1$ units) in

$$8\pi \ G \ \rho_{vacuum} \ = \ \Lambda \ = \ 3\lambda \ = \ \frac{3}{R(r=\infty)^2} = \frac{3}{R^2_H} \Rightarrow$$

$$\rho_{vacuum} = \frac{3}{8\pi} \frac{1}{L^2_P} \frac{1}{R^2_H} = \frac{3}{8\pi} \frac{1}{L^4_P} (\frac{L_P}{R_H})^2 \ \sim \ 10^{-123} \ (M_{Planck})^4. \quad when \ R_H \sim 10^{61} L_P. \qquad (2.8)$$

we obtain a result which agrees with the experimental observations. Notice the importance of using the radial function $R = R(r)$ in eqs-(2.6, 2.8). Had one used $R = r$ in eq-(2.6) one would have obtained a *zero* value for the cosmological constant when $r = \infty$. Thus, the presence of the radial function $R(r)$ is essential to understand *why* the cosmological constant is not *zero* and why it is so *tiny* . In the next section we shall study the case of a *time* dependent Hubble parameter $H(t) = \frac{1}{a}(da/dt) = H_o \ tanh \ (H_0 t)$ corresponding to the scaling function $a(t) = cosh \ (H_0 t)$ and which tends to a constant H_0 in the asymptotic late time limit $H(t \rightarrow \infty) \rightarrow H_0$. We shall study also the pure inflationary de Sitter phase associated with a constant Hubble parameter H_0 corresponding to $a(t) = e^{H_0 t}$.

We continue with a relevant analysis of the UV/IR (ultraviolet-infrared) entanglement involving the interaction of small-large scales within the context of the cosmological constant problem. The transformation

$$r \rightarrow \frac{1}{\lambda r}; \quad \lambda \neq 0. \qquad (2.9)$$

exchanges *small* distances with *large* distances and vice versa, reminiscent of the T-duality in string theory compactifications, and leads to a dual radial function of the form

$$\frac{1}{\tilde{R}^2} = (\lambda r)^2 + \lambda. \qquad (2.10a)$$

where now one has the reciprocal ("dual") behaviour as that of eq-(2.7)

$$\tilde{R}(r = \infty) = 0; \quad \tilde{R}(r = 0) = \frac{1}{\sqrt{\lambda}}. \qquad (2.10b)$$

and the horizon condition $g_{00}(R_{Horizon}) = 0$ is now attained at $r = 0$ (due to the small-large scales exchange)

$$g_{00}(r = 0) = 0 \ \Rightarrow \ 1 - \lambda \ \tilde{R}(r = 0)^2 = 0 \ \Rightarrow \tilde{R}(r = 0) = \sqrt{1/\lambda} \ = R_{Horizon}. \qquad (2.11)$$

and once again we get the same result as in (2.8).

It is clear now why if one had written $\tilde{R}(r) = r$ in eq-(2.11) and introduced the Planck scale as an ultraviolet cutoff, instead of setting $r = 0$, one would have obtained an answer in eq-(2.11) that is off by 122 orders of magnitude (which *is* the cosmological constant problem) . What the dual radial function $\tilde{R}(r)$ achieves in eqs-(2.10a, 2.11) is to map the extreme ultraviolet (UV) region $r = 0$ onto the infrared (IR) region $\tilde{R}(r = 0) = R_{Hubble}$. Hence, the presence of the dual radial function $\tilde{R}(r)$ implements the necessary UV/ IR entanglement associated with the resolution of the cosmological constant problem.

The reason one can invoke the use of the *dual* radial function $\tilde{R}(r)$, as well as $R(r)$, is because there is an infinite family of admissible radial functions associated with the SSS solutions to Einstein's equations with a comological constant $\Lambda = 3\lambda$, given by eq-(2.1) when $M = 0$. The choice $R(r) = r$ yields the familiar solution we have been accustomed to all these years. However, as we have shown, the correct choice of the admissible radial functions displayed in eqs-(2.3a, 2.10a) is what allows us to obtain the correct value of the vacuum energy density consistent with the astrophysical observations ! In the same vein, the genuine and original Schwarzschild solution that truly describes the gravitational field due to a point mass source at $r = 0$ required Schwarschild to choose $R = [r^3 + (2M)^3]^{1/3}$. The Hilbert choice $R = r$ has been shown to be incorrect by [5], [6], [19], [18] among many others (it does not describe the gravitational field of a mass point at $r = 0$ and does not provide a consistent arrow of time [19]).

In [15], where *no* assumptions whatsoever were made, we have shown why AdS_4 gravity with a topological term; i.e. an Einstein-Hilbert action with a cosmological constant plus Gauss-Bonnet terms can be obtained from the vacuum state of a **BF**-Chern-Simons-Higgs theory *without* introducing by *hand* the zero torsion condition imposed in the MacDowell-Mansouri-Chamsedine-West construction. One of the most salient features of [15] was that a *geometric mean* relationship was *derived* (from scratch, instead of postulating it) among the vacuum energy density ρ , the Planck area L_P^2 and the AdS_4 throat size squared R^2 given by $\rho = (L_P)^{-2}\ R^{-2}$. Upon setting the throat size to coincide with the Hubble scale $R_H \sim 10^{61} L_P$ (since the throat size of de Sitter and Anti de Sitter is the same) one obtains the observed value of the vacuum energy density $\rho = L^{-2}_{Planck} R_H^{-2} = L_P^{-4} (L_P/R_H)^2 \sim 10^{-122} (M_{Planck})^4$.

For example, the calculation based on the model of vacuum fluctuations of a scalar field by [20] relies on several assumptions and leads to a numerical result that coincides with the Casimir energy density $(1/R_H^4)$ (associated with an infrared scale R_{Hubble}) times an overall numerical factor related to the sum over all the radial modes $\sum_1^N n$. So the $\rho_{vacuum} \sim (1/R^4_{Hubble}) \sum_1^N n \sim\ \frac{1}{2}(N^2/R^4_{Hubble})$. The value of N was set to be of the order of R_H/L_P . This is consistent with the Scale Relativity theory [26] and its connection to Yang's Noncommutative spacetimes and QM in Clifford spaces [14], where there are both an UV and IR cutoffs related to the Planck L_P and Hubble scale R_H, respectively. Therefore, there is a maximum $\hbar/L_P$ and minimum $\hbar/R_H$ momentum which determines the value of the maximum mode number N in the sum $\sum_1^N n$ given by $N = (R_H/L_P)$. Hence, the $\rho_{vacuum} \sim (N^2/R^4_{Hubble}) = L_P^{-2} R_H^{-2} \sim 10^{-122} M_P^4$, when $R_H \sim 10^{61} L_P$.

It is poignant to mention that related to the issue of T-duality (the UV/IR entanglement displayed by the radial function $R(r)$ and its dual $\tilde{R}(r)$ above) the analog of S-duality for linearized gravity in $4D$ was developed by [24] where the strong-weak coupling duality is an exact symmetry which implies a small-large duality for the cosmological constant. A quantization of the cosmological constant was another implication of a duality symmetry [24]. This interplay between S and T dualities has to be investigated further.

2.2 An estimate of the Mass of the Universe and Dirac-Eddington-Weyl Large Numbers Coincidences

To finalize this section we should add that an estimate of a lower mass bound of the Universe can also be attained from studying the location of the horizon corresponding to the Anti de Sitter-Schwarzschild generalized metric solutions of eq-(2.1) .

The cubic equation that sets the location R_* of the horizon in the Anti de Sitter case $g_{00}(R = R_*) = 0$ is given by

$$g_{00}(r_*) = 0 \Rightarrow 1 - \frac{2M}{R(r_*)} + \lambda\ R(r_*)^2 = 0 \tag{2.12}$$

and whose unique real positive solution is [16] :

$$R_1 = [\ \frac{M}{\lambda} + \sqrt{\frac{M^2}{\lambda^2} + \frac{1}{27\lambda^3}}\ ]^{1/3} + [\ \frac{M}{\lambda} - \sqrt{\frac{M^2}{\lambda^2} + \frac{1}{27\lambda^3}}\ ]^{1/3} > 0. \tag{2.13}$$

We must disregard the two complex roots. There are *no* double roots in the AdS case because

$$\frac{M^2}{\lambda^2} + \frac{1}{27\lambda^3} \neq 0. \tag{2.14}$$

It is very important to emphasize that one has already taken into account the fact $\lambda_{AdS} = -\lambda_{dS}$ in the root of eq-(2.13). Therefore in eq-(2.13), and all the expressions that follow, when we write λ it should be understood as $|\lambda|$ and hence it is a *positive* quantity. The radial function $R(r)$ in the Anti de Sitter case must $differ$ from the de Sitter case and is obtained from eq-(2.2) by replacing $\lambda \rightarrow -\lambda$

$$\frac{1}{R^2 - (2M)^2} = \frac{1}{r^2} - \lambda \Rightarrow R(r=0) = 2M; \quad R(r=\infty) = \sqrt{(2M)^2 - \frac{1}{\lambda}} < 2M. \tag{2.15}$$

and it leads to the inequality $2M > R_* > R(r = \infty)$ because it is a *decreasing* function of r and which can be recast explicitly as

$$2M > [\ \frac{M}{\lambda} + \sqrt{\frac{M^2}{\lambda^2} + \frac{1}{27\lambda^3}}\ ]^{1/3} + [\ \frac{M}{\lambda} - \sqrt{\frac{M^2}{\lambda^2} + \frac{1}{27\lambda^3}}\ ]^{1/3} > \sqrt{(2M)^2 - \frac{1}{\lambda}} \geq 0 \tag{2.16}$$

The implementation of the UV/IR entanglement map $r \rightarrow 1/\lambda r$ in eq-(2.15) yields the dual version of the radial function $\tilde{R}(r)$

$$\frac{1}{\tilde{R}^2 - (2M)^2} = (\lambda r)^2 \ - \ \lambda \ \Rightarrow \ \tilde{R}(r = \infty) = 2M; \quad \tilde{R}(r = 0) = \sqrt{(2M)^2 - \frac{1}{\lambda}} \ < \ 2M. \tag{2.17}$$

which is an *increasing* function of r, instead of a decreasing function like $R(r)$ in eq-(2.15).

From eq-(2.17) one can infer from the condition

$$\sqrt{(2M)^2 - \frac{1}{\lambda}} \ = \ real - valued \ \Rightarrow 2M \ \geq \ \frac{1}{\sqrt{\lambda}}. \tag{2.18}$$

Hence, if one were to *equate* the quantity $2M = \frac{1}{\sqrt{\lambda}} = R_{Hubble}$ with the net mass of the galaxies, stars, ...inside that region of the observable universe enclosed by the Hubble radius R_H, and take a value of $R_{Hubble} \sim 10^{61} L_{Planck}$, one would have in the appropriate units the following

$$2M \sim 10^{61} \ M_{Planck} \sim 10^{80} \ m_{proton}. \tag{2.19}$$

that agrees with the Dirac-Eddington-Weyl large number coincidences

$$N = 10^{80} \sim (\frac{F_e}{F_G})^2 \sim (\frac{R_{Hubble}}{r_e})^2. \tag{2.20}$$

where $F_e = e^2/r$ is the electrostatic force between an electron and a proton; $F_G = G m_e m_p / r^2$ is the corresponding gravitational force and $r_e = e^2/m_e \sim 10^{-13} cm$ is the classical electron radius in natural units of $\hbar = c = 1$. Of course, this is not to say that the AdS-Schwarzchild case is the same as the Friedman-Robertson-Walker model, but only that one could equate the net mass (inside R_H) of the latter with the $2M$ parameter of the former to get an estimate of the *lower* bound of the mass of the observable universe. To match the observational data requires further work since it is more likely that $2M > \frac{1}{\sqrt{\lambda}} = R_{Hubble}$ due to the presence of dark matter. Recently, the authors [25] have studied the finite-action solutions (square integrable) of the Klein-Gordon equation on Lorentzian manifolds (Friedman type and de Sitter) and have found a discrete mass spectrum that could help answer why elementary particles have a discrete spectrum. Thus this interplay between cosmology and particle physics needs to be explored further.

By inspection one can verify that the lower bound $2M = \frac{1}{\sqrt{\lambda}}$ obeys the condition given by eq-(2.18). The latter becomes

$$2M = \frac{1}{\sqrt{\lambda}} \ > \ R_* = \ (\ [\ \frac{1}{2} + \sqrt{\frac{31}{108}} \ ]^{1/3} \ + \ [\ \frac{1}{2} - \sqrt{\frac{31}{108}} \ ]^{1/3} \ ) \ \frac{1}{\sqrt{\lambda}} = 0.6823 \ \frac{1}{\sqrt{\lambda}}. \tag{2.21}$$

It is clear that a lot of work and re-thinking remains to be done pertaining the proper use of the radial functions $R(r)$ in the class of SSS solutions to Einstein's equations with and without a cosmological constant. The fact that we were able to obtain the correct magnitude of the observed cosmological constant and the correct lower estimate of the mass of the universe related to the Dirac-Eddington's large number $N = 10^{80}$ is a positive sign that one should use the solutions displayed in this work based on a suitable class of radial functions $R(r)$ rather than the naive choice $R = r$ we have been familiar with during all these decades . The presence of the radial function $R(r)$ was instrumental to understand *why* the cosmological constant is not *zero* and why it is so *tiny*.

2.3 The Pioneer Anomaly, Mach's Principle and Modified Newtonian Dynamics

To finalize this section we will present a derivation of the anomalous Pioneer acceleration based on a very specific choice of the radial function $R(r)$ corresponding to the (Anti) de Sitter-Schwarzschild generalized solutions given in eq-(2.1). We will choose the specific radial function to be of the form (after reintroducing the Newtonian coupling G_N which was set to unity)

$$\frac{1}{R - (2G_N M)} = \frac{1}{r} + \sqrt{\lambda} \Rightarrow R = (2G_N\ M) + \frac{r}{1 + \sqrt{\lambda}\ r}. \tag{2.22a}$$

$$\frac{1}{R - (2G_N M)} = \frac{1}{r} - \sqrt{\lambda} \Rightarrow R = (2G_N\ M) + \frac{r}{1 - \sqrt{\lambda}\ r}. \tag{2.22b}$$

Let us choose the plus sign in front of the square root $+\sqrt{\lambda}$ (2.22a) first and then we will focus on the negative sign of the square root $-\sqrt{\lambda}$ in eq-(2.22b). For distances r of the order of the solar system that are much smaller than the cosmological scales $\lambda\ R(r)^2 << 1$ we can approximate the metric component in eq-(2.1) by :

$$g_{tt} = -(1 - \frac{2\ G_N\ M}{R(r)} - \lambda\ R(r)^2) \sim -(1 - \frac{2\ G_N\ M}{R(r)}). \tag{2.23}$$

and in the weak field approximation when M is of the order of the solar mass (solar system) the $modified$ Newtonian potential V_N can be read from the condition

$$g_{tt} \sim -(1 + 2V_N) \Rightarrow V_N = -\frac{G_N\ M}{R(r)}. \tag{2.24}$$

and the force per unit mass, acceleration, is then :

$$a = \frac{F}{m} = -\frac{\partial V_N}{\partial r} = -\frac{G_N\ M}{R(r)^2} [\frac{1 + \sqrt{\lambda}\ r - \sqrt{\lambda}\ r}{(1 + \sqrt{\lambda}\ r)^2}] = -\frac{G_N\ M}{R(r)^2} [\frac{1}{(1 + \sqrt{\lambda}\ r)^2}]. \tag{2.25}$$

By approximating $R(r)^2 \sim r^2$ and performing a binomial (Taylor) expansion leads to

$$a \sim -\frac{G_N\ M}{r^2} (1 + \sqrt{\lambda}\ r)^{-2} \sim -\frac{G_N\ M}{r^2} (1 - 2\sqrt{\lambda}\ r + ...). \tag{2.26}$$

Thus the first order corrections to the Newtonian acceleration in powers of $\sqrt{\lambda}\ r$ are :

$$\Delta\ a\ =\ +\ \frac{2\ G_N\ M}{r^2}\ \sqrt{\lambda}\ r = \ =\ +\ \frac{2\ G_N\ M}{r^2}\ \frac{r}{R_{Hubble}}\ =\ +\ \frac{2\ G_N\ M}{r\ R_{Hubble}}. \qquad (2.27)$$

The positive sign means that the corrections point outwards away from the sun. This is because we chose the radial function given by eq-(2.22a). The choice of the negative sign of the square root in eq-(2.22b) would have reversed the signs and one would have obtained corrections pointing *towards* the sun.

$$\Delta\ a\ =\ -\ \frac{2\ G_N\ M}{r^2}\ \sqrt{\lambda}\ r = \ =\ -\ \frac{2\ G_N\ M}{r^2}\ \frac{r}{R_{Hubble}}\ =\ -\ \frac{2\ G_N\ M}{r\ R_{Hubble}}. \qquad (2.28)$$

and the factional corrections relative to the standard Newtonian acceleration $-G_N M/r^2$ are

$$|\frac{\Delta a}{a}| = \frac{2\ r}{R_{Hubble}}\ \sim\ c\ \frac{Time\ of\ flight}{Hubble\ Scale} \qquad (2.29)$$

where the time of flight of the photons in their return trip from the earth to the spacecraft and back is of the order of $T = 2r/c$.

Let us set the scale r and speed v of the spacecraft to be such that

$$\frac{1}{2} m_{Pioneer}\ v^2\ = \frac{G_N\ M\ m_{Pioneer}}{r}\ \Rightarrow\ v^2\ =\ \frac{2\ G_N\ M}{r}. \qquad (2.30)$$

meaning that its speed at the location r is of the same order as its escape velocity that allows the spacecraft to leave the gravitational bounds of the solar system. Therefore, by plugging-in the value of $v^2 = (2G_N M/r)$ into eq-(2.27) we get the anomalous corrections to the Pioneer spacecraft acceleration

$$\Delta a\ \sim\ +\ \frac{2\ G_N\ M}{r\ R_{Hubble}} =\ +\ \frac{v^2}{R_{Hubble}}. \qquad (2.31a)$$

The positive sign means that it points away from the sun. The choice of the *negative* sign in front of the square root in eq-(2.22b) yields a change in sign :

$$\Delta a\ \sim\ -\ \frac{2\ G_N\ M}{r\ R_{Hubble}} =\ -\ \frac{v^2}{R_{Hubble}}. \qquad (2.31b)$$

and the corrections to the acceleration point *towards* the sun which is what is observed, there is an anomalous Doppler blueshift which was first predicted by [42] (to our knowledge). Had we used a different expression for the radial function, like that in eq-(2.2), one would have obtained a different result

$$when\ R(r)\ =\ \sqrt{(2G_N M)^2\ +\ \frac{r^2}{1 \pm \lambda\ r^2}}\ \Rightarrow\ \Delta a\ \sim\ \pm\ \frac{2G_N\ M}{R^2_{Hubble}}. \qquad (2.32)$$

and a fractional change of

$$|\frac{\Delta a}{a}| = \frac{2\ r^2}{R^2_{Hubble}}. \tag{2.33}$$

To finalize we may invoke Mach's principle [39] [41] by equating the rest mass (energy) of the Pioneer spacecraft $m_{Pioneer}$ to the gravitational potential energy it experiences due to the net mass M_U of the universe at a scale R_{Hubble}

$$m_{Pioneer} c^2 \ = \ \frac{G_N \ M_{Universe} \ m_{Pioneer}}{R_{Hubble}} \Rightarrow a_P = \frac{c^2}{R_{Hubble}} \ = \ \frac{G_N \ M_{Universe}}{R^2_{Hubble}}. \tag{3.34}$$

that has the same order of magnitude as the observed value for the anomalous Pioneer acceleration. At the end of the next section we will revisit the anomalous Pioneer acceleration within the framework of Weyl's geometry.

3 WEYL GEOMETRY SOLVES THE RIDDLE OF DARK ENERGY

The problem of dark energy is one of the most challenging problems facing us today, see [21], [23] for a review. In this section we will show how Weyl's geometry (and its scaling symmetry) is instrumental to solve this dark energy riddle. Before starting we must emphasize that our procedure is quite different than previous proposals [29] to explain dark matter (instead of dark energy) in terms of Brans-Dicke gravity. It is not only necessary to include the Jordan-Brans-Dicke scalar field ϕ but it is essential to have a Weyl geometric extension and generalization of Riemannian geometry (ordinary gravity). It will be shown why the scalar ϕ has a *nontrivial* energy density despite having *trivial* dynamics due entirely to its potential energy density $V(\phi = \phi_o)$ and which is precisely equal to the observed vacuum energy density of the order of $10^{-123} M^4_{Planck}$. For other approaches to solve the riddle of dark energy and dark matter based on modifications of gravity by starting with Lagrangians of the type $f(\mathcal{R})$ see [31] and references therein.

Weyl's geometry main feature is that the norm of vectors under parallel infinitesimal displacement going from x^μ to $x^\mu + dx^\mu$ change as follows :

$$\delta ||V|| \ \sim \ ||V|| A_\mu dx^\mu$$

where A_μ is the Weyl gauge field of scale calibrations that behaves as a connection under Weyl transformations :

$$A'_\mu = A_\mu \ - \ \partial_\mu \ \Omega(x). \quad g_{\mu\nu} \ \rightarrow \ e^{2\Omega} \ g_{\mu\nu}. \tag{3.1}$$

involving the Weyl scaling parameter $\Omega(x^\mu)$.

The Weyl covariant derivative operator acting on a tensor $\mathbf{T}$ is defined by $D_\mu \mathbf{T} = (\ \nabla_\mu \ + \ \omega(\mathbf{T} \ ) \ A_\mu)\mathbf{T}$; where $\omega(\mathbf{T})$ is the Weyl weight of the tensor $\mathbf{T}$ and the derivative operator $\nabla_\mu \ = \ \partial_\mu + \Gamma_\mu$ involves a connection Γ_μ which is comprised of the ordinary

Christoffel symbols plus extra A_μ terms in order for the metric to obey the condition $D_\mu(g_{\nu\rho}) = 0$. The Weyl weight of the metric $g_{\nu\rho}$ is 2. The meaning of $D_\mu(g_{\nu\rho}) = 0$ is that the angle formed by two vectors remains the same under parallel transport despite that their lengths may change. This also occurs in conformal mappings of the complex plane.

The Weyl covariant derivative acting on a scalar ϕ of Weyl weight $\omega(\phi) = -1$ is defined by

$$D_\mu \phi = \partial_\mu \ \phi \ + \ \omega(\phi) A_\mu \ \phi = \partial_\mu \ \phi \ - \ A_\mu \ \phi. \quad (3.2)$$

The Weyl scalar curvature in D dimensions and signature $(+, -, -, -....)$ is

$$\mathcal{R}_{Weyl} = \mathcal{R}_{Riemann} - (D-1)(D-2) A_\mu A^\mu + 2(D-1) \nabla_\mu A^\mu. \quad (3.3)$$

For a signature of $(-, +, +, +, ...)$ there is a *sign* change in the second and third terms due to a sign change of $\mathcal{R}_{Riemann}$.

The Jordan-Brans-Dicke action is

$$S = - \int d^4x \ \sqrt{|g|} \ [\ \phi^2 \ \mathcal{R}_{Weyl} \ ]. \quad (3.4)$$

Under Weyl scalings,

$$\mathcal{R}_{Weyl} \ \rightarrow \ e^{-2\Omega} \ \mathcal{R}_{Weyl}; \quad \phi^2 \ \rightarrow \ e^{-2\Omega} \ \phi^2. \quad (3.5)$$

to compensate for the Weyl scaling (in $4D$) of the measure $\sqrt{|g|} \rightarrow e^{4\Omega} \ \sqrt{|g|}$ in order to render the action (3.4) Weyl invariant.

When the Weyl integrability condition is imposed $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0 \ \Rightarrow \ A_\mu = \partial_\mu \Omega$, the Weyl gauge field A_μ does not have dynamical degrees of freedom; it is pure gauge and barring global topological obstructions, one can choose the gauge in eq-(3.4)

$$A_\mu = 0; \quad \phi_0^2 = \frac{1}{16\pi G_N} = constant. \quad (3.6)$$

such that the action (3.4) reduces to the standard Einstein-Hilbert action of Riemannian geometry

$$S = - \frac{1}{16\pi G_N} \int d^4x \ \sqrt{|g|} \ [\mathcal{R}_{Riemann}(g)]. \quad (3.7)$$

The Weyl integrability condition $F_{\mu\nu} = 0$ means physically that if we parallel transport a vector under a closed loop, as we come back to the starting point, the *norm* of the vector has not changed; i.e, the rate at which a clock ticks does not change after being transported along a closed loop back to the initial point; and if we transport a clock from A to B along different paths, the clocks will tick at the same rate upon arrival at the same point B. This will ensure, for example, that the observed spectral lines of identical atoms will not change when the atoms arrive at the laboratory after taking different paths (histories) from their coincident starting point. If $F_{\mu\nu} \neq 0$ Weyl geometry may be responsible for the alleged variations of the physical constants in recent Cosmological observations.

Our starting action is

$$S = S_{Weyl}(g_{\mu\nu}, A_\mu) \ + S(\phi).. \quad (3.8)$$

with

$$S_{Weyl}(g_{\mu\nu}, A_\mu) \ = \ - \int d^4x \ \sqrt{|g|} \ \ \phi^2 \ \ [\ \mathcal{R}_{Weyl}(g_{\mu\nu}, A_\mu) \ ]. \tag{3.9}$$

where we define $\phi^2 = (1/16\pi G)$. The Newtonian coupling G is *spacetime* dependent in general and has Weyl weight equal to 2. A different approach to the problem of dark energy where the Newtonian coupling is time dependent has been undertaken by [35].

The term $S(\phi)$ involving the Jordan-Brans-Dicke scalar ϕ is

$$S_\phi = \int d^4x \ \sqrt{|g|} \ [\ \frac{1}{2} g^{\mu\nu} \ (D_\mu\phi)(D_\nu\phi) \ - \ V(\phi) \ ]. \tag{3.10}$$

where $D_\mu\phi = \partial_\mu\phi - A_\mu\phi$.

The FRW metric is

$$ds^2 = dt^2 - a^2(t) \ (\frac{dr^2}{1 - k(r/R_0)^2} + r^2(d\Omega)^2). \tag{3.11}$$

where $k = 0$ for a 3-dim spatially flat region; $k = \pm 1$ for regions of positive and negative constant spatial curvature, respectively. The de Sitter metric belongs to a special class of FRW metrics and it admits different forms depending on the coordinates chosen. In particular when $a(t) = cosh(H_0 t) = cosh(t/R_0)$; $k = 1$, the de Sitter metric is

$$ds^2 = dt^2 \ - \ cosh^2(H_0 t) \ (\frac{dr^2}{1 - (r/R_0)^2} + r^2(d\Omega)^2) =$$

$$ds^2 = cosh^2(H_0 t) \ [\ \frac{dt^2}{cosh^2(H_0 t)} \ - \ (\frac{dr^2}{1 - (r/R_0)^2} + r^2(d\Omega)^2) \ ] \tag{3.12}$$

and which can also be recast in terms of the conformal factor $a^2(\tau)$ and the conformal time τ, respectively,

$$a^2(\tau) \ = \ cosh^2(H_0 t) = cosh^2(t/R_0); \quad (d\tau)^2 = \frac{dt^2}{cosh^2(H_0 t)}$$

as

$$ds^2 = a^2(\tau) \ d\eta^2 = a^2(\tau) \ [d\tau^2 \ - (\frac{dr^2}{1 - (r/R_0)^2} + r^2(d\Omega)^2) \ ] =$$

$$a^2(\tau) \ [\ d\tau^2 \ - \ R_0^2 (d\Omega_3)^2 \ ] \ = \ a^2(\tau) \ [\ d\tau^2 \ - \ R_0^2 \ d\chi^2 - R_0^2 \ sin^2\chi \ (d\Omega)^2 \ ]. \tag{3.13}$$

where Ω is the two-dim solid angle corresponding to the sphere S^2 and Ω_3 is the 3-dim solid angle corresponding to the 3-sphere S^3 . The third angle coordinate χ (besides θ, ϕ) of the S^3 is defined by $sin(\chi) = (r/R_0)$.

When $a(t) = cosh(H_0 t) = cosh(t/R_o)$, the $4D$ spacetime Riemannian scalar curvature $\mathcal{R}_{Riemann}$ for $k = 1$ can be shown to be *constant* despite the *temporal* dependence of $a(t)$:

$$\mathcal{R}_{Riemann} = - \ 6 \ [\ (\frac{(d^2a/dt^2)}{a}) + (\frac{(da/dt)}{a})^2 + \frac{1}{a^2 R_0^2} \ ] =$$

$$-6\ [\ H_0^2 + H_0^2\ tanh^2(H_0 t) + \frac{H_0^2}{cosh^2(H_0 t)}\ ] = -12\ H_0^2. \qquad (3.14a)$$

notice that the negative sign of $\mathcal{R}_{Riemann}$ was due to the chosen signature. The following identities of hyperbolic functions are employed :

$$1 + tanh^2(H_0 t) + \frac{1}{cosh^2(H_0 t)} = \frac{cosh^2(H_0 t) + sinh^2(H_0 t) + 1}{cosh^2(H_0 t)} = \frac{2cosh^2(H_0 t)}{cosh^2(H_0 t)} = 2. \qquad (3.14b)$$

$$cosh^2(H_0 t) - sinh^2(H_0 t) = 1. \quad tanh^2(H_0 t) + \frac{1}{cosh^2(H_0 t)} = 1. \qquad (3.14c)$$

de Sitter space is geometrically a 4-dim hyperboloid embedded in $5D$ and can be seen as being the $3+1$ dimensional world-volume spanned by the motion of a 3-brane of topology S^3 in a flat target $5D$ embedding spacetime background. We should emphasize as well that the Hubble parameter $H(t) = \frac{1}{a}(da/dt) = H_o\ tanh\ (H_0 t)$ is *time* dependent when $a(t) = cosh\ (H_0 t)$ and tends to a constant H_0 in the asymptotic late time limit $H(t \rightarrow \infty) \rightarrow H_0$. We shall study this case and the pure inflationary de Sitter phase associated with a constant Hubble parameter H_0 corresponding to $a(t) = e^{H_0 t}$.

The Friedman-Einstein-Weyl equations in the gauge $A_\mu = (0,0,0,0)$ (in units of $c = 1$) :

$$G_{\mu\nu} = 8\pi G\ T_{\mu\nu}; \quad \phi^2 = \frac{1}{16\pi\ G}. \quad T_{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}}. \qquad (3.15)$$

read

$$3(\frac{(da/dt)}{a})^2 + (\frac{3k}{a^2 R_0^2}) = 8\pi G(t)\rho. \qquad (3.16)$$

and

$$-2\ (\frac{(d^2a/dt^2)}{a}) - (\frac{(da/dt)}{a})^2 - (\frac{k}{a^2 R_0^2}) = 8\pi G(t)\ p. \qquad (3.17a)$$

From eqs-(3.16-3.17a) one can infer the important relation :

$$-\ (\frac{(d^2a/dt^2)}{a}) = \frac{4\pi G(t)}{3}\ (\rho + 3p). \qquad (3.17b)$$

Eqs-(3.16-3.17) are the ones one must use.

If one had partially fixed the gauge $A_\mu = (A_t, 0, 0, 0)$ and try to identify the Hubble variable $H(t)$ with $A_t = H(t)$ like the author [33] did these equations would have been

$$3(\frac{(da/dt)}{a})^2 + (\frac{3k}{a^2 R_0^2}) =$$

$$-\ 9(\ A_t(x)\ A^t(x) - \frac{1}{\sqrt{|g|}}\partial_t(\sqrt{|g|}A^t)\ ) + 8\pi G(t)\rho. \qquad (3.18a)$$

and

$$-2\ (\frac{(d^2a/dt^2)}{a}) - (\frac{(da/dt)}{a})^2\ -\ (\frac{k}{a^2R_0^2}) =$$

$$3\ (\ A_t(x)\ A^t(x) - \frac{1}{\sqrt{|g|}}\partial_t(\sqrt{|g|}A^t)\ )\ +\ 8\pi G(t)\ p. \tag{3.18b}$$

Notice the presence of a crucial and net factor of 9 in eq-(3.18a) due to the contribution of the variation of the $\sqrt{g}g^{tt}A_tA_t...$ terms w.r.t the g_{tt} metric component, compared to the factor of 3 in eq- (3.18b) because $A_i = 0$. One can infer from eqs-(3.18) that

$$(\frac{(da/dt)}{a})^2\ =\ H^2(t)\ =\ -\ (\frac{k}{a^2R_0^2})\ - 3(\ A_t(x)\ A^t(x)\ -\ \frac{1}{\sqrt{|g|}}\partial_t(\sqrt{|g|}A^t)\ ) +$$

$$\frac{8\pi G(t)}{3}\rho. \tag{3.19a}$$

and

$$-\ (\frac{(d^2a/dt^2)}{a})\ =\ -\ (\ H^2(t) + \frac{dH}{dt}\ )\ =\ \frac{4\pi G(t)}{3}\ (\rho + 3p). \tag{3.19b}$$

The density and pressure terms associated with the scalar field ϕ are given by eqs-(3.24) below . The scalar ϕ must be chosen to depend solely on time , $\phi(t)$, because this is the relevant case suitable for the FRW cosmologies due to the fact that the geometry is spatially homogeneous and isotropic .

The gauge choice condition imposed by [33] : $A_t = H(t)$; $A_i = 0$, $i = 1, 2, 3$ is compatible with the spatial isotropy and homogeneity of the FRW models. However, despite that a non-zero value A_t was chosen by [33] there is a *residual* symmetry that is still available to gauge A_t to zero. As mentioned earlier, Weyl's integrability condition $F_{\mu\nu} = 0$ when the Weyl gauge field A_μ is both closed $F_{\mu\nu} = 0$ and exact (a total derivative) physically means that A_μ is pure gauge, a total derivative, whence it does not have true dynamical degrees of freedom and *all* of its components can be gauged to zero $A_\mu = (0, 0, 0, 0)$ barring global topological obstructions.

However, if one partially fixes the gauge $A_t = H(t); A_i = 0$ like it was done in [33], one arrives at a *caveat* that was overlooked by [33] . One would arrive at a deep *contradiction* and *inconsistency* between the left hand side (l.h.s) and the right hand side (r.h.s) of the Friedman-Einstein-Weyl equations (for example in eq-(3.19b)) in the partially fixed gauge $A_t = H(t)$ because the l.h.s does *not* transform homogeneously under Weyl scalings, whereas the r.h.s does; if the quantities ρ and p were to transform properly under Weyl scalings, homogeneously, this behaviour would be *incompatible* with the transformation properties of the $A_t = H(t)$ terms appearing in the l.h.s of eqs-(3.19b).

In order to *reconcile* this *incompatibility* between the *inhomogeneous* transformation properties of the l.h.s of eq-(3.19b) with the homogeneous transformation properties of the r.h.s of (3.19b), *one must fix* the gauge $A_\mu = 0$ fully in the Einstein-Friedman-Weyl equations as shown in eqs-(3.16-3.17). The latter equations *are* the physically relevant and *not* eqs-(3.18-3.19). One may be inclined to say : if one is going to fix the gauge $A_\mu = 0$ anyway, then *what* is the role of Weyl's geometry and symmetry in all of this ? We will show below why despite fixing the gauge $A_\mu = 0$ one cannot forget the *constraint* which arises from the variations of the action w.r.t the Weyl's field A_μ ! This constraint holds the key to see why the density and pressure associated with the scalar ϕ obey the sought after relation $\rho(\phi) = -p(\phi)$ (which is the hallmark of dark energy) as we intend to prove next.

The Jordan-Brans-Dicke scalar ϕ must obey the generalized Klein-Gordon equations of motion

$$(D_\mu D^\mu \ + \ 2\mathcal{R}_{Weyl} \ ) \ \phi \ + \ (\frac{dV}{d\phi}) = 0 \tag{3.20}$$

notice that because the Weyl covariant derivatives obey the condition $D_\mu(g_{\nu\rho}) = 0 \ \Rightarrow D_\mu(\sqrt{|g|}) = 0$ there are no terms of the form $(D_\mu\sqrt{|g|})(D^\mu\phi)$ in the generalized Klein-Gordon equation like it would occur in ordinary Riemannian geometry $(\partial_\mu\sqrt{|g|})(\partial^\mu\phi) \neq 0$. In addition, we have the crucial *constraint* equation obtained from the variation of the action w.r.t to the A^μ field :

$$\frac{\delta S}{\delta A^\mu} \ = \ 0 \ \Rightarrow \ 6 \ (A_\mu\phi^2 \ + \ \partial_\mu(\phi^2)) \ + \ \frac{1}{2} \ (A_\mu\phi^2 \ - \ \partial_\mu(\phi)^2 \ ) \ = \ 0. \tag{3.21}$$

The last constraint equation in the gauge $A_\mu = 0$, then forces $\partial_\mu\phi = 0, \ \ \Rightarrow \phi = \phi_o =$ *constant*. Consequently $G \sim \phi^{-2}$ is also constrained to a constant G_N and one may set $16\pi \ G_N \ \phi_o^2 = 1$, where G_N is the observed Newtonian constant today.

Furthermore, in the gauge $A_\mu = 0$, due to the constraint eq-(3.21), one can infer that $D_\mu\phi = 0, \ \ \Rightarrow D^\mu D_\mu\phi = 0$ because $D_t \ \phi(t) = \partial_t \ \phi - A_t \ \phi = \partial_t \ \phi = 0$, and $D_i\phi(t) = -A_i\phi(t) = 0$. These results will be used in the generalized Klein-Gordon equation.

To sum up, the solution to the constraint equation (3.21) in the gauge $A_\mu = (0,0,0,0)$ leads to the result $\partial_\mu\phi = 0$, which in turn, is equivalent to

$$D_t \ \phi(t) = (\partial_t - A_t)\phi = \partial_t \ \phi(t) = 0; \quad D_i \ \phi(t) = -A_i\phi(t) = 0; \tag{3.22}$$

$$\phi = \phi_o = constant \ = \ \sqrt{\frac{1}{16\pi \ G_N}}. \tag{3.23}$$

Therefore, the stress energy tensor $T^\mu_\mu = diag \ (\rho, -p, -p, -p)$ corresponding to the constant scalar field configuration $\phi(t) = \phi_o$, in the $A_\mu = 0$ gauge, becomes :

$$\rho_\phi = \frac{1}{2}(\partial_t \ \phi - A_t \ \phi)^2 + V(\phi) \ = \ V(\phi); \quad p_\phi = \frac{1}{2}(\partial_t \ \phi - A_t \ \phi)^2 - V(\phi) \ = \ -V(\phi). \tag{3.24}$$

$$\rho + \ 3p = 2 \ (\partial_t \ \phi - A_t \ \phi)^2 \ - \ 2V(\phi) \ = -2V(\phi). \tag{3.25}$$

Thus, after imposing the result $\partial_\mu\phi = \partial_t \ \phi(t) = 0$, derived from eq-(3.21), in the gauge $A_\mu = 0$, the above ρ and p terms (3.24) become $\rho(\phi) = V(\phi) = -p(\phi)$ such that

$\rho + 3p = -2V(\phi)$ (that will be used in the Einstein-Friedman-Weyl equations (3.17b)). This is the *key* reason why Weyl's geometry and symmetry is essential to explain the origins of a $non - vanishing$ vacuum energy (dark energy). The latter relation $\rho(\phi) = V(\phi) = -p(\phi)$ is the *key* to derive the vacuum energy density in terms of $V(\phi = \phi_o)$!., because such relation resembles the dark energy relation $p_{DE} = -\rho_{DE}$. Had one not had the constraint condition $D_t\ \phi(t) = (\partial_t - A_t)\phi = \partial_t\ \phi = 0$, and $D_i\phi(t) = -A_i\phi(t) = 0$, in the gauge $A_\mu = 0$, enforcing $\phi = \phi_o$, one would not have been able to deduce the crucial condition $\rho(\phi = \phi_o)\ =\ -p(\phi = \phi_o)\ =\ V(\phi = \phi_o)$ that will furnish the observed vacuum energy density today.

We will find now solutions of the Einstein-Friedman-Weyl equations in the gauge $A_\mu = (0,0,0,0)$ after having explained why A_μ can (and must) be gauged to zero. There are three relevant cases that pop up immediately :

• Case **1** is the trivial case corresponding to a static flat Minkowski spacetime :

$$a(t) = 1;\quad A_\mu = (0,0,0,0);\quad k = 0;\ \mathcal{R}_{Weyl} = R_{Riemann} = 0;\quad V(\phi) \equiv 0. \tag{3.26}$$

that solves *trivially* the Einstein-Friedmann-Weyl equations. This case corresponds to a *vanishing* vacuum energy since $V(\phi) \equiv 0$ (flat potential).

• Case **2** : is the one corresponding to the late time pure inflationary de Sitter space (where only $one - half$ of de Sitter space is covered) :

$$a(t) = e^{H_o t};\quad A_\mu = (0,0,0,0);\quad k = 0;\ \mathcal{R}_{Weyl} = \mathcal{R}_{Riemann} = -12\ H_0^2;$$

where we will show that the potential is

$$V(\phi) = 12 H_0^2 \phi^2\ + V_o. \tag{3.27}$$

one learns in this case that $V(\phi = \phi_o) \neq 0$ since this non-vanishing value is precisely the one that shall furnish the observed vacuum energy density today (as we will see below) .

• Case **3** : belonging to a different parametrization of de Sitter space (all of the de Sitter space is covered in this case) when $k = 1$ (spatially closed universe of constant positive spatial curvature) and a *time* dependent Hubble parameter $H(t) = \frac{1}{a}(da/dt) = H_0\ tanh\ (H_0 t)$ such that $H(t \to \infty) \to H_0$:

$$a(t) = cosh[H_o t];\quad A_\mu = (0,0,0,0);\quad k = 1;\ \mathcal{R}_{Weyl} = \mathcal{R}_{Riemann} = -12\ H_0^2;$$

with the same potential $V(\phi) = 12H_0^2\phi^2\ + V_o$.

We shall begin by solving the Einstein-Friedman-Weyl equations eq-(3.16-3.17) in the gauge $A_\mu = (0,0,0,0)$ for a spatially flat universe $k = 0$ and $a(t) = e^{H_0 t}$, corresponding to the inflationary de Sitter metric :

$$ds^2 = dt^2\ -\ e^{2H_o t}\ \ (dr^2 + r^2 (d\Omega)^2). \tag{3.28}$$

the Riemannian scalar curvature when $k = 0$ is

$$\mathcal{R}_{Riemann} = -\ 6\ [\ (\frac{(d^2a/dt^2)}{a}) + (\frac{(da/dt)}{a})^2\ ]\ = -12\ H_0^2 \tag{3.29}$$

(the negative sign is due to the chosen signature $+,-,-,-$).

To scalar Weyl curvature $\mathcal{R}_{Weyl}$ in the gauge $A_\mu = (0,0,0,0)$ is the same as the Riemannian one $\mathcal{R}_{Weyl} = \mathcal{R}_{Riemann} = -12\ H_0^2$. Inserting the condition $D_\mu \phi = D_t \phi(t) = (\partial_t \phi - A_t \phi) = \partial_t\ \phi = 0$, in the gauge $A_\mu = 0$, the generalized Klein-Gordon equation (3.20) will be satisfied if, and only if, the potential density $V(\phi)$ is chosen to satisfy

$$(12\ H_0^2)\ \phi\ =\ \frac{1}{2}\ (\frac{dV}{d\phi})\ \Rightarrow\ V(\phi)\ =\ 12\ H_0^2\ \phi^2\ +\ V_o \tag{3.30}$$

As stated earlier, one must firstly differentiate w.r.t the scalar ϕ , and only afterwards, one may set $\phi = \phi_o$. $V(\phi)$ has a Weyl weight equal to -4 under Weyl scalings in order to ensure that the full action is Weyl invariant. H_0^2 and ϕ_o^2 have both a Weyl weight of -2, despite being constants, because as one performs a Weyl scaling of these quantities (a change of a scales) they will acquire then a spacetime dependence. H_0^2 is a masslike parameter, one may interpret H_0^2 (up to numerical factors) as the "mass" squared of the Jordan-Brans-Dicke scalar. We will see soon why the integration constant V_o plays the role of the "cosmological constant".

An important remark is in order. Even if we included other forms of matter in the Einstein-Fredmann-Weyl equations, in the very large t regime, their contributions will be *washed* away due to their scaling behaviour. We know that ordinary matter ($p = 0$); dark matter ($p_{DM} = w\rho_{DM}$ with $-1 < w < 0$) and radiation terms ($p_{rad} = \frac{1}{3}\rho_{rad}$) are all *washed* away due to their scaling behaviour :

$$\rho_{matter} \sim R(t)^{-3}. \quad \rho_{radiation} \sim R(t)^{-4}. \quad \rho_{DM} \sim R(t)^{-3(1+w)}. \tag{3.31}$$

where $R(t) = a(t)R_0$. The dark energy density remains *constant* with scale since $w = -1$ and the scaling exponent is zero, $\rho_{DE} \sim R^0 = costant$. For this reason it is the only contributing factor at very large times.

Now we are ready to show that eqs-(3.16-3.17) are indeed satisfied when $a(t) = e^{H_0 t}$; $k = 0$; $A_\mu = 0$; $\phi = \phi_o \neq 0$. Eq-(3.17b), due to the conditions $\rho + 3p = -2V(\phi)$ and $\phi(t) = \phi_o$ (resulting from the constraint eq-(3.21) in the $A_\mu = 0$ gauge) gives :

$$-\ (\frac{(d^2a/dt^2)}{a})\ =\ -\ H_0^2 = \ \frac{4\pi G_N}{3}\ (\rho + 3p) =$$

$$-\ \ (\frac{8\pi\ G_N\ V(\phi = \phi_o)}{3}) = -\ (\frac{8\pi\ G_N\ 12\ H_0^2\ \phi_o^2}{3})\ -\ \frac{8\pi G_N\ V_o}{3}. \tag{3.32}$$

Eq-(3.16) (with $k = 0$) is just the same as eq-(3.17b) but with an overall *change* of sign because $\rho(\phi = \phi_o) = V(\phi = \phi_o)$. Using the definition $16\pi\ G_N\ \phi_o^2 = 1$ in (3.32) one gets

$$-\ H_0^2\ = -\ (\frac{8\pi\ G_N\ 12\ H_0^2\ \phi_o^2}{3})\ -\ \frac{8\pi\ G_N\ V_o}{3}\ = -2\ H_0^2\ -\ \frac{8\pi\ G_N\ V_o}{3}\ \Rightarrow$$

$$-\frac{8\pi\ G_N\ V_o}{3}\ =\ H_0^2\ \Rightarrow\ -\ 8\pi\ G_N\ V_o = 3\ H_0^2 \tag{3.33}$$

Therefore, we may identify the term $-\ V_o$ with the vacuum energy density so the quantity $3H_0^2 = -8\pi\ G_N\ V_o = \Lambda$ is nothing but the cosmological constant. It is *not surprising* at all to obtain $\Lambda = 3\ H_0^2$ in de Sitter space . One knew it long ago. What is most *relevant* about eq-(3.33) is that the observed vacuum energy density is minus the *constant* of integration V_o corresponding to the potential density $V(\phi) = 12H^2\phi^2 + V_o$!. Hence one has from the last term of eq-(3.33) :

$$-V_o = \rho_{vacuum} = \frac{3H_0^2}{8\pi\ G_N}. \quad (3.34)$$

and finally, when we set $H_0^2 = (1/R_0^2) = (1/R_{Hubble}^2)$ and $G_N = L_{Planck}^2$ in the last term of eq-(3.34), as announced, the vacuum density ρ_{vacuum} observed today is *precisely* given by :

$$-V_o = \rho_{vacuum} \ = \ \frac{3H_0^2}{8\pi\ G_N} \ = \ \frac{3}{8\pi}\ (L_{Planck})^{-2}\ (R_{Hubble})^{-2} \ =$$

$$\frac{3}{8\pi}\ (\frac{1}{L_{Planck}})^4\ (\frac{L_{Planck}}{R_{Hubble}})^2\ \sim 10^{-123}\ (M_{Planck})^4. \quad (3.35)$$

This completes our third derivation of the vacuum energy density given by the formula (3.34-3.35). The first derivation was attained in [15]. The second derivation in section **2** and the third derivation in this last section.

Concluding this analysis of the Einstein-Friedman-Weyl eqs-(3.16-3.17) : By invoking the principle of Weyl scaling symmetry in the context of Weyl's geometry; when $k = 0$ (spatially flat Universe), $a(t) = e^{H_0 t}$ (de Sitter inflationary phase) ; H_o = Hubble constant today; $\phi(t) = \phi_o = constant$, such $16\pi G_N\ \phi_o^2 = 1$, one finds that

$$V(\phi = \phi_o) = 12\ H_0^2\ \phi_o^2\ +\ V_o =\ 2\rho_{vacuum}\ -\ \rho_{vacuum}\ =\ \rho_{vacuum} =$$

$$6H_0^2\phi_o^2 = \frac{3H_0^2}{8\pi\ G_N} \sim 10^{-123}\ M_{Planck}^4. \quad (3.36)$$

is precisely the observed vacuum energy density (3.34) . Therefore, the observed vacuum energy density is intrinsically and inexorably linked to the potential density $V(\phi = \phi_o)$ corresponding to the Jordan-Brans-Dicke scalar ϕ required to build Weyl invariant actions and evaluated at the special point $\phi_o^2 = (1/16\pi G_N)$.

There is the trivial solution to the Einstein-Friedman-Weyl equations given by

$$k = 0\ (flat);\ \mathcal{R}_{Weyl} = \mathcal{R}_{Riemann} = 0;\ a(t) = 1;\ \ A_\mu = 0;\ V(\phi) \equiv 0 \quad (3.37)$$

and the relationship between Λ and $V(\phi)$ when $a(t) = 1$ is inferred from

$$0\ =\ -\ (\frac{8\pi G_N\ V(\phi)}{3}),\ since\ V(\phi) \equiv 0. \quad (3.38)$$

As expected, in this trivial flat universe with a zero flat potential one must have $\Lambda = 0$.

It is straightforward to verify due to the identity $tanh^2(H_0 t) + 1/cosh^2(H_0 t) = 1$ that the case of a *time* dependent Hubble parameter $H(t) = H_0\ tanh\ (H_0 t)$ when $k = 1$; $a(t) = cosh[H_0 t]$ and $\mathcal{R}_{Weyl} = -12H_0^2$ as derived in eqs-(3.14) also solves the Einstein-Friedman-Weyl equations in the gauge $A_\mu = 0$ for the *same* potential

$V(\phi) = 12H_0^2\phi^2 + V_o$ when $\phi(t) = \phi_o$. For this reason, these nontrivial solutions for an *ever expanding accelerating universe* (consistent with observations) is so promising because it incorporates the presence of the Hubble and Planck scales into the expression for the observed vacuum energy density via the Jordan-Brans-Dicke scalar field ϕ needed to implement Weyl invariance of the action. It is warranted to study the connection between this Weyl geometric approach versus the Scale Relativity Theory [26] and the Extended Relativity Theory in Clifford spaces [28] where there is *scale motion* without the need to introduce a Weyl gauge field.

Concluding, Weyl's scaling symmetry principle permits us to explain *why* the observed value of the vacuum energy density ρ_{vacuum} is *precisely* given by the expression (3.35). This completes our third derivation of the vacuum energy density. The first derivation was attained in [15] while the second derivation was attained in section **2**.

In order to introduce true dynamics to the Weyl gauge field, one must add the kinetic term for the Weyl gauge field $F_{\mu\nu}F^{\mu\nu}$. In this case, the integrability condition $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0$ is *no* longer obeyed in general and the rate at which clocks tick may depend on their worldline history. This could induce a variation of the physical constants (even dimensionless constants like the fine structure constant $\alpha = 1/137$). For instance, as the size of the universe grows, ($a(t) = e^{H_0 t}$ increases with time) the variable speed of light, Newtonian coupling and cosmological constant , may vary according to the law $[G(t)/c^4(t)\ \Lambda(t)] \ \sim\ (1/\rho_{vacuum})$ if the vacuum energy density ρ_{vacuum} would remain constant. Many authors have speculated about this last behaviour among c, G, Λ.

The most general Lagrangian involving dynamics for A_μ is

$$\mathcal{L} = -\phi^2 \mathcal{R}_{Weyl}(g_{\mu\nu}, A_\mu) + \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} g^{\mu\nu}(D_\mu\phi)(D_\nu\phi) - V(\phi) + L_{matter} + ... \quad (3.39)$$

The L_{matter} must involve the full fledged Weyl gauge covariant derivatives acting on scalar and spinor fields contrary to the Cheng-Weyl models of [36] where there is no Weyl gauge field in the derivatives. $L_{radiation}$ terms may be included involving the Maxwell field $\mathcal{A}_\mu$ which must not be confused with the Weyl gauge field A_μ. Once could also add Yang-Mills fields $\mathcal{A}_\mu^a$ and kinetic and potential terms for the Higgs scalars as well.

The simplest scenario, of course, was the one given at the beginning of this section when the Weyl field A_μ was both closed $F_{\mu\nu} = 0$ and exact (a total derivative) that allowed us to gauge it to zero barring global topological obstructions. The latter topological obstructions deserved to be investigated further. Thus the simplest modifications to the equations studied in this section are obtained when A_μ cannot be gauged to zero *globally* due to topological obstructions. Therefore, by setting $A_\mu = (\ A_t(t) \neq 0,\ \ 0, 0, 0)$ and reminding the reader that one cannot equate the Hubble parameter $H(t) = a^{-1}(da/dt)$ to the temporal component $A_t(t)$ yields the modified Friedmann-Einstein-Weyl equations

$$3(\frac{(da/dt)}{a})^2 \ + \ (\frac{3k}{a^2 R_0^2}) \ = \ - 9(\ A_t(x)\ A^t(x) \ - \ \frac{1}{\sqrt{|g|}} \partial_t(\sqrt{|g|} A^t)\ ) \ + \ 8\pi G(t)\rho \ =$$

$$- 9(\ A_t(x)\ A^t(x) \ - \ \frac{1}{\sqrt{|g|}} \partial_t(\sqrt{|g|} A^t)\ ) \ + \ \frac{1}{2\phi^2}\ [\ \frac{1}{2}(\partial_t\ \phi - A_t\ \phi)^2 \ + \ V(\phi)\ ]. \quad (3.40a)$$

$$-2\,(\frac{(d^2a/dt^2)}{a})-(\frac{(da/dt)}{a})^2-(\frac{k}{a^2R_0^2}) = 3\;(\;A_t(x)\;A^t(x) - \frac{1}{\sqrt{|g|}}\partial_t(\sqrt{|g|}A^t)\;) + 8\pi G(t)\;p\;=$$

$$3\;(\;A_t(x)\;A^t(x) - \frac{1}{\sqrt{|g|}}\partial_t(\sqrt{|g|}A^t)\;)\;+\;\frac{1}{2\phi^2}\;[\;\frac{1}{2}(\partial_t\;\phi - A_t\;\phi)^2\;-\;V(\phi)\;]. \qquad (3.40b)$$

$$\frac{\delta S}{\delta A^\mu}\;=\;0\;\Rightarrow\;6\;(A_\mu\;\phi^2\;+\;\partial_\mu(\phi^2))\;+\;\frac{1}{2}\;(A_\mu\;\phi^2\;-\;\partial_\mu(\phi)^2\;)\;=$$

$$6\;(A_t\;\phi^2\;+\;\partial_t(\phi^2))\;+\;\frac{1}{2}\;(A_t\;\phi^2\;-\;\partial_t(\phi)^2\;)\;=\;0;\quad A_t(t)\neq 0. \qquad (3.40c)$$

$$(\;D_\mu\;D^\mu\;+\;2\mathcal{R}_{Weyl}\;)\;\phi\;+\;(\frac{dV}{d\phi})\;=\;(\;D_t\;D^t\;+\;2\mathcal{R}_{Weyl}\;)\;\phi\;+\;(\frac{dV}{d\phi})\;=\;0 \qquad (3.40d)$$

leading to a system of $four$ differential equations to solve for the $four$ unknown functions $a(t), \phi(t), A_t(t)$ and the potential $V(\phi)$.

One may add ordinary matter terms (pressureless matter) and radiation terms to the right hand side of eqs-(3.40a, 3.40b) of the form $\rho_m = \rho_o^m\;(R_0/R)^{-3} = \rho_o^m\;a(t)^{-3}$ and $\rho_{rad} = \rho_o^{rad}\;(R_0/R)^{-4} = \rho_o^{rad}\;a(t)^{-4}$, $p_{rad} = \frac{1}{3}\rho_{rad}$, respectively. Upon doing so one needs to know what are the values of the constants $\rho_o^m;\;\rho_o^{rad}$ based on the present estimates of the cosmological density parameters in order to solve for the system of $four$ differential equations (3.40). What is to be expected is that in the late times regime one should reproduce the prior solutions when $k = 0$ given by $A_t(t\to\infty)\to 0;\;\phi(t\to\infty)\to\phi_o;\;a(t\to\infty)\to e^{H_o t}; V(\phi)\to 12H_0^2\phi^2 + V_o$. And when $k = 1;\;a(t)\to cosh(H_0 t)$. It is a nontrivial task to solve the system of equations (3.40) with and without adding matter and radiation terms. The most general cosmological scenario is when the homogeneity and isotropy of spacetime is broken. In this case one will have a full fledge spacetime dependence on all the physical quantities $g_{\mu\nu}(x^\rho), A_\mu(x^\rho), \phi(x^\rho)$ (a spacetime dependent Newtonian coupling). This most general case scenario warrants further investigation to explore the early stages of the Universe.

There are many differences among our approach and that of [36]. The Cheng-Weyl approach [36] to account for dark energy and matter (including phantom) does *not* use the Weyl scalar curvature with a variable Newtonian coupling $16\pi\;G = \phi^{-2}$ for the gravitational part of the action, but the ordinary Riemannian scalar curvature with the standard Newtonian gravitational constant . One does *not* use Weyl covariant derivatives in the matter terms. The Weyl covariant derivative is *only* used in the kinetic $(D_\mu\phi)^2$ terms for the Jordan-Brans-Dicke scalar ϕ . And the authors [36] introduced a triplet of Cheng-Weyl gauge fields $A_\mu^1, A_\mu^2, A_\mu^3$ whereas here we have only one A_μ. The role of conformal transformations in accelerated cosmologies has bee studied by [30]. Weyl invariance has been used in [32] to construct Weyl-Conformally Invariant Light-Like p-Brane Theories with numerous applications in Astrophysics, Cosmology, Particle Physics Model Building, String theory,..... Concerning Weyl geometry and matter creation in the universe see the work of [34]. A thorough study of the unification of geometric and random structures in Physics within the framework of Riemann-Cartan-Weyl spacetimes has been performed by [37]. The role of scaling in cosmology and the arrow of time to explain the large number coincidences was discussed by [38].

Finally, a study of the Pioneer anomaly was made by [33] based on Weyl's geometry (due to the Weyl gauge field modifications to the connection) corrections for the low velocity orbits where he showed that the low velocity effect of the expanding space cosmology on the spacecraft is :

$$\frac{d^2x^\mu}{dt^2} \sim \; - g^{\mu\nu}\frac{\partial \; V_{Newton}}{\partial x^\nu} \; - 2H_0 \; \frac{dx^\mu}{dt}. \tag{4.41}$$

thus it consists of an acceleration proportional (but opposite in sign) to the velocity with a factor depending on the Hubble parameter H_0. Despite the problem described earlier with the identification of the A_t component of the Weyl gauge potential $A_\mu = (A_t, 0, 0, 0)$ with the Hubble parameter $H(t)$ this result deserves further investigation. The literature on many different approaches to the Pioneer anomaly is extensive, see for example [39], [40], [41], [35], [42].

To end this work, we just point out the known fact that the electron neutrino mass $m_\nu \sim 10^{-3} \; eV$ is of the same order as $(m_\nu)^4 \sim 10^{-123} \; M^4_{Planck}$ and that the SUSY breaking scale in many models is given by a geometric mean relation : $m^2_{SUSY} = m_\nu \; M_{Planck} \sim (5 \; TeV)^2$. We hope that the contents of this work will help us elucidate further the connection between the microscopic and macroscopic world.

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APPENDIX A : SCHWARZSCHILD-LIKE SOLUTIONS IN $D \geq 4$

Let us start with the line element [17]

$$ds^2 = -e^{\mu(r)}(dt)^2 + e^{\nu(r)}(dr)^2 + R^2(r)\tilde{g}_{ij}d\xi^i d\xi^j. \tag{A.1}$$

Here, the metric $\tilde{g}_{ij}$ corresponds to a homogeneous space and $i, j = 3, 4, ..., D-2$. The only nonvanishing Christoffel symbols are

$$\begin{aligned} &\Gamma^1_{21} = \tfrac{1}{2}\mu', \qquad &&\Gamma^2_{22} = \tfrac{1}{2}\nu', \qquad &&\Gamma^2_{11} = \tfrac{1}{2}\mu'' e^{\mu-\nu}, \\ &\Gamma^2_{ij} = -e^{-\nu}RR'\tilde{g}_{ij}, \qquad &&\Gamma^i_{2j} = \tfrac{R'}{R}\delta^i_j, \qquad &&\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk}, \end{aligned} \tag{A.2}$$

and the only nonvanishing Riemann tensor are

$$\begin{aligned} &\mathcal{R}^1_{212} = -\tfrac{1}{2}\mu'' - \tfrac{1}{4}\mu'^2 + \tfrac{1}{4}\nu'\mu', \qquad &&\mathcal{R}^1_{i1j} = -\tfrac{1}{2}\mu' e^{-\nu}RR'\tilde{g}_{ij}, \\ &\mathcal{R}^2_{121} = e^{\mu-\nu}(\tfrac{1}{2}\mu'' + \tfrac{1}{4}\mu'^2 - \tfrac{1}{4}\nu'\mu'), \qquad &&\mathcal{R}^2_{i2j} = e^{-\nu}(\tfrac{1}{2}\nu'RR' - RR'')\tilde{g}_{ij}, \\ &\mathcal{R}^i_{jkl} = \tilde{R}^i_{jkl} - R'^2 e^{-\nu}(\delta^i_k\tilde{g}_{jl} - \delta^i_l\tilde{g}_{jk}). \end{aligned} \tag{A.3}$$

The field equations are

$$\mathcal{R}_{11} = e^{\mu-\nu}(\frac{1}{2}\mu'' + \frac{1}{4}\mu'^2 - \frac{1}{4}\mu'\nu' + \frac{(D-2)}{2}\mu'\frac{R'}{R}) = 0, \tag{A.4}$$

$$\mathcal{R}_{22} = -\frac{1}{2}\mu'' - \frac{1}{4}\mu'^2 + \frac{1}{4}\mu'\nu' + (D-2)(\frac{1}{2}\nu'\frac{R'}{R} - \frac{R''}{R}) = 0, \tag{A.5}$$

and

$$\mathcal{R}_{ij} = e^{-\nu}(\frac{1}{2}(\nu' - \mu')RR' - RR'' - (D-3)R'^2)\tilde{g}_{ij} + k(D-3)\tilde{g}_{ij} = 0, \tag{A.6}$$

where $k = \pm 1$, depending if $\tilde{g}_{ij}$ refers to positive or negative curvature. From the combination $e^{-\mu+\nu}R_{11} + R_{22} = 0$ we get

$$\mu' + \nu' = \frac{2R''}{R'}. \tag{A.7}$$

The solution of this equation is

$$\mu + \nu = \ln R'^2 + a, \tag{A.8}$$

where a is a constant.

Substituting (A.7) into the equation (A.6) we find

$$e^{-\nu}(\nu'RR' - 2RR'' - (D-3)R'^2 = -k(D-3) \tag{A.9}$$

or

$$\gamma' R R' + 2\gamma R R'' + (D-3)\gamma R'^2 = k(D-3), \qquad (A.10)$$

where

$$\gamma = e^{-\nu}. \qquad (A.11)$$

The solution of (A.10) for an ordinary D-dim spacetime (one temporal dimension) corresponding to a $D-2$-dim sphere for the homogeneous space can be written as

$$\gamma = (1 - \frac{8\pi G_D M}{(D-3)\Omega_{D-2} R^{D-3}}) \ (\frac{dR}{dr})^{-2} \Rightarrow$$

$$g_{rr} = e^{\nu} = (1 - \frac{8\pi G_D M}{(D-3)\Omega_{D-2} R^{D-3}})^{-1} \ (\frac{dR}{dr})^2. \qquad (A.12)$$

where Ω_{D-2} is the appropriate solid angle in $D-2$-dim and G_D is the D-dim gravitational constant whose units are $(length)^{D-2}$. Thus $G_D M$ has units of $(length)^{D-3}$ as it should. When $D=4$ as a result that the 2-dim solid angle is $\Omega_2 = 4\pi$ one recovers from eq-(A.12) the 4-dim Schwarzchild solution. The solution in eq-(A.12) is consistent with Gauss law and Poisson's equation in $D-1$ spatial dimensions obtained in the Newtonian limit.

For the most general case of the $D-2$-dim homogeneous space we should write

$$-\nu = \ln(k - \frac{\beta_D G_D M}{(D-3)R^{D-3}}) - 2\ln R'. \qquad (A.13)$$

where β_D is a constant. Thus, according to (A.8) we get

$$\mu = \ln(k - \frac{\beta_D G_D M}{(D-3)R^{D-3}}) \ + \ constant. \qquad (A.14)$$

we can set the constant to zero, and this means the line element (A.1) can be written as

$$ds^2 = -(k - \frac{\beta_D G_D M}{(D-3)R^{D-3}})(dt)^2 \ + \ \frac{(dR/dr)^2}{(k - \frac{\beta_D G_D M}{(D-3)R^{D-3}})}(dr)^2 \ + \ R^2(r)\tilde{g}_{ij} d\xi^i d\xi^j. \qquad (A.15)$$

One can verify, taking for instance (A.5), that the equations (A.4)-(A.6) do *not* determine the form $R(r)$. It can be arbitrary. It is also interesting to observe that the only effect of the homogeneous metric $\tilde{g}_{ij}$ is reflected in the $k = \pm 1$ parameter, associated with a positive (negative) constant scalar curvature of the homogeneous $D-2$-dim space.

APPENDIX B : GENERALIZED (ANTI) de SITTER-SCHWARZSCHILD SOLUTIONS

We wish to solve now the 4-dim Einstein's equations with a cosmological constant

$$\mathcal{R}_{\mu\nu} \ = \ \Lambda \ g_{\mu\nu} \ = \ 3\lambda \ g_{\mu\nu}. \qquad (B.1)$$

we will write the same ansatz for the metric

$$ds^2 = -e^{\mu(r)}(dt)^2 + e^{\nu(r)}(dr)^2 + R(r)^2 \ (d\Omega)^2. \qquad (B.2)$$

and write now

$$\mathcal{R}_{11} = e^{\mu-\nu}(\frac{1}{2}\mu'' + \frac{1}{4}\mu'^2 - \frac{1}{4}\mu'\nu' + \frac{(D-2)}{2}\mu'\frac{R'}{R}) = 3\lambda \ g_{11}. \qquad (B.3)$$

$$\mathcal{R}_{22} = -\frac{1}{2}\mu'' - \frac{1}{4}\mu'^2 + \frac{1}{4}\mu'\nu' + (D-2)(\frac{1}{2}\nu'\frac{R'}{R} - \frac{R''}{R}) = 3\lambda \ g_{22} \qquad (B.4)$$

and

$$\mathcal{R}_{ij} = e^{-\nu}(\frac{1}{2}(\nu' - \mu')RR' - RR'' - (D-3)R'^2) \ g_{ij} + (D-3) \ g_{ij} = 3\lambda \ g_{ij} \qquad (B.5)$$

One can verify that in the $D = 4$ case when the mass parameter $M = 0$ the solutions of the above equations corresponding to the metric eq-(B.2) are :

$$g_{tt} = -e^{\mu(r)} \ = \ - \ (1 - \lambda \ R^2(r)) \ ; \quad g_{rr} = e^{\nu(r)} \ = \ \frac{(dR/dr)^2}{1 - \lambda \ R^2(r)} \qquad (B.6)$$

where $R(r)$ is an arbitrary radial function.

By inserting the following expressions

$$\frac{d\mu}{dr} = -\frac{2\lambda R(dR/dr)}{1-\lambda R^2}; \quad \frac{d\nu}{dr} \ = \ \frac{2(d^2R/dr^2)}{(dR/dr)} \ + \ \frac{2\lambda R(dR/dr)}{1-\lambda R^2}. \qquad (B.7)$$

$$\frac{d^2\mu}{dr^2} \ = \ - \ \frac{4\lambda^2 R^2(dR/dr)^2 \ + \ (1-\lambda R^2) \ [\ 2\lambda(dR/dr)^2 + 2\lambda R(d^2R/dr^2) \ ]}{(1-\lambda R^2)^2}. \qquad (B.8)$$

$$\frac{d\mu}{dr}\ \frac{d\nu}{dr} \ = \ - \ \frac{2\lambda R(dR/dr)}{1-\lambda R^2} \ [\ 2\frac{(d^2R/dr^2)}{(dR/dr)} + \frac{2\lambda R(dR/dr)}{1-\lambda R^2} \ ]. \qquad (B.9)$$

$$(\frac{d\mu}{dr})^2 \ = \ \frac{4\lambda^2 R^2(dR/dr)^2}{(1-\lambda R^2)^2}. \qquad (B.10)$$

$$e^{\mu-\nu} \ = \ \frac{(1-\lambda R^2)^2}{(dR/dr)^2}. \qquad (B.11)$$

we can verify that eqs-(B.3, B.4, B.5) are indeed satisfied for any *arbitrary* radial function $R(r)$. For example, after plugging in the values of eqs-(B.7-B.11) into eq-(B.3) one gets when $D = 4$:

$$- [\frac{1 - \lambda\ R^2}{(dR/dr)^2}]\ [\ \lambda\ (dR/dr)^2 + \lambda\ R\ (d^2R/dr^2)\ ]\ -\ \frac{2\lambda^2\ R^2\ (dR/dr)^2}{(dR/dr)^2} +$$

$$[\frac{1 - \lambda R^2}{(dR/dr)^2}]\ [\lambda R\ (d^2R/dr^2)]\ +\ \frac{2\lambda^2\ R^2\ (dR/dr)^2}{(dR/dr)^2}\ -\ \frac{2\lambda\ (dR/dr)^2\ (1 - \lambda R^2)}{(dR/dr)^2} =$$

$$-3\lambda\ (1 - \lambda R^2) = \Lambda\ g_{tt}. \tag{B.12}$$

Similary, eqs-(B.4,B.5) are also satisfied

When the mass parameter $M \neq 0$ we can also verify that the solutions

$$e^{\mu(r)}\ =\ 1 - \frac{2M}{R} - \lambda\ R^2(r); \quad e^{\nu(r)}\ =\ \frac{(dR/dr)^2}{1 - (2M/R) - \lambda R^2(r)} \tag{B.13}$$

obey eqs-(B.3, B.4, B.5) as well for an arbitrary radial function $R(r)$. Furthermore, one can extend these generalized de Sitter-Schwarzcshild solutions to higher dimensions $D > 4$ as well.

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