RELATIVISTIC OSCILLATOR
IN QUATERNION RELATIVITY

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Abstract

In the framework of Quaternion (Q-) Theory of Relativity implying invariance of the 6D-space-time vector "interval" the kinematics of two frames is considered under condition that one frame is inertial and the other is subject to action of harmonic force. Using mathematical tools of Q-relativity the cinematic problem is completely solved from the viewpoint of each frame, i.e. distance, velocity and acceleration are found as functions of observers' time. Majority of cinematic relations are revealed to be represented by exact expressions: elementary functions and series; some relations though are found only approximately. Observed motions are of course not harmonic functions. Clock paradox is discussed.

I. Introduction: Q-relativity in short

There are many types of physics theories based on more than three space-time dimensions, but the only one, Einstein-Minkowski 4D theory, has comprehensive reasons for number of its dimensions. All others, beginning from 5D Kaluza-Klien theory up to 21D supergravity or to 2nD Calabi-Yau string theory spaces, are heuristically postulated. It is worth mentioning that several attempts to build "symmetric" 6D-relativities (3D-space + 3D-time) were made by Cole, Starr, Pavshic, Recami and others (see e.g. [1] and ref. therein). But the symmetry introduced also "ad hoc" together with abelian character of multiplication inherited from Einstein's relativity lead in these theories to a series of interpretational difficulties.

Differently from these patterns 6D-theory of Q-relativity (or Rotational Relativity) suggested in 1996 [2,3] does not result from phenomenological considerations but is extracted from quaternion mathematics as its modest but quite natural part. The extraction goes through following six steps. First, basic multiplication rule for Q-numbers is discovered to be form-invariant under Q-units transformations composing rotational group SO(3,C). Second, and this was pointed out by W.Hamilton, three "imaginary" Q-units, behave exactly as a Cartesian vector triad. Fourth, it is shown that real rotations from SO(3,C) save form of Q-vector (with real components) defined in a Q-triad. Fifth, similar form-invariance property is observed for biquaternionic (BQ) vectors under mixed real-imaginary rotations reducing the initial group to SO(1,2); this distinguishes the set of BQ-vectors with definable norm. All these facts have purely mathematical nature with no evident relevance to physics. But knowledge that SO(3,C) and its subgroup SO(1,2) are closely related to Lorents group hints to make the sixth "physical" step: the BQ-vector components are taken for space and time "displacements", space-time acquires 3+3-symmetric geometry, and the basic BQ-vector turns out nothing else but a specific 6D Q-square root of the interval of Einstein's relativity. Since no limitations are found for rotation parameters one is free to operate with inertial as well with non-inertial Q-frames.

So, Q-Relativity exploits the fact of SO(1,2)-invariance of 6D-space-time biquaternionic vector "interval"

$$d\mathbf{z} = (idt_k + dx_k)\,\mathbf{q}_k$$

with definable real norm

$$dz^2 = dt^2 - dx^2 \,.$$

If a Q-frame composed of "imaginary" vector Q-units

$$\Sigma' \equiv \mathbf{q}_{k'} = \{\mathbf{q}_{1'}, \mathbf{q}_{2'}, \mathbf{q}_{3'}\}^{*}$$

is observed from another analogous Q-frame Σ, then the $d\mathbf{z}$-invariance results in a simple relation for time and space "displacement" vectors

$$idt'\mathbf{q}_{1'} = idt\,\mathbf{q}_1 + dx\,\mathbf{q}_2,$$

vectors $d\mathbf{t}$ and $d\mathbf{r}$ obviously orthogonal to each other. The last condition naturally distinguishes scalar time out of 6-dimensionality, and allows regarding physical situations. The Q-frames may depend in general on 6 real parameters representing spatial rotations and boosts; in their turn the parameters are not banned to be variable e.g. dependent on time of observers hiding at the origins of the frames which are in this case non-inertial but nevertheless well described in the Q-approach. Technological tool of the theory is a Rotational Equation (RE) of the type

$$\Sigma' = O\,\Sigma$$

where O is a combination of real R and hyperbolic H rotations from SO(1,2) "converting" the frame of the observer Σ into the observed frame Σ'. From the RE cinematic effects of Q-frames relative motion are easily calculated, among them all effects of Einstein's Special Relativity and a number of non-inertial motion effects, e.g. hyperbolic motion and Thomas precession [4].

The Q-relativity also represents a good mean to study non-inertial clock behavior once largely discussed [5]. A desirable model to illustrate the problem is a "fast" linear harmonic oscillator. In Sect.2 definition of a relativistic harmonic oscillator is given and full cinematic problem is solved from the viewpoint of inertial and oscillating observers. Sect.3 is devoted to discussion of twin paradox issues associated with the solution.

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* The multiplication rule for Q-units is $\mathbf{q}_k\mathbf{q}_n = -\delta_{kn} + \varepsilon_{knj}\mathbf{q}_j$ where δ_{kn}, ε_{knj} are Kroneker and Levi-Civita 3D symbols, summation convention is assumed.

II. Linear harmonic oscillator problem in Q-relativity

Mathematical tool of Q-relativity allows studying behavior of non-inertially moving clock. A natural model of such a clock is a harmonic oscillator, "spring pendulum", arranged so that initial and final positions of its «massive body» (too, a body of reference of non-inertial harmonically moving observer) precisely coincide with position of immobile inertial observer, and relative velocity of the two observers at these moments is zero.

Let Σ be inertial frame and Σ' represent non-inertial frame whose body of reference is subject to action of a periodical harmonic force along a straight line. Since kinematics of the system is the focus of this study nature of the force here is of no importance.

CASE A. Σ' IS OBSERVED FROM Σ

If inertial frame Σ is modeled by a constant Q-triad $\mathbf{q}_k$ whose vector $\mathbf{q}_2$ is aligned with frames relative velocity, then rotational equation interconnecting two frames in question has the form

$$\Sigma' = H_3^{\psi(t')}\,\Sigma \tag{1}$$

with $H_3^{\psi(t')}$ being 3×3-matrix of simple hyperbolic rotation (about axis parallel to $\mathbf{q}_3$) and variable parameter $\psi(t')$ depending on time of moving observer

$$H_3^{\psi} = \begin{pmatrix} \cosh\psi & -i\sinh\psi & 0 \\ i\sinh\psi & \cosh\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The first row of Eq.1

$$\mathbf{q}_{1'} = \cosh\psi\,\mathbf{q}_1 - i\sinh\psi\,\mathbf{q}_2\,,$$

under standard conditions

$$\cosh\psi = \frac{dt'}{dt}\,, \tag{2}$$

$$V = \tanh\psi \tag{3}$$

is equivalent to biquaternionic vector basic in Q-theory of relativity (fundamental velocity equals unity: $c=1$)

$$d\mathbf{z} = idt'\mathbf{q}_{1'} = idt\,\mathbf{q}_1 + dx\,\mathbf{q}_2 \tag{4}$$

so that Σ-time and Σ′-time are aligned respectively with $\mathbf{q}_1$, and $\mathbf{q}_{1'}$. Eq.4 yields main cinematic vector characteristics, i.e. for Σ′-observer one readily finds relative Q-vector velocity

$$\mathbf{v}' \equiv \frac{d\mathbf{z}}{idt'} = \mathbf{q}_{1'}$$

and Q-vector acceleration

$$\mathbf{a}' \equiv \frac{d\mathbf{v}'}{idt'} = \frac{d\mathbf{q}_{1'}}{idt'} = -i\omega_{1'2'}\mathbf{q}_{2'} \equiv a'\,\mathbf{q}_{2'}\,. \tag{5}$$

where the only non-vanishing component of Q-connection [3]

$$\omega_{1'2'} = i\frac{d\psi}{dt'}$$

is computed as

$$\omega' = \frac{dH}{dt'}H^{-1}\,.$$

Thus value of Q-acceleration (5) aligned with $\mathbf{q}_{2'}$ is simply expressed through velocity parameter

$$a'(\Sigma') = \frac{d\psi}{dt'}\,.$$

It is natural to attribute to Q-frame a type of motion corresponding to the type of acceleration "felt" by observer in this frame. Well known example is the hyperbolic motion where non-inertial observer subject to constant acceleration feels constant force acting on him [6]. Similarly motion of Σ′ is represented as harmonic one if Σ′-acceleration, force per unit mass, obeys harmonic, e.g. cosine, law measured in its own time[**]

$$a'(\Sigma') = \frac{d\psi}{dt'} = \Omega'\beta\cos\Omega' t' ; \tag{6}$$

here β is a real constant (amplitude is chosen in this form for future commodity reasons), acceleration is maximal for $t' = 0$. Integration of Eq.6 gives dependence of hyperbolic parameter on proper Σ′-time

$$\psi(t') = \beta\sin\Omega' t' . \tag{7}$$

Constant of integration, initial phase, is chosen zero so that at the beginning and the end of oscillation period relative velocity vanish. It is worth noting here that the hyperbolic parameter, not velocity itself, has to be a harmonic function.

Σ′(Σ) time ratio

Now complete cinematic problem for the regarded mechanical system can be solved, i.e. coordinate, velocity and acceleration of Σ′ are to be found as functions of Σ-observer's time. But preliminary integration of the time-correlation equation resulting from Eq.2

$$dt' = dt\cosh\psi(t') ,$$

is necessary to determine Σ′-Σ observers' times interdependence

$$t = \int\cosh\psi(t')\,dt' = \int\cosh(\beta\sin\Omega' t')\,dt' \ . \tag{8}$$

An easy analysis shows that the integral can be computed exactly, not in elementary functions but as series. First, one applies well-known development of hyperbolic cosine

$$\cosh u = 1 + \sum_{n=1}^{\infty}\frac{1}{(2n)!}u^{2n} , \quad |u| < \infty ,$$

[**] Phase of the oscillation is chosen so that at initial and final moments of oscillation period velocity vanishes.

last condition being always fulfilled since $u \equiv \beta \sin \Omega' t' < \infty$. Second, one uses the following table integral

$$\int \sin^{2n} y\, dy = \frac{1}{2^{2n}} \binom{2n}{n} y + \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n} (-1)^k \binom{2n}{k} \frac{\sin(2n-2k)y}{2n-2k}. \quad (9)$$

And third, the substitution $y \equiv \Omega' t'$ in Eq.9 gives the sought for result of integration in Eq.8

$$t = t' + \sum_{n=1}^{\infty} \frac{\beta^{2n}}{(2n)!} \left[\frac{1}{2^{2n}} \binom{2n}{n} t' + \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \frac{\sin(2n-2k)\Omega' t'}{(2n-2k)\Omega'} \right]. \quad (10)$$

This result compels to recall that obtaining exact solutions in framework of relativity theory is a remarkable feature of simple and correctly formulated physical problems such as hyperbolic or circular motion. The relativistic oscillator problem seems to belong to the distinguished set.

One oscillation is completed when $\Omega' T' = 2\pi$, T' being oscillation period measured in Σ'. Corresponding Σ-time interval, "period" T, is straightforwardly found from Eq.10

$$T = T' \left(1 + \sum_{n=1}^{\infty} \frac{\beta^{2n}}{(2n)!} \frac{1}{2^{2n}} \binom{2n}{n} \right), \quad (11)$$

as well as respective cycle frequencies ratio

$$\Omega' = \Omega \left(1 + \sum_{n=1}^{\infty} \frac{\beta^{2n}}{(2n)!} \frac{1}{2^{2n}} \binom{2n}{n} \right). \quad (12)$$

Eq.12 tells that Σ-observed periodic motion possesses less frequency than oscillations felt by Σ'-observer; the fact sounds conventionally in relativity: moving clock is apparently slow.

Inversion of Eq.10 is evidently hardly possible, so expression of Σ'-time as a function of Σ-time $t'(t)$ is looked for in approximation. First several terms of series in Eq.10 are written as

$$t = t' + \frac{\beta^2}{4}\left(t' - \frac{1}{2\Omega'}\sin 2\Omega' t'\right) + \\ + \frac{\beta^4}{8 \cdot 4!}\left[3t' - \frac{1}{\Omega'}\left(\frac{1}{4}\sin 4\Omega' t' - 2\sin 2\Omega' t'\right)\right] + \ldots \tag{10a}$$

Dimensionless factor β may be given in the form

$$\beta = V_0 / c << 1,$$

where V_0 can be any characteristic value of relative velocity, e.g. mean value for ½ of period. Then the following ratios connecting frequencies and times are written up to the terms including β^2

$$\Omega = \frac{\Omega'}{1 + \beta^2 / 4}, \tag{13a}$$

$$t' = t - \frac{\beta^2}{4}\left(t - \frac{1}{2\Omega}\sin 2\Omega t\right). \tag{13b}$$

Substitution of Eqs.13 into expression for velocity parameter (Eq.3) allows to find approximate solution of the cinematic problem for Σ-observer.

Σ′(Σ) Velocity

Velocity value of the frame Σ′ observed from Σ is (fundamental velocity c is explicitly shown)

$$V(t) = c \tanh\left(\beta \sin \Omega' t'\right) \cong V_0 \sin \Omega t \left(1 - \frac{1}{3}\beta^2 + \frac{7}{12}\beta^2 \cos^2 \Omega t\right).$$

At the beginning and at the end of period velocity acquires minimal value $V(T) = 0$, i.e. at these moments the two frames are really immobile relative to each other. Maximal value of the velocity

$$V(T/4) = V_0\left(1 - \frac{1}{3}\beta^2\right)$$

may be regarded as “V-amplitude”

$$\tilde{V} = V_0\left(1 - \frac{1}{3}\beta^2\right),$$

and final expression has the form

$$V(t) \cong \tilde{V}\sin\Omega t\left(1 + \frac{7}{12}\beta^2\cos^2\Omega t\right) \tag{14}$$

meaning that the periodic process observed from Σ definitely has no harmonic character.

<u>Σ′(Σ) Acceleration</u>

Value of Σ-observed acceleration of Σ′ is found as

$$a(t) = \frac{dV(t)}{dt} \cong \Omega\tilde{V}\cos\Omega t\left(1 - \frac{7}{6}\beta^2 + \frac{7}{4}\beta^2\cos^2\Omega t\right).$$

Minimal value acceleration acquires at the middle of period

$$a(T/4) = 0,$$

and maximal value, "*a*-amplitude", at its end

$$a(T) = \Omega\tilde{V}\left(1 - \frac{7}{12}\beta^2\right) = \tilde{A};$$

the final expression is

$$a(t) \cong \tilde{A}\cos\Omega t\left(1 - \frac{7}{4}\beta^2\sin^2\Omega t\right). \tag{15}$$

<u>Σ′(Σ) Coordinate</u>

Σ-coordinate of Σ′ is computed as result of integration

$$x(t) = \int V(t)dt \cong x_0 - \frac{\tilde{V}}{\Omega}\cos\Omega t\left(1 + \frac{7}{36}\beta^2\cos^2\Omega t\right), \tag{16}$$

the integration constant

$$x_0 = \frac{\tilde{V}}{\Omega}\left(1 + \frac{7}{36}\beta^2\right)$$

is chosen to satisfy the following initial conditions

$$x(0) = x(T) = 0\,, \quad x(T/4) = x_0\,, \quad x(T/2) = 2\,x_0\,,$$

meaning, that at the initial and final moments of period the two frames are not only relatively immobile but too are found at the same point in space.

Thus the $\Sigma'(\Sigma)$-cinematic problem is solved in approximation $\beta << 1$; the result is given in Eqs.10, 14, 15, 16.

CASE B. Σ IS OBSERVED FROM Σ'

Rotational equation for this case

$$\Sigma = H_3^{-\psi(t')}\,\Sigma'\,, \tag{17}$$

has the same parameter

$$\psi(t') = \beta \sin \Omega' t'$$

describing harmonic oscillations of Σ'. The first row of Eq.17 represent space-time vector "interval"

$$idt\mathbf{q}_1 = idt'\,\mathbf{q}_{1'} - dx'\,\mathbf{q}_{2'} \tag{18}$$

where dx' is space displacement of Σ and dt' is respective time interval, both measured by non-inertially moving Σ'-observer. Calculated from Eq.18 proper Q-vector acceleration of Σ is naturally zero

$$\mathbf{a} \equiv \frac{d\mathbf{q}_1}{idt} = 0\,,$$

so study of this case considers cinematic magnitudes only as they are seen from genuinely accelerated frame Σ'.

<u>Σ (Σ')-time ratio</u>

Following from Eq.18 standard relativistic expression

$$dt' = dt \cosh \psi(t')$$

leads to integral determining $t(t')$ functional dependence

$$t = \int \frac{dt'}{\cosh \psi(t')} = \int \operatorname{sech}(\beta \sin \Omega' t')\, dt' . \tag{19}$$

This integration also can be performed exactly due to existence of development

$$\operatorname{sech} u = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{E_n}{(2n)!} u^{2n}, \quad |u| < \pi / 2 \tag{20}$$

and table integral already used in the Case A and given by Eq.9. Series in Eq.20 includes Euler numbers

$$E_n \equiv \frac{2^{2n+2}(2n)!}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^{2n+1}} ,$$

and its convergence condition

$$|u| \equiv |\beta \sin \Omega' t'| < \pi / 2 ,$$

is always satisfied since

$$\beta = V_0 / c < 1 , \qquad \sin \Omega' t' < 1 .$$

Resulting formula of integration in Eq.20 has the form

$$t = t' + \sum_{n=1}^{\infty} \frac{(-1)^n E_n \beta^{2n}}{(2n)!} \left[\frac{1}{2^{2n}} \binom{2n}{n} t' + \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \frac{\sin(2n-2k)\Omega' t'}{(2n-2k)\Omega'} \right]. \tag{21}$$

Eq.21 permits to Σ'-observer to measure real period of T' and cycle frequency Ω' and also to calculate similar characteristics T, Ω theoretically attributed to the frame Σ and to find respective correlations

$$T = T' \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n E_n \beta^{2n}}{(2n)! 2^{2n}} \binom{2n}{n} \right), \tag{22}$$

$$\Omega' = \Omega \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n E_n \beta^{2n}}{(2n)! 2^{2n}} \binom{2n}{n} \right) \tag{23}$$

of course different from analogous Eqs.11, 12 of the Case A due to different inertiality properties of the observers. But computing frequency ratio in Eq.23 up to first approximation in β^2

$$\Omega' \cong \Omega\left(1+\frac{-E_1\beta^2}{2\cdot 2!}\binom{2}{1}\right),$$

where

$$E_1=\frac{2^5}{\pi^3}\left(1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-...\right)\cong 1,$$

gives result similar to Case A analog Eq.13a

$$\Omega' \cong \Omega\left(1-\frac{\beta^2}{4}\right), \qquad (24)$$

i.e. Σ'-frequency, normally for relativity, is smaller than that of Σ from the viewpoint of Σ'-observer.

All cinematic functions in the Case B are to depend on time t', hence there is no need to inverse variables in Eq.21, giving relation of time-lines length: proper one, t', and "observed" one, t. Nonetheless it seems useful to put down several first terms of the development

$$t=t'-\frac{\beta^2}{4}\left(t'-\frac{1}{2\Omega'}\sin 2\Omega' t'\right)+\frac{5\beta^4}{16\cdot 4!}\left[t'+\frac{1}{\Omega'}\left(2\sin 4\Omega' t'-4\sin 2\Omega' t'\right)\right]+... \quad (21a)$$

Comparing Eq.21a with its analogue Eq.10a one notes symmetry of time-functions in the least β^2-approximation for Cases A and B; this is too an expected relativistic result of exchanging observation bases.

In the Case B whole of cinematic problem has exact solution.

<u>Σ (Σ') Velocity</u>

$$V'(t')=c\tanh\left(\beta\sin\Omega' t'\right). \qquad (24)$$

For practical purposes it is useful to consider approximation up to small $\beta^2 = (V_0 / c)^2$

$$V'(t') \cong V_0 \sin\Omega' t' \left(1 - \frac{\beta^2}{3}\sin^2\Omega' t'\right).$$

Minimal value velocity $V(T') = 0$ acquires at the beginning and the end of each oscillation, at these moments the two frames are immobile to each other. Maximal value (V-amplitude) velocity has at quarter of period

$$\tilde{V}'(T'/4) = c \tanh\frac{V_0}{c}.$$

<u>Σ (Σ′) Acceleration</u>

$$a'(t') = \frac{dV(t')}{dt'} = \frac{V_0\Omega' \cos\Omega' t'}{\cosh^2(\beta \sin\Omega' t')}; \qquad (25)$$

its β^2 -approximation is

$$a'(t') = \frac{dV(t')}{dt'} \cong \Omega\tilde{V} \cos\Omega t (1 - \beta^2 \sin^2\Omega' t').$$

Minimal and maximal values respectively are

$$a'(T'/4) = 0, \qquad a'(T') = V_0\Omega'.$$

<u>Σ (Σ′) Coordinate</u>

$$x'(t') = \int V'(t')dt' = \int c \tanh(\beta \sin\Omega' t')dt'.$$

This function too can be integrated exactly due to (i) existence of divergent series

$$\tanh u = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)B_n}{(2n)!} u^{2n-1}, \qquad |u| \equiv |\beta \sin\Omega' t'| < \pi/2,$$

whose coefficients are Bernoulli numbers tied by recurrent formula

$$B_n \equiv (-1)^n \left[\frac{1}{2n+1} - \frac{1}{2} + \sum_{k=1}^{n-1} (-1)^{k-1} \frac{B_k (2n)(2n-1)(2n-2)...(2n-k+2)}{(2k)!}\right]$$

so that $B_1 = 1/6$, $B_2 = -1/30$, and (ii) existence of table integral

$$\int \sin^{2n-1} y\, dy = \frac{(-1)^n}{2^{2n-2}} \sum_{k=0}^{n-1} (-1)^k \binom{2n-1}{k} \frac{\cos(2n-1-2k)y}{2n-1-2k} \;;$$

here $y \equiv \Omega' t'$. Resulting coordinate function of time has the form

$$x'(t') = x_0 + \frac{c}{\Omega'} \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)B_n}{(2n)!} \beta^{2n-1} \frac{(-1)^n}{2^{2n-2}} \sum_{k=0}^{n-1} (-1)^k \binom{2n-1}{k} \frac{\cos(2n-1-2k)\Omega' t'}{2n-1-2k}\,, \quad (26)$$

its β^2-approximation is

$$x'(t') = x_0' - \frac{V_0}{\Omega'} \cos\Omega' t' \left[1 - \frac{1}{3}\beta^2 \left(1 - \frac{1}{3}\cos^2 \Omega t \right) \right] .$$

Integration constant

$$x_0' = \frac{V_0}{\Omega'} \left(1 - \frac{2}{3}\beta^2 \right)$$

is chosen so that the following conditions are satisfied

$$x'(0) = x'(T') = 0\,, \quad x'(T'/4) = x_0'\,, \quad x'(T'/2) = 2\,x_0'$$

meaning that at the beginning and the end of oscillation relatively immobile frames are found at the same space point.

Thus the cinematic problem for Σ'-observer is shown to have exact solution, it is represented by Eqs.24, 25, 26, and their week-relativity approximations with $\beta \ll 1$ are given.

Clock paradox discussion

Specific features of the discussed relativistic oscillator model make it an appropriate cinematic system for discussion of famous clock paradox formulated in Special Relativity (SR) a centaury ago. First, one of the two involved frames of the system is always immobile (inertial) while the other is accelerated hence obviously non-inertial. Second, at starting and final moments the initial points of the frames spatially coincide while the frames are reciprocally at rest. Third, oscillating frame itself can serve as a clock for both observers. And last but not least, the key formulae describing time correlations are exact solutions.

Frequent explanation of the paradox relates clock delay to non-inertiality of motion (e.g. [7,8]). But if one considers two identical non-inertial frames moving in opposite directions the paradox seems to arise again: alleged non-inertial ("gravitational") delay of the both clocks should be exactly the same, but accordingly to SR locally, at any moment, each observer detects his/her partner's time slowing down. Conventional SR seems not to be able to cope with the problem.

The clock paradox can be regarded as a result of "one-side" measurement procedure, when two cinematically different time intervals of the same observation are measured by a time-unit of one observer; in this case they obviously will have different "length". But if each interval is measured by time-unit of its own observer the lengths should be equal similar to the case of distance measurement: indeed, moving ruler seems to immobile observer shorter but it has the same "number of centimeters" as an identical ruler at rest.

Discussed above oscillating frame seems to be a successful illustration to the explanation of paradox given further in terms of space-traveling twins. Fig.1 shows Minkowski diagram of the oscillation process form the viewpoint of inertial ("immobile") Σ-twin. Let time-segments subject to measurement be periods T and T' "observed" from Σ, hence interconnected by Eq.11, while a "Σ-second" (Σ-time-unit) is

$$\tau \equiv T / 4 = \pi / 2\Omega \,.$$

Then the "length" of segment T is 4 sec. Period T' measured in Σ-twin time-units τ appears obviously shorter, so that returning home twin-traveler is allegedly younger than his brother-observer. This "one-side" measurement gives incorrect result since for Σ-twin not only Σ′-time-segment contracts but Σ′-time-unit too, adequate change of "Σ′-seconds" (in this case all equal) extracted from Eq.11

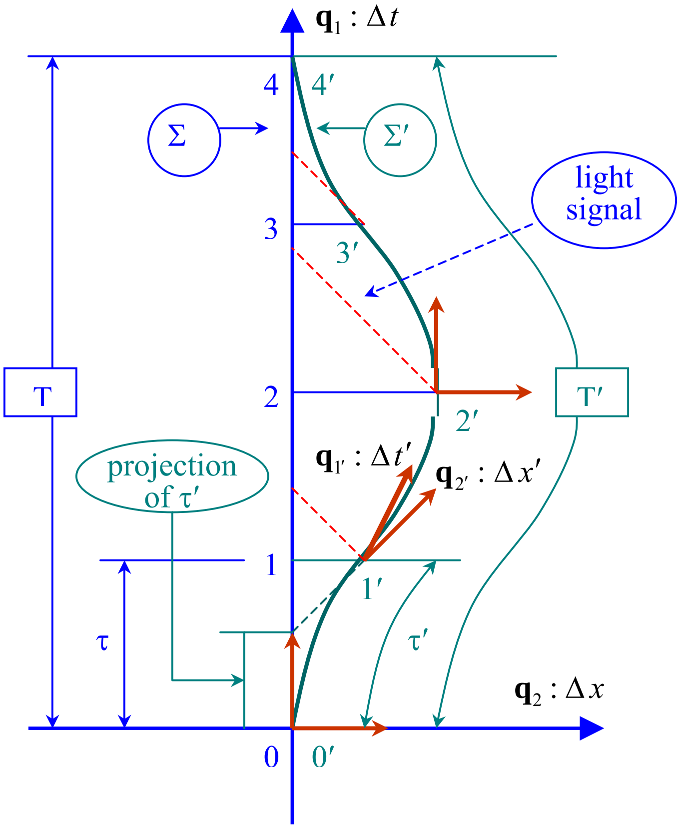

Fig.1. Minkowski diagram for relativistic oscillator.

$$\tau' = \tau\left(1 + \sum_{n=1}^{\infty} \frac{\beta^{2n}}{(2n)!} \frac{1}{2^{2n}} \binom{2n}{n}\right)^{-1};$$

using the units one finds the length of T' also to be 4 seconds. If Σ′-twin on his way sends regular (at each his second) light signals then Σ-twin receives them irregularly, but during traveling period Σ-twin will count exactly four such signals. Exchange of the last signal occurs at the very end of mission, at the same space point, at zero relative velocity, and the age of twins at the meeting point remains equal.

Besides, Fig.1 shows that projections of equal in length but different Σ'-time segments onto Σ-time (straight) line are also different, but they change steadily as smooth functions, and in three points of the twins relative immobility (0-0′, 2-2′, 4-4′) their units smoothly became equal. This means that this model is free of "lost time" or "unit gap" features sometimes present in discussions of non-inertial approach to the paradox from Special Relativity position.

There are two final remarks.

1. Found relativistic solution for oscillator system frequent in nature and in description of many physical processes remarkably incorporates to the set of non-inertial cinematic problems already solved in the framework of Q- relativity theory from the viewpoint of all involved observers: hyperbolic motion, circular motion and Thomas-like precession [9].

2. In the study an accent may be made on the method used to endow a frame with definite non-inertial character. The method can serve as an instructive example helpful for construction of any non-inertial frame provided the acceleration law is given; this allows easier formulation and solution of new relativistic problems involving non-inertially moving observers.

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