An Identity Crisis for the Casimir Operator

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Abstract

The Casimir operator of a Lie algebra $\mathcal{L}$ is $C^2 = \sum g^{ij} X_i X_j$ and the action of the Casimir operator is usually taken to be $C^2 Y = \sum g^{ij} X_i X_j Y$, with ordinary matrix multiplication. With this definition, the eigenvalues of the Casimir operator depend upon the representation showing that the action of the Casimir operator is not well defined. We prove that the action of the Casimir operator should be interpreted as $C^2 Y = \sum g^{ij}[X_i, [X_j, Y]]$. This intrinsic definition does not depend upon the representation. Similar results hold for the higher order Casimir operators. We construct higher order Casimir operators which do not exist in the standard theory including a new type of Casimir operator which defines a complex structure and third order intrinsic Casimir operators for $so(3)$ and $so(3,1)$. These operators are not multiples of the identity. The standard theory of Casimir operators predicts neither the correct operators nor the correct number of invariant operators. The quantum theory of angular momentum and spin, Wigner's classification of elementary particles as representations of the Poincaré Group and quark theory are based on faulty mathematics. The "no-go theorems" are shown to be invalid.

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1 Introduction

Lie groups and Lie algebras play a fundamental role in classical mechanics, electrodynamics, quantum mechanics, relativity, and elementary particle physics. Many hope that the Lie group/algebra setting will provide an appropriate framework for the unification of quantum theory, general relativity and particle physics. Within the unification via group theory program, the so-called Casimir operators or invariant operators play a pivotal role. In quantum mechanics, the quadratic Casimir operator of $so(3)$ is either L^2, the total angular momentum or J^2, the total spin. In the program of dynamical groups or spectrum generating algebras, the eigenvalues of the Casimir operators can be interpreted as mass, energy, momentum, or other dynamical quantities. H. Schwartz [28] emphasized the role of the Casimir operator in Relativity, while W. Greiner and B. Muller [7] emphasized the role of the Casimir operator in quantum mechanics. Thus it seems likely that the Casimir operators of some Lie algebra will play a major role in the unification of the two theories. The author [17] suggested that $u(3,2)$ is the unique Lie algebra capable of such a unification. A Theory of Matter based on the geometry of $u(3,2)$ was developed in Love [18, 19, 20]. In this program, the field equations arise as eigenvalue equations involving the Casimir operators of $u(3,2)$, with the conserved quantities as the eigenvalues. Thus identification of the proper operators is essential to progress in the Theory Of Matter.

A Casimir operator, C, of a Lie Algebra $\mathcal{L}$ is an operator constructed as a polynomial in the elements of $\mathcal{L}$ which commutes with every element of $\mathcal{L}$. With an abuse of notation, this is written as

$$[C, X] = 0 \qquad \forall X \in \mathcal{L}.$$

This is an abuse of notation because the bracket is use to denote the operation defined on the Lie algebra so writing $[A, B] = 0$ implies that both A and B are in the Lie algebra. The Casimir operator is not in the Lie algebra itself, rather the Casimir operator is in the Enveloping algebra of the Lie algebra. So

$$[C, X] = 0$$

really means that $CXY = XCY$, but this equation makes no sense in a Lie algebra since the product XY is not defined, only the bracket is defined. Putting brackets in, we have: $[CX, Y] = [X, CY]$, or should it be $C[X, Y] = [X, CY]$? We need to examine this issue.

We begin with Schur's lemma as phrased in Proposition 2 of Chevalley [3]:

> Let P be an irreducible representation of a group G in an algebraically closed field K. The only matrices which commute simultaneously with all matrices $P(\sigma)$, $\sigma \in G$ are the scalar multiples of the unit matrix. (page 183)

In many treatments of the Casimir operator, an appeal is made to Schur's Lemma to show that the Casimir operator (and every generalized Casimir operator) is a multiple of the identity matrix. As we will show, this statement as it stands is not true. In the context of representation by differential operators, the phrase doesn't even make sense. We will show that Schur's Lemma is not true for differential operator representations of Lie algebras.

Suppose that each element of a Lie algebra is an eigenvector of the Casimir operator C of the Lie algebra $\mathcal{L}$, thus:

$$CX = \alpha X \qquad \forall X \in \mathcal{L} \tag{1}$$

Now let ρ be an isomorphism of the Lie Algebra $\mathcal{L}$ and apply ρ to both sides of (1) to obtain:

$$\rho(CX) = \rho(\alpha X) \tag{2}$$

Commutivity of the diagram:

$$\begin{array}{ccc} \mathcal{L} & \stackrel{C}{\longrightarrow} & \mathcal{L} \\ \rho\downarrow & & \rho\downarrow \\ \rho(\mathcal{L}) & \stackrel{\rho(C)}{\longrightarrow} & \rho(\mathcal{L}) \end{array}$$

requires that:

$$\rho(CX) = \rho(C)\rho(X)$$

In the representation space we have:

$$\rho(C)\rho(X) = \alpha\rho(X).$$

In order to be a scalar, α must be the same in all representations (that is the definition of scalar). In the standard approach, with

$$C = \sum g^{ij} X_i X_j$$

and

$$CY = \sum g^{ij} X_i X_j Y$$

for $Y \in \mathcal{L}$ for consistency we must have:

$$\rho(C^2)\rho(Y) = \sum g^{ij} \rho(X_i)\rho(X_j)\rho(Y)$$

But this is not the case, as the examples considered by Schiff [26] show. Schiff asserts (p. 199): "Direct substitution from the matrices (27.11) shows that

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

is equal to $2\hbar^2$ times the unit matrix". This is indeed true, if we just multiply the matrices (recall that physicists put in a factor of i to make the matrix hermitean):

$$S_x = i\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$S_y = i\hbar \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$S_z = i\hbar \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then we have:

$$S^2 A = S_x^2 A + S_y^2 A + S_z^2 A = 2\hbar A$$

$$A \in so(3)$$

This calculation also 'proves' that $S^2 A = 2\hbar A$ for any three by three matrix, and in particular for $A \in su(3)$. Consequently, if this proof were valid, S^2 would be a Casimir operator for $su(3), sl(3)$ and any other Lie algebra of 3 by 3 matrices which contains $so(3)$. It is not. Thus, the 'proof' is not valid.

Switching representations, Schiff contends on page 203 that

$$J_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$J_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$J_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Summing, we obtain:

$$J^2 = J_x^2 + J_y^2 + J_z^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Schiff continues with two more representations and finds:

$$J^2 = \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or

$$J^2 = \frac{1}{2}\hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now we ask the question "What is $J^2 A$ for $A \in so(3)$?." And the standard answer is: "That depends on which representation you are in". In the standard approach, the eigenvalue of the Casimir operator changes with each representation, thus it is a "varying invariant", the ultimate oxymoron. If the eigenvalue of an operator changes from representation to representation, the operator cannot be an invariant of the Lie algebra. From the viewpoint of differential geometry, a representation is essentially a coordinate system. Exponentiate a representation of the Lie algebra and you have a local coordinate system for the manifold underlying the Lie group. Differential Geometry requires that in order to be well defined, "geometric objects" be independent of the coordinate system. Thus the dependence of the eigenvalue of the Casimir operator on the representation shows that with the standard definition the Casimir operator is not well defined as a geometric object. In the parlance of classical Differential geometry, it does not transform properly. The reason for this is clear: in the standard approach, the Casimir operator is defined in terms of matrix multiplication and a Lie algebra isomorphism does not preserve matrix multiplication. Fulton and Harris [6] (page 108) observe '. . . that the "composition" $X \circ Y$ of elements of a Lie algebra is not well defined.' In order to be well defined within the category of Lie algebras, the action of the Casimir operator cannot be defined in terms of matrix multiplication, it must be defined in terms of the Lie bracket.

Consequently, although the standard results about the Casimir operator follow from direct calculation, those calculations are meaningless from a geometric (or a categorical) viewpoint. Our immediate goal must be to find a way of defining the Casimir operator in a way which is geometrically and categorically satisfactory.

We begin by looking at the geometric origin of the Lie bracket. Let $F(t)$ be the flow of the vector field X and F^* the pullback map under the diffeomorphism induced by that flow, then the Lie derivative of a tensor field K with respect to the vector field X is defined by

$$L_X K(p) = \lim_{t \to 0} (K(p) - F^*(t)K(p))/t$$

It is a standard exercise in differential geometry to prove that the Lie derivative of a vector field Y with respect to another vector field X is given by:

$$L_X Y = [X, Y] = XY - YX$$

(Kobayashi and Nomizu [14],p.29).

Within differential geometry, the commutator is then a secondary tool used to *compute* the Lie brackets. Any computations using a matrix representation of a Lie algebra which comes from a Lie group must be consistant with the geometric origin of the Lie derivative, matrix multiplication is not. The reader interested in more detail should read the entire discussion of the matter in Fulton and Harris [6].

In the standard approach, the Casimir operator for a Lie algebra $\mathcal{L}$, is

$$C^2 = \sum g^{ij} X_i X_j \qquad (3)$$

and the action of the Casimir operator is

$$C^2 Y = \sum g^{ij} X_i X_j Y \qquad (4)$$

with ordinary matrix multiplication. However, matrix multiplication is not defined in a Lie algebra, only the Lie bracket, addition and scalar multiplication are defined. Since a Lie Algebra with the bracket operation is nonassociative, this expression is meaningless. Since the action (4) is not well defined, we look for other ways to interpret the symbols. In another standard treatment, the Casimir operator is taken to be an element of the Enveloping Algebra in which case

$$C^2 Y = \sum g^{ij} X_i \otimes X_j \otimes Y$$

with the tensor product as the multiplication. We will look at this treatment after discussing yet another way to interpret the symbols.

Since the Casimir operator is defined in terms of the generators of the Lie algebra and the generators are defined in terms of the Lie derivative, it seems appropriate to have the Casimir operators defined in terms of the Lie derivative.

Kobayashi and Nomizu [15](p.128) prove that a vector field X is an infinitesimal automorphism of an almost complex structure J iff

$$J[X, Y] = [X, JY]$$

Phrased another way, this condition insures the almost complex structure J is invariant under the flow generated by the vector field X. This theorem is relevant since we will construct a Casimir operator which defines a complex structure. We can use their proof verbatim, replacing J by C to prove:

Theorem:

Given a Lie Algebra $\mathcal{L}$, if $C :\rightarrow L$ is a linear operator, then C is invariant under the flow generated by the vector field X, iff

$$C[X, Y] = [X, CY] = [CX, Y] \tag{5}$$

for all $X, Y \in \mathcal{L}$.

Proof:

Consider the Lie algebra as the tangent space of M, the manifold underlying the Lie group. Then the Casimir operator is a mapping

$$C : TM \rightarrow TM.$$

Let X and Y be any vector fields on M. Then

$$[X, CY] = L_X(CY) = (L_X C)Y + CL_X(Y) = (L_X C)Y + C[X, Y]$$

Hence, $L_X C = 0$ iff

$$[X, CY] = C[X, Y].$$

The condition that the Lie derivative of C with respect to X is zero, $L_X C = 0$ is the *definition* of invariant.

Then we also have

$$C[X,Y] = -C[Y,X] = -[Y,CX] = [CX,Y]$$

The standard approach requires that $CX = XC$, or putting in another element Y for these operators to act on, the standard approach requires that C commutes with X under the operation of matrix multiplication:

$$CXY = XCY \tag{6}$$

In the approach taken here, we require that C interacts with X under the operation of the Lie algebra, the Lie bracket and thus:

$$C[X,Y] = [X,CY] \tag{7}$$

Again, the standard approach cannot be correct simply because matrix multiplication is not defined in a Lie Algebra. This is a different interpretation of the phrase “commutes with X” than the standard theory and is justified because it is local invariance under the action of the Lie algebra which leads to global invariance under the action of the Lie Group.

Theorem

In any representation, the action of the Casimir operator C is given by

$$CY = \sum g^{ij}[X_i,[X_j,Y]] \tag{8}$$

Proof:

In the standard treatment of the Casimir operator, C ,

$$CX_k = \sum_{ij} g^{ij} X_i X_j X_k$$

In the adjoint representation, we have the Casimir operator acting on an element of the Lie algebra:

$$CA = \sum_{ij} g^{ij} ad(X_i) ad(X_j) A$$

then, by definition of the adjoint representation,

$$(adC)(adA) = \sum_{ij} g^{ij}[adX_i,[adX_j,adA]]$$

Now, let ρ be a representation of the Lie algebra, and note that any representation can be factored through the adjoint representation:

$$\begin{array}{ccc} \mathcal{L} & \stackrel{ad}{\longrightarrow} & \hom(\mathcal{L}) \\ \rho \searrow & & \swarrow \rho' \\ & \hom(V) & \end{array}$$

Then

$$\rho = \rho' ad.$$

We have the corresponding diagram for C:

$$\begin{array}{ccc} C & \stackrel{ad}{\longrightarrow} & ad(C) \\ \rho \searrow & & \swarrow \rho' \\ & \rho(C) & \end{array}$$

Then commutivity of the diagram requires that

$$\rho(C) = \rho'(adC)$$

$$\rho(CA) = \rho(C)\rho(A) = \rho'(adC)\rho'(adA) = \rho' \sum_{ij} g^{ij}[adX_i, [adX_j, adA]]$$

$$= \sum_{ij} g^{ij}[\rho'(adX_i), [\rho'(adX_j), \rho'(adA)]]$$

$$= \sum_{ij} g^{ij}[\rho(X_i), [\rho(X_j), \rho(A)]]$$

This last equality holds because a Lie algebra isomorphism preserves the bracket. Thus proving that the Casimir operator must be interpreted as an intrinsic Casimir operator.

In the expansion of the intrinsic Casimir operator, C, define the coefficients g^{ij} by:

$$CX_k = \sum_{ij} g^{ij}[X_i, [X_j, X_k]]$$

In the standard treatment of the Casimir operator the matrix g^{ij} is the inverse of the Killing form:

$$g_{ij} = Tr(adX_i)(adX_j) = \sum_{km} C^m_{ik} C^k_{jm}$$

However, with the reinterpretation of the Casimir operator, this relationship is not valid. As a direct calculation with the normalized intrinsic Casimir operator shows:

$$CX_k = \sum_{ij} g^{ij}[X_i, [X_j, X_k]]$$

$$= \sum_{ij} g^{ij}[X_i, \sum_l C^l_{jk} X_l]$$

$$= \sum_{ijl} g^{ij}[X_i, C^l_{jk} X_l]$$

$$= \sum_{ijl} g^{ij} C^l_{jk}[X_i, X_l]$$

$$= \sum_{ijlm} g^{ij} C^l_{jk} C^m_{il} X_m$$

Denote the inverse of g^{ij} by h_{jk} then the relation between a matrix and its inverse is given by

$$g^{ij} h_{jk} = \delta^i_k$$

and is summed over the index j only. Since the expression

$$\sum_{ijlm} g^{ij} C^l_{jk} C^m_{il} X_m \tag{9}$$

is summed over both i and j, the relation between g^{ij} and $C^l_{jk} C^m_{il}$ is not that between a matrix and its inverse.

O'Raifeartaigh [24] claimed that the Casimir operator could be written as

$$CX_k = \sum_{ij} Tr(adX_i)(adX_j)[X_i, [X_j, X_k]].$$

However, this expression cannot be correct since $Tr(adX_i)(adX_j)$ is the Killing form and according to the standard formula, it is the inverse of the Killing form which defines the coefficients of the Casimir operator. But (9) shows that this doesn't work. Instead of taking the inverse of the matrix, we need to take the inverse of each term individually and define:

$$CX_k = \sum_{ij} (\ Tr(adX_i)(adX_j)\ )^{-1}\,[X_i,[X_j,X_k]]. \qquad (10)$$

(The sum is over nonzero traces)

O'Raifeartaigh's calculations work out because he only considers cases where $Tr(adX_i)(adX_j)$ is 1 or -1.

Note that $(adX_i)(adX_j) = [X_i, [X_j,$; so we could write:

$$CX_k = \sum_{ij} (Tr[X_i, [X_j,)\ ^{-1}[X_i, [X_j, X_k]] \qquad (11)$$

(The sum is over nonzero traces)

We will call the operator defined in (10) the normalized intrinsic Casimir operator. With a slight change of notation, this agrees with the definition of the Casimir operator given by Knapp [13]. This operator is scale invariant, that is, if each X_i is multiplied by a scalar a^i, the normalized intrinsic Casimir operator is unchanged (in the standard picture, this is only true for an orthogonal basis). Such rescalings will be necessary to properly scale field strengths [31].

The intrinsic Casimir operator is one of the invariants of the Lie algebra. When we speak of invariants in differential geometry, we mean a differential operator which is invariant under the action of the Lie Group. This immediately leads to the criteria that in the representation of the Lie algebra as differential operators the invariant operator commutes with elements of the Lie algebra as in Theorem 5. But here arises the main point of confusion when we work with a matrix representation. If we want an operator which commutes with every element of the Lie algebra, does that mean that the operator commutes with the matrices in some representation under the action of ordinary matrix multiplication or that the operator commutes with the action of that matrix as an element of the Lie algebra? The standard approach takes the first route and requires that the operator commutes with every matrix in the representation. Then, supposedly Schur's lemma requires that these "invariant operators" are just multiples of the unit matrix. But then the "invariants" are a different multiple of the identity for each representation and can hardly be called "invariant". Noting that matrix multiplication is not defined in a Lie algebra, we take the second tack and require that the operator commute with the action of the matrices as elements of the Lie algebra, as required by Theorem 5. The eigenvalue of these operators does not change with the representation and they are truly invariant. We will also show that the invocation of Schur's lemma is not justified since these operators are not representable by matrix operators.

Since this approach is such a break with tradition, perhaps further discussion is warranted. If A is an element of the Lie algebra of a Lie group, then A is the infinitesimal generator of a one parameter subgroup with the group action on $B \in \mathcal{L}$ given by: $exp(tA)Bexp(-tA)$

In order to prove the invariance of an operator C we need to show that:

$$Cexp(tA)Bexp(-tA) = exp(tA)(CB)exp(-tA)$$

Using the Baker-Campbell-Hausdorff identity, we can expand the group action in terms of the Lie algebra action:

$$exp(tA)Bexp(-tA) = (B + t[A,B] + \frac{t^2}{2!}[A,[A,B]] + \frac{t^3}{3!}[A,[A,[A,B]]] + \ldots$$

Then

$$exp(tA)(CB)exp(-tA) = (CB+t[A,CB]+\frac{t^2}{2!}[A,[A,CB]]+\frac{t^3}{3!}[A,[A,[A,CB]]]+\ldots$$

$$= (CB + Ct[A,B] + C\frac{t^2}{2!}[A,[A,B]] + C\frac{t^3}{3!}[A,[A,[A,B]]] + \ldots$$

$$= C(B + t[A,B] + \frac{t^2}{2!}[A,[A,B]] + \frac{t^3}{3!}[A,[A,[A,B]]] + \ldots)$$

$$= Cexp(tA)Bexp(-tA)$$

An invariant operator is one whose action commutes with that of the Lie Group. The above calculation shows that in order for an operator to commute with the action of the Lie group, it must commute with the action of the Lie algebra in the sense of 5.

2 The Lie Algebra of Vectors and The Intrinsic Casimir Operator of $so(3)$

Consider the Lie Algebra of vector fields on $\mathcal{R}^3$ with bracket defined by cross product:

$$\begin{array}{ccc} \vec{i}\times\vec{i}=\vec{0} & \vec{j}\times\vec{i}=-\vec{k} & \vec{k}\times\vec{i}=\vec{j} \\ \vec{i}\times\vec{j}=\vec{k} & \vec{j}\times\vec{j}=\vec{0} & \vec{k}\times\vec{j}=-\vec{i} \\ \vec{i}\times\vec{k}=-\vec{j} & \vec{j}\times\vec{k}=\vec{i} & \vec{k}\times\vec{k}=\vec{0} \end{array}$$

It follows immediately from the cross product that:

$$\vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}$$

$$\vec{i} \times (\vec{i} \times \vec{k}) = \vec{i} \times (-\vec{j}) = -\vec{k}$$

Thus $Tr(\vec{i} \times (\vec{i} \times) = -2$.
Likewise, $Tr(\vec{j} \times (\vec{j} \times) = -2$ and $Tr(\vec{k} \times (\vec{k} \times) = -2$
On the space of 3 dimensional vectors, define the operator C by

$$CA = \vec{i} \times (\vec{i} \times A) + \vec{j} \times (\vec{j} \times A) + \vec{k} \times (\vec{k} \times A) \tag{12}$$

Clearly, C is linear, so it suffices to compute C on the basis $\vec{i}, \vec{j}, \vec{k}$:

$$C\vec{i} = \vec{i} \times (\vec{i} \times \vec{i}) + \vec{j} \times (\vec{j} \times \vec{i}) + \vec{k} \times (\vec{k} \times \vec{i}) = \vec{0} + \vec{j} \times (-\vec{k}) + \vec{k} \times \vec{j} = -\vec{i} - \vec{i} = -2i$$

Likewise, $C\vec{j} = -2\vec{j}$ and $C\vec{k} = -2\vec{k}$. Thus, $C\vec{A} = -2\vec{A}$ for any $\vec{A} \in R^3$.

At some point,we must confront some of the misunderstandings which abound in the physics literature. Let's begin with a comment by Sudarshan [30](page 170):

> ...notions such as "the square x^2" of an element x, -and, more generally, any power, polynomial or power series in one or more elements, - are not defined in a Lie-algebra. In fact, if the existing definition of a product is used,-i.e., the Lie-bracket $[x, y]$- , x^2 etc. would be identically zero!

We confront the same problem addressed by Dirac in his treatment of Bras and Kets, $[x, y]$ represents the operator $[x$, acting on the vector $y]$ Thus "the square x^2" is not defined by $[x, x]$, rather "the square x^2" is $[x, [x$, and it is not necessarily zero. We work towards an example.

The Adjoint representation is:

$$Ad(i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = X_1$$

$$Ad(j) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = X_2$$

$$Ad(k) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = X_3$$

Thus the adjoint representation of the cross product algebra is

$$so(3) = \{X_1, X_2, X_3\}$$

The Adjoint representation, applied to 12, yields for $A \in so(3)$:

$$CA = [X_1, [X_1, A]] + [X_2, [X_2, A]] + [X_3, [X_3, A]] = -2A \tag{13}$$

for any $A \in so(3)$. The placement of brackets in 13 is dictated by the isomorphism between 12 and 13. The operator C is clearly a multiple of the intrinsic Casimir operator since it is quadratic in the generators of $so(3)$ and C commutes with all $X \in so(3)$:

$$[X, CY] = [X, Y] = C\,[X, Y]$$

Since the trace of each term is -2, the normalized intrinsic Casimir operator is:

$$\frac{1}{-2}CA = \frac{1}{-2}([X_1, [X_1, A]] + [X_2, [X_2, A]] + [X_3, [X_3, A]]) = A$$

Suppose C'' is an intrinsic Casimir operator of $so(3)$, then C'' is a linear operator on $so(3)$ and there are coefficients satisfying:

$$C''X_1 = a_{11}X_1 + a_{12}X_2 + a_{13}X_3$$

$$C''X_2 = a_{21}X_1 + a_{22}X_2 + a_{23}X_3$$

$$C''X_3 = a_{31}X_1 + a_{32}X_2 + a_{33}X_3$$

Determining the possible coefficients will allow us to determine which eigenvalues are possible although it will not allow us to determine which brackets actually yield those eigenvalues. We can greatly simplify our calculations once we note a few limitations on the possible images of any intrinsic Casimir operator.

The first limitation on the image of the intrinsic Casimir operator is:

$$[CX, X] = C\,[X, X] = 0 \tag{14}$$

Thus, CX commutes with X.

The second and third go together:

Theorem

If D is an element of the Cartan subalgebra, so is CD.

Proof:

Let D_I be a maximal commuting subalgebra (the Cartan subalgebra), then $C[D_I, D_J] = [D_I, CD_J] = 0$ Thus CD_J commutes with all the D_I and is thus in the Cartan subalgebra.

Theorem

If X is not in the Cartan subalgebra, neither is CX.

Proof: If X is not in the Cartan subalgebra, then there is a $D \in$ the Cartan Subalgebra such that $[D, X] = Y \neq 0$ Then $C[D, X] = [D, CX] = CY \neq 0$

then: $C\,[D, X] = \alpha CX$

$C\,[D, X] = [CD, X] = [D, CX] = \alpha CX$

From $[D, CX] = \alpha CX$ we conclude that CX is not in the Cartan subalgebra since such operators commute. Furthermore, CX is an eigenvector of D with the same eigenvalue as X.

From the first limitation, $[CX, X] = C\,[X, X] = 0$, we see that we can easily find the possible coefficients of an intrinsic Casimir operator of $so(3)$:

$$C''X_1 = aX_1 \tag{15}$$
$$C''X_2 = aX_2$$
$$C''X_3 = aX_3$$

This calculation would make it appear that the operator C'' is merely a multiple of the identity matrix. We have the operator C, so we must ask if C is merely a multiple of the identity matrix as is often claimed and as the above would indicate? To answer this question, let us compute CM where M is a generic 3×3 matrix.

$$[X_1, M] = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}\right]$$

$$\begin{pmatrix} 0 & 0 & 0 \\ -m_{31} & -m_{32} & -m_{33} \\ m_{21} & m_{22} & m_{23} \end{pmatrix} - \begin{pmatrix} 0 & m_{13} & -m_{12} \\ 0 & m_{23} & -m_{22} \\ 0 & m_{33} & -m_{32} \end{pmatrix}$$

$=$

$$\begin{pmatrix} 0 & -m_{13} & m_{12} \\ -m_{31} & -m_{23} - m_{32} & m_{22} - m_{33} \\ m_{21} & m_{22} - m_{33} & m_{23} + m_{32} \end{pmatrix}$$

Then

$$[X_1, [X_1, M]] = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -m_{13} & m_{12} \\ -m_{31} & -m_{23} - m_{32} & m_{22} - m_{33} \\ m_{21} & m_{22} - m_{33} & m_{23} + m_{32} \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & -m_{12} & -m_{13} \\ -m_{21} & -2m_{22} + 2m_{33} & -2m_{23} - 2m_{32} \\ -m_{31} & -2m_{23} - 2m_{32} & 2m_{22} - 2m_{33} \end{pmatrix}$$

Likewise:

$$[X_2, [X_2, M]] = \begin{pmatrix} 2m_{33} - 2m_{11} & -m_{12} & -2m_{13} - 2m_{31} \\ -m_{21} & 0 & -m_{23} \\ -2m_{31} - 2m_{13} & -m_{32} & -2m_{33} + 2m_{11} \end{pmatrix}$$

$$[X_3, [X_3, M]] = \begin{pmatrix} -2m_{11} + 2m_{22} & -2m_{12} - 2m_{21} & -m_{13} \\ -2m_{12} - 2m_{21} & 2m_{11} - 2m_{22} & -m_{23} \\ -m_{31} & -m_{32} & 0 \end{pmatrix}$$

Summing, we obtain:

$$CM = \begin{pmatrix} -4m_{11} + 2m_{22} + 2m_{33} & -4m_{12} - 2m_{21} & -4m_{13} - 2m_{31} \\ -4m_{12} - 2m_{12} & 2m_{11} - 4m_{22} + 2m_{33} & -4m_{23} - 2m_{32} \\ -4m_{31} - 2m_{13} & -2m_{23} - 4m_{32} & 2m_{11} + 2m_{22} - 4m_{33} \end{pmatrix}$$

Thus C is not a multiple of the identity matrix, but the operator C has an eigenvalue of -2 which we need to investigate. Requiring that $CM = -2M$, we have for the off diagonal terms:

$$2m_{12} + m_{21} = m_{12}$$

Therefore,

$$m_{12} = -m_{21}$$

$$2m_{13} + m_{31} = m_{13}$$

Therefore,

$$m_{13} = -m_{31}$$

$$2m_{23} + m_{32} = m_{23}$$

It follows that

$$m_{23} = -m_{32}$$

For the diagonal terms:

$$2m_{11} - m_{22} - m_{33} = m_{11}$$

$$-m_{11} + 2m_{22} - m_{33} = m_{22}$$

$$-m_{11} - m_{22} + 2m_{33} = m_{33}$$

This system has only the solution $m_{11} = m_{22} = m_{33} = 0$

Thus $CM = -2M$ iff M is skew-symmetric, i.e. iff $M \in so(3)$! Hence, in the defining representation, the intrinsic Casimir operator of $so(3)$ characterizes the Lie algebra. This also happens to be the adjoint representation, so it is not clear which is important.

Writing

$$C = ([X_1)^2 + ([X_2)^2 + ([X_3)^2$$

Then it is clear that

$$C = ([X_1)^{2n} + ([X_2)^{2n} + ([X_3)^{2n}$$

is also a Casimir operator for any n. Similiar remarks hold for every Casimir operator we will construct.

3 Third Order Intrinsic Casimir Operators for $so(3)$

In this section we construct third order intrinsic Casimir operators for $so(3)$, contrary to the standard wisdom in which $so(3)$ has no third order Casimir operators. Once again, we begin with the Lie Algebra of vectors on R^3 with the cross product as bracket.

$$\vec{i} \times (\vec{j} \times (\vec{k} \times \vec{i})) = 0 \tag{16}$$

$$\begin{aligned} &\vec{k} \times (\vec{i} \times (\vec{j} \times \vec{i})) \\ &= \vec{k} \times (\vec{i} \times (-\vec{k})) \\ &= \vec{k} \times \vec{j} = -\vec{i} \end{aligned}$$

$$\vec{j} \times (\vec{k} \times (\vec{i} \times \vec{i})) = 0$$

$$\begin{aligned} &\vec{i} \times (\vec{j} \times (\vec{k} \times \vec{j})) \\ &= \vec{i} \times (\vec{j} \times (-\vec{i})) \\ &= \vec{i} \times \vec{k} = -\vec{j} \end{aligned} \tag{17}$$

$$\vec{k} \times (\vec{i} \times (\vec{j} \times \vec{j})) = 0$$

$$\vec{j} \times (\vec{k} \times (\vec{i} \times \vec{j})) = 0$$

$$\vec{i} \times (\vec{j} \times (\vec{k} \times \vec{k})) = 0 \tag{18}$$

$$\vec{k} \times (\vec{i} \times (\vec{j} \times \vec{k})) = 0$$

$$\begin{aligned} \vec{j} \times (\vec{k} \times (\vec{i} \times \vec{k})) &= \vec{j} \times (\vec{k} \times (-\vec{j})) \\ &= \vec{j} \times \vec{i} = -\vec{k} \end{aligned}$$

Define for $\vec{A} \in R^3$,

$$C_-^3 \vec{A} = \vec{i} \times (\vec{j} \times (\vec{k} \times \vec{A})) + \vec{k} \times (\vec{i} \times (\vec{j} \times \vec{A})) + \vec{j} \times (\vec{k} \times (\vec{i} \times \vec{A}))$$

It follows immediately that

$$C_-^3 \vec{A} = -\vec{A}.$$

Now reverse the order of $\vec{i}, \vec{j}$, and $\vec{k}$:

$$\vec{k} \times (\vec{j} \times (\vec{i} \times \vec{i})) = 0 \tag{19}$$

$$\vec{i} \times (\vec{k} \times (\vec{j} \times \vec{i})) = 0$$

$$\begin{aligned} \vec{j} \times (\vec{i} \times (\vec{k} \times \vec{i})) &= \vec{j} \times (\vec{i} \times \vec{j}) \\ &= \vec{j} \times \vec{k} = \vec{i} \end{aligned}$$

$$\vec{k} \times (\vec{j} \times (\vec{i} \times \vec{j})) = \vec{k} \times (\vec{j} \times \vec{k}) \tag{20}$$

$$= \vec{k} \times \vec{i} = \vec{j}$$

$$\vec{i} \times (\vec{k} \times (\vec{j} \times \vec{j})) = 0$$
$$\vec{j} \times (\vec{i} \times (\vec{k} \times \vec{j})) = 0$$

$$\begin{aligned} \vec{k} \times (\vec{j} \times (\vec{i} \times \vec{k})) &= 0 \\ \vec{i} \times (\vec{k} \times (\vec{j} \times \vec{k})) &= \vec{i} \times (\vec{k} \times \vec{i}) \\ &= \vec{i} \times \vec{j} = \vec{k} \\ \vec{j} \times (\vec{i} \times (\vec{k} \times \vec{k})) &= 0 \end{aligned} \tag{21}$$

Define for $\vec{A} \in R^3$

$$C_+^3 \vec{A} = \vec{k} \times (\vec{j} \times (\vec{i} \times \vec{A})) + \vec{i} \times (\vec{k} \times (\vec{j} \times \vec{A})) + \vec{j} \times (\vec{i} \times (\vec{k} \times \vec{A})) \tag{22}$$

It follows immediately that $C_+^3 \vec{A} = \vec{A}$. Furthermore, note that the same calculations show that the three terms of this operator are the projection operators, if $\vec{A} = A_1 \vec{i} + A_2 \vec{j} + A_3 \vec{k}$ then

$$\vec{j} \times (\vec{i} \times (\vec{k} \times \vec{A})) = A_1 \vec{i}$$

$$\vec{k} \times (\vec{j} \times (\vec{i} \times \vec{A})) = A_2 \vec{j}$$
$$\vec{i} \times (\vec{k} \times (\vec{j} \times \vec{A})) = A_3 \vec{k}$$

Passing to the adjoint representation, we have for $A \in so(3)$

$$C^3_- A = [X_1, [X_2, [X_3, A]]] + [X_3, [X_1, [X_2, A]]] + [X_2, [X_3, [X_1, A]]] = -A$$

and

$$C^3_+ A = [X_3, [X_2, [X_1, A]]] + [X_1, [X_3, [X_2, A]]] + [X_2, [X_1, [X_3, A]]] = A$$

Thus C^3_- and C^3_+ are third order intrinsic Casimir operators of eigenvalue type for $so(3)$. But according to the standard treatment of higher order Casimir elements, $so(3)$ should not have any third order Casimir operators which are not multiples of the second order operator. Thus the standard treatment of higher order Casimir operators is fundamentally flawed.

The discovery of these third order operators was accomplished by what appears to be a systematic way to construct candidate third order invariants for any Lie algebra, beginning with the second order invariant:

$$L^2 Y = [X_1, [X_1, Y]] + [X_2, [X_2, Y]] + [X_3, [X_3, Y]]$$

In the term $[X_1, [X_1, Y]]$, for the first X_1, use the substitution: $X_1 = [X_2, X_3]$

Thus

$$[X_1, [X_1, Y]] = [[X_2, X_3], [X_1, Y]]$$

Now when we ask how this term could have arisen from the Jacobi identity in a third order operation we are led to consider the operator:

$$[X_2, [X_3, [X_1, Y]]]$$

In the term $[X_2, [X_2, Y]]$, for the first$[X_2$, use the substitution:

$$X_2 = [X_3, X_1]$$

Thus

$$[X_2, [X_2, Y]] = [[X_3, X_1], [X_2, Y]]$$

This term could have arisen from the Jacobi identity in a third order operation:

$$[X_3, [X_1, [X_2, Y]]] = [[X_3, X_1], [X_2, Y]] + [X_1, [X_3, [X_2, Y]]]$$

$$= [X_2, [X_2, Y]] + [X_1, [X_3, [X_2, Y]]]$$

In the term $[X_3, [X_3, Y]]$, for the first $[X_3$, use the substitution:

$$X_3 = [X_1, X_2]$$

Thus

$$[X_3, [X_3, Y]] = [[X_1, X_2], [X_3, Y]]$$

This term could have arisen from the Jacobi identity from an expansion of the operator:

$$[X_1, [X_2, [X_3, Y]]] = [[X_1, X_2], [X_3, Y]] + [X_2, [X_1, [X_3, Y]]]$$

$$= [X_3, [X_3, Y]] + [X_2, [X_1, [X_3, Y]]]$$

Now we sum the three operators:

$$[X_1, [X_2, [X_3, Y]]] + [X_2, [X_3, [X_1, Y]]] + [X_3, [X_1, [X_2, Y]]]$$

Which we recognize as the operator:

$$C_-^3 Y = [X_1, [X_2, [X_3, Y]]] + [X_3, [X_1, [X_2, Y]]] + [X_2, [X_3, [X_1, Y]]]$$

which we have already shown is an invariant. We could also use this method to arrive at:

$$C_+^3 Y = [X_3, [X_2, [X_1, Y]]] + [X_2, [X_1, [X_3, Y]]] + [X_1, [X_3, [X_2, Y]]]$$

We compute the action on a generic three by three matrix:

$$C_+^3 M = [X_3, [X_2, [X_1, M]]] + [X_2, [X_1, [X_3, M]]] + [X_1, [X_3, [X_2, M]]]$$

by calculating each term separately.

For the first term, we first calculate

$$[X_2, [X_1, M]]$$

$$= \left[\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -m_{13} & m_{12} \\ -m_{31} & -m_{23} - m_{32} & m_{22} - m_{33} \\ m_{21} & m_{22} - m_{33} & m_{23} + m_{32} \end{pmatrix} \right]$$

$=$

$$\begin{pmatrix} m_{21}+m_{12} & m_{22}-m_{33} & m_{23}+m_{32} \\ m_{22}-m_{33} & 0 & m_{31} \\ m_{23}+m_{32} & m_{13} & -m_{21}-m_{12} \end{pmatrix}$$

Then

$$[X_3,[X_2,[X_1,M]]]$$

$$= \left[\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} m_{21}+m_{21} & m_{22}-m_{33} & m_{23}+m_{32} \\ m_{22}-m_{33} & 0 & m_{31} \\ m_{23}+m_{32} & m_{13} & -m_{21}-m_{12} \end{pmatrix} \right]$$

$$= \begin{pmatrix} -2m_{22}+2m_{33} & m_{21}+m_{12} & -m_{13} \\ m_{21}+m_{12} & 2m_{22}-2m_{33} & m_{23}+m_{32} \\ -m_{13} & m_{23}+m_{32} & 0 \end{pmatrix}$$

The second and third terms are calculated in the same manner:

$$[X_1,[X_3,[X_2,M]]]$$

$$= \begin{pmatrix} 0 & -m_{21} & m_{13}+m_{31} \\ -m_{12} & 2m_{11}-2m_{33} & m_{23}+m_{32} \\ m_{13}+m_{31} & m_{23}+m_{32} & -2m_{11}+2m_{33} \end{pmatrix}$$

$$[X_2,[X_1,[X_3,M]]]$$

$$= \begin{pmatrix} 2m_{11}-2m_{22} & m_{21}+m_{12} & m_{13}+m_{31} \\ m_{21}+m_{12} & 0 & -m_{32} \\ m_{13}+m_{31} & -m_{23} & -2m_{11}+2m_{22} \end{pmatrix}$$

Summing the three terms we obtain:

$$C_+^3 M = \begin{pmatrix} 2m_{11}-4m_{22}+2m_{33} & 2m_{12}+m_{21} & 2m_{13}+m_{31} \\ 2m_{21}+m_{12} & 2m_{11}+2m_{22}-4m_{33} & 2m_{23}+m_{32} \\ m_{13}+2m_{31} & m_{23}+2m_{32} & -4m_{11}+2m_{22}+2m_{33} \end{pmatrix}$$

First we note that although $C_+^3 A = A, \forall A \in so(3), C_+^3$ is not just a multiple of the identity matrix.

If it were, we would have $C_+^3 M = M$ as an identity. Also, note that $C_+^3 M$ is not a multiple of CM as calculated in section II. To solve the eigenvalue equation $C_+^3 M = M$ we have 9 equations.

For the diagonal terms:

$$2m_{11} - 4m_{22} + 2m_{33} = m_{11}$$

$$2m_{11} + 2m_{22} - 4m_{33} = m_{22}$$

$$-4m_{11} + 2m_{22} + 2m_{33} = m_{33}$$

which has the unique solution $m_{11} = m_{22} = m_{33} = 0$.

For the off diagonal terms, the equations are the same as $C(M)$ from above. Thus $C_+^3 M = M$ iff $M \in so(3)$.

By the Jacobi identity:

$$[X_1, [X_2, [X_3, Y]]] = [[X_1, X_2], [X_3, Y]] + [X_2, [X_1, [X_3, Y]]]$$

$$= [X_3, [X_3, Y]] + [X_2, [X_1, [X_3, Y]]]$$

Likewise,

$$[X_2, [X_3, [X_1, Y]]] = [[X_2, X_3], [X_1, Y]] + [X_3, [X_2, [X_1, Y]]]$$

$$= [X_1, [X_1, Y]] + [X_3, [X_2, [X_1, Y]]]$$

and

$$[X_3, [X_1, [X_2]]] = [X_2, [X_2, Y]] + [X_1, [X_3, [X_2, Y]]]$$

Summing the three terms we obtain:

$$C_-^3 Y = C^2 Y + C_+^3 \tag{23}$$

The calculations above show that C_+^3 and C_-^3 are each intrinsic Casimir operators, contrary to the standard claim (e.g. Wybourne [34]) that any Casimir operator is a *multiple* of C^2.

The general eigenvalue equation $C_-^3 M = -M$ again holds iff $M \in so(3)$. Wybourne (p. 141) calculates the third order Casimir element of $so(3)$ and claims that C^3 is proportional to C^2. The expression Wybourne calculates is our $C_+^3 + C_-^3$. While it is true that, when operating on elements of the Lie algebra, the eigenvalues of the operators are multiples of each other, when we apply these results to differential geometry, we will be dealing with representations by differential operators where C_+^3 and C_-^3 will be third order differential operators and C^2 will be a second order differential operator.

Now $(C_+^3 + C_-^3)A = 0$ for all $A \in so(3)$, while for an arbitrary matrix M:

$$(C_+^3 + C_-^3)M =$$

$$\begin{pmatrix} -6m_{22} + 6m_{33} & 0 & 0 \\ 0 & 6m_{11} - 6m_{33} & 0 \\ 0 & 0 & -6m_{11} + 6m_{22} \end{pmatrix}$$

The equation $(C_+^3 + C_-^3)M = 0$ has the solution $m_{11} = m_{22} = m_{33} =$ arbitrary constant, while the other entries of M are arbitrary. Thus C_+^3 and C_-^3 each independently characterize the Lie algebra $so(3)$ but the standard Casimir operator $C_+^3 + C_-^3$ does not.

4 Higher Powers and the Theory of Angular Momentum

Revisiting the cross product algebra, we will look at higher powers of the operators. Let's start with one multiplication:

$$\vec{k} \times \vec{i} = \vec{j}$$

Now, let's remultiply by $\vec{k}\times$:

$$\vec{k} \times (\vec{k} \times \vec{i}) = \vec{k} \times \vec{j} = -\vec{i}$$

$$\vec{k} \times (\vec{k} \times (\vec{k} \times \vec{i})) = \vec{k} \times (\vec{k} \times \vec{j}) = \vec{k} \times (-\vec{i}) = \vec{j}$$

$$\vec{k} \times (\vec{k} \times (\vec{k} \times (\vec{k} \times \vec{i}))) = \vec{k} \times (\vec{k} \times (\vec{k} \times \vec{j})) = \vec{k} \times (\vec{k} \times (-\vec{i})) = \vec{k} \times (-\vec{j}) = \vec{i}$$

We need a short hand:

$$(\vec{k}\times)^2(\vec{i}) = -\vec{i}$$

$$(\vec{k}\times)^3(\vec{i}) = \vec{j}$$

$$(\vec{k}\times)^4(\vec{i}) = \vec{i}$$

Then we see that multiplication by $\vec{k}\times$ satisfies an algebraic equation.

$$(\vec{k}\times)^4(\vec{i}) + (\vec{k}\times)^2(\vec{i}) = 0$$

$$(\vec{k}\times)^4(\vec{j}) + (\vec{k}\times)^2(\vec{j}) = 0$$

$$(\vec{k}\times)^4(\vec{k}) + (\vec{k}\times)^2(\vec{k}) = 0$$

This defines a second Casimir operator, with eigenvalue zero. Although a third order equation exists, we go to the fourth power. The motivation for going to the fourth power will be obvious momentarily.

The passage from Lie Algebra to Lie Group requires the exponential function. Having calculated the powers of $\vec{k}\times$, we can exponentiate it:

$$\exp(\gamma\vec{k}\times)\vec{i} = \sum_{n=0}^{\infty} \frac{(\gamma\vec{k}\times)^n}{n!}\vec{j}$$

$$= \vec{i} + \gamma(\vec{j}) + \frac{\gamma^2}{2!}(-\vec{i}) + \frac{\gamma^3}{3!}(-\vec{j}) + \frac{\gamma^4}{4!}(\vec{i}) + \frac{\gamma^5}{5!}(\vec{j}) + \frac{\gamma^6}{6!}(-\vec{i}) + \frac{\gamma^7}{7!}(-\vec{j}) + \frac{\gamma^8}{8!}(\vec{i})\ldots$$

$$= \vec{i}(1 - \frac{\gamma^2}{2!} + \frac{\gamma^4}{4!} - \frac{\gamma^6}{6!} + \frac{\gamma^8}{8!}\ldots) + \vec{j}(\gamma - \frac{\gamma^3}{3!} + \frac{\gamma^5}{5!} - \frac{\gamma^7}{7!}\ldots)$$

$$= \cos(\gamma)\vec{i} + \sin(\gamma)\vec{j}$$

Thus, exponentiating $\vec{k}$ yields a circle in the same way that exponentiating the square root of -1 via de Moivre's formula:

$$\exp(i\theta) = \cos\theta + i\sin\theta$$

Thus, the group action can be calculated without a matrix representation.

Now, define x as the coefficient of $\vec{i}$ and y as the coefficient of $\vec{j}$:

$$x = \cos\gamma$$

$$y = \sin\gamma$$

By the chain rule, we obtain:

$$\frac{\partial}{\partial\gamma} = \frac{\partial x}{\partial\gamma}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\gamma}\frac{\partial}{\partial y}$$

$$= -\sin\gamma\frac{\partial}{\partial x} + \cos\gamma\frac{\partial}{\partial y} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

We can repeat the calculation for $\vec{i}\times$

$$\vec{i} \times \vec{j} = \vec{k}$$
$$\vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}$$
$$(\vec{i}\times)^3\vec{j} = \vec{i} \times (-\vec{j}) = -\vec{k}$$
$$(\vec{i}\times)^4\vec{j} = \vec{i} \times (-\vec{k}) = \vec{j}$$
$$(\vec{i}\times)^5\vec{j} = \vec{i} \times \vec{j} = \vec{k}$$
$$(\vec{i}\times)^6\vec{j} = \vec{i} \times \vec{k} = -\vec{j}$$
$$(\vec{i}\times)^7\vec{j} = \vec{i} \times (-\vec{j}) = -\vec{k}$$
$$(\vec{i}\times)^8\vec{j} = \vec{i} \times (-\vec{k}) = \vec{j}$$

Then we see that multiplication by $\vec{i}\times$ satisfies an algebraic equation.

$$(\vec{i}\times)^4(\vec{i}) + (\vec{i}\times)^2(\vec{i}) = 0$$

$$(\vec{i}\times)^4(\vec{j}) + (\vec{i}\times)^2(\vec{j}) = 0$$

$$(\vec{i}\times)^4(\vec{k}) + (\vec{i}\times)^2(\vec{k}) = 0$$

This defines a third Casimir operator, with eigenvalue zero.

Exponentiating $\vec{i}\times$:

$$\exp(\alpha\vec{i}\times)\vec{j} = \sum_{n=0}^{\infty} \frac{(\alpha\vec{i}\times)^n}{n!}\vec{j}$$

$$= \vec{j} + \alpha(\vec{k}) + \frac{\alpha^2}{2!}(-\vec{j}) + \frac{\alpha^3}{3!}(-\vec{k}) + \frac{\alpha^4}{4!}(\vec{j}) + \frac{\alpha^5}{5!}(\vec{k}) + \frac{\alpha^6}{6!}(-\vec{j}) + \frac{\gamma^7}{7!}(-\vec{k}) + \frac{\gamma^8}{8!}(\vec{j}) \ldots$$

$$= \vec{j}(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \frac{\alpha^8}{8!} \ldots) + \vec{k}(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} \ldots)$$

$$= \cos(\alpha)\vec{j} + \sin(\alpha)\vec{k}$$

Let $y = \cos(\alpha)$, the coefficient of $\vec{j}$ and $z = \sin(\alpha)$. By the chain rule, we obtain:

$$\frac{\partial}{\partial \alpha} = \frac{\partial x}{\partial \alpha}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \alpha}\frac{\partial}{\partial y}$$

$$= -\sin\alpha\frac{\partial}{\partial y} + \cos\alpha\frac{\partial}{\partial z} = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$$

Finally, we repeat the calculation for $\vec{j}\times$

$$\vec{j} \times \vec{k} = \vec{i}$$
$$\vec{j} \times (\vec{j} \times \vec{k}) = \vec{j} \times \vec{i} = -\vec{k}$$
$$(\vec{j}\times)^3\vec{k} = \vec{j} \times (-\vec{k}) = -\vec{i}$$
$$(\vec{j}\times)^4\vec{k} = \vec{j} \times (-\vec{i}) = \vec{k}$$
$$(\vec{j}\times)^5\vec{k} = \vec{j} \times \vec{k} = \vec{i}$$
$$(\vec{j}\times)^6\vec{k} = \vec{j} \times \vec{i} = -\vec{k}$$
$$(\vec{j}\times)^7\vec{k} = \vec{j} \times (-\vec{k}) = -\vec{i}$$
$$(\vec{j}\times)^8\vec{k} = \vec{j} \times (-\vec{i}) = \vec{k}$$

Then we see that multiplication by $\vec{j}$ satisfies the same algebraic equation.

$$(\vec{j}\times)^4(\vec{i}) + (\vec{j}\times)^2(\vec{i}) = 0$$

$$(\vec{j}\times)^4(\vec{j}) + (\vec{j}\times)^2(\vec{j}) = 0$$

$$(\vec{j}\times)^4(\vec{k}) + (\vec{j}\times)^2(\vec{k}) = 0$$

This defines a fourth Casimir operator, with eigenvalue zero.

Now to exponentiate $\vec{j}\times$:

$$\exp(\beta\vec{j}\times)\vec{k} = \sum_{n=0}^{\infty} \frac{(\beta\vec{j}\times)^n}{n!}\vec{k}$$

$$= \vec{k} + \beta(\vec{i}) + \frac{\beta^2}{2!}(-\vec{k}) + \frac{\beta^3}{3!}(-\vec{i}) + \frac{\beta^4}{4!}(\vec{k}) + \frac{\beta^5}{5!}(\vec{i}) + \frac{\beta^6}{6!}(-\vec{k}) + \frac{\gamma^7}{7!}(-\vec{i}) + \frac{\gamma^8}{8!}(\vec{j})\ldots$$

$$= \vec{k}(1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \frac{\beta^6}{6!} + \frac{\beta^8}{8!}\ldots) + \vec{i}(\beta - \frac{\beta^3}{3!} + \frac{\beta^5}{5!} - \frac{\beta^7}{7!}\ldots)$$
$$= \cos(\beta)\vec{k} + \sin(\beta)\vec{i}$$

Let $z = \cos(\beta)$, the coefficient of $\vec{k}$ and $x = \sin(\beta)$. Again using the chain rule, we obtain:

$$\frac{\partial}{\partial\beta} = \frac{\partial z}{\partial\beta}\frac{\partial}{\partial z} + \frac{\partial x}{\partial\beta}\frac{\partial}{\partial x}$$

$$= -\sin\beta\frac{\partial}{\partial z} + \cos\beta\frac{\partial}{\partial x}$$

$$= -x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}$$

We can write the quadratic Casimir as:

$$(\vec{i}\times)^2\vec{A} + (\vec{j}\times)^2\vec{A} + (\vec{k}\times)^2\vec{A} = -2\vec{A}$$

It is easy to prove that:

$$(\vec{i}\times)^{2n}\vec{A} + (\vec{j}\times)^{2n}\vec{A} + (\vec{k}\times)^{2n}\vec{A} = (-1)^n 2\vec{A}$$

Beginning with the cross product, we wound up with three partial derivatives:
$\partial_\alpha = -z\partial_y + y\partial_z$
$\partial_\beta = -x\partial_z + z\partial_x$
$\partial_\gamma = -y\partial_x + x\partial_y$
These operators generate rotations in three dimensions.
The Lie bracket for these operators is

$$[\partial_\alpha, \partial_\beta]f = (\partial_\alpha\partial_\beta - \partial_\beta\partial_\alpha)f = \partial_\alpha\partial_\beta f - \partial_\beta\partial_\alpha f$$

$$= (-z\partial_y + y\partial_z)(-x\partial_z + z\partial_x)f - (-x\partial_z + z\partial_x)(-z\partial_y + y\partial_z)f$$

$$= (-z\partial_y + y\partial_z)(-x\partial_z f + z\partial_x f) - (-x\partial_z + z\partial_x)(-z\partial_y f + y\partial_z f)$$

$$= -z\partial_y(-x\partial_z f) + y\partial_z(-x\partial_z f) - z\partial_y(z\partial_x f) + y\partial_z(z\partial_x f)$$

$$+x\partial_z(-z\partial_y f) - z\partial_x(-z\partial_y f) + x\partial_z(y\partial_z f) - z\partial_x(y\partial_z f)$$

$$= xz\partial_y\partial_z f - xy\partial_z^2 f - z^2\partial_y\partial_x f + y\partial_x f + yz\partial_z\partial_x f$$

$$-x\partial_y f - xz\partial_z\partial_y f - z^2\partial_x\partial_y f + xy\partial_z^2 f) - yz\partial_x\partial_z f$$

$$= -x\partial_y f + y\partial_x f = -\partial_\gamma$$

In the differential operator formalism, the quadratic Casimir operator becomes a second order differential operator:

$$j^2 = \partial_\alpha^2 + \partial_\beta^2 + \partial_\gamma^2$$

The eigenvalue of this differential operator acting on either elements of the Lie algebra or on functions is a conserved quantity.

Likewise, the equations for $\vec{i}$,$\vec{j}$ and $\vec{k}$ become differential operators:

$$\partial_\alpha^4 + \partial_\alpha^2$$

$$\partial_\beta^4 + \partial_\beta^2$$

$$\partial_\gamma^4 + \partial_\gamma^2$$

These are Casimir operators of $so(3)$. The eigenvalue of each of these differential operators acting on either elements of the Lie algebra or on functions is a conserved quantity.

In the standard theory of angular momentum [35] one goes on to

> ...construct states $|jm>$ that are simultaneously eigenfunctions of j^2 and any one component of j, say, j_z...

This procedure is not acceptable considering the arbitrariness of picking one component when classically all three components of angular momentum are conserved. The new formalism introduced here allows for a new theory of angular momentum. The three operators ∂_α^2, ∂_β^2 and ∂_γ^2 mutually commute. Thus a theory of angular momentum based on these operators would allow for the conservation of angular momentum in three directions.

The fourth order differential operators are constructed from these second order mutually commuting operators. They are also invariant operators and a theory of angular momentum built on three invariant operators would have to be better than the current theory of angular momentum which is built on one invariant operator plus a noninvariant operator. Obviously putting in the details of such a theory is a long term project and is not attempted here.

5 The Intrinsic Casimir Operators of $so(2,1)$

All of the constructions for $so(3)$ can also be done for $so(2,1)$. A basis for $so(2,1)$ is:

$$Y_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Y_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The commutation relations for $so(2,1)$ are:

$$[Y_1, Y_2] = -Y_3$$

$$[Y_1, Y_3] = Y_2$$

$$[Y_2, Y_3] = Y_1$$

Computing the traces:

$$[Y_1, [Y_1, Y_2]] = [Y_1, -Y_3] = -Y_2$$

$$[Y_1, [Y_1, Y_3]] = [Y_1, Y_2] = -Y_3$$

Thus, $tr([Y_1, [Y_1,) = -2$

$$[Y_2, [Y_2, Y_1]] = [Y_2, Y_3] = Y_1$$

$$[Y_2, [Y_2, Y_3]] = [Y_2, Y_1] = Y_3$$

Thus, $tr([Y_2, [Y_2,) = 2$

$$[Y_3, [Y_3, Y_1]] = [Y_3, -Y_2] = Y_1$$

$$[Y_3, [Y_3, Y_2]] = [Y_3, -Y_1] = Y_2$$

Thus, $tr([Y_3, [Y_3,) = 2$

The normalized intrinsic Casimir operator for $so(2,1)$ is then:

$$2C^2A = -[Y_1, [Y_1, A]] + [Y_2, [Y_2, A]] + [Y_3, [Y_3, A]]$$

We will work with $2C^2$ instead of C^2 because the lack of fractions makes the calculations easier to follow:

$$2C^2Y_1 = ([Y_2, [Y_2, Y_1] + [Y_3, [Y_3, Y_1]])$$

$$= ([Y_2, Y_3] + [Y_3, -Y_2])$$

$$= (Y_1 + Y_1) = 2Y_1$$

$$2C^2Y_2 = (-[Y_1, [Y_1, Y_2] + [Y_3, [Y_3, Y_2]])$$

$$= (-[Y_1, -Y_3] + [Y_3, -Y_1]) = 2Y_2$$

$$2C^2Y_3 = (-[Y_1,[Y_1,Y_3]] + [Y_2,[Y_2,Y_3]])$$

$$= (-[Y_1,-Y_2] + [Y_2,Y_1]) = 2Y_3$$

Thus, the eigenvalue of C^2 is 1.

Acting on the generic 3×3 matrix, M we leave it to the reader to verify that

$$2C^2M =$$

$$\begin{pmatrix} 4m_{11} - 2m_{22} - 2m_{33} & 4m_{12} + 2m_{21} & 4m_{13} - 2m_{31} \\ 4m_{21} + 2m_{12} & -2m_{11} + 4m_{22} - 2m_{33} & 4m_{23} - 2m_{32} \\ 4m_{13} - 2m_{31} & -2m_{23} + 4m_{32} & -2m_{11} - 2m_{22} + 4m_{33} \end{pmatrix}$$

There are three things to note about this calculation. Acting on generic matrices, the intrinsic Casimir operators of $so(3)$ and $so(2,1)$ give different results. The actions of C and C^2 on matrices are not representable by matrix multiplication as 3×3 matrices. Also, if $2C^2M = 2M$, we can solve the resulting equations just as we did for so(3) .

The equations for the diagonal terms of $2C^2M = 2M$ are identical to those for $CM = -2M$, thus the diagonal terms are zero. The off diagonal equations lead to

$$m_{12} + m_{21} = 0$$

$$m_{13} - m_{31} = 0$$

$$m_{23} - m_{32} = 0$$

Thus $C^2(M) = M$ iff $M \in so(2,1)$.

Hence, as was the case for so(3), in the defining representation, the intrinsic Casimir operator of $so(2,1)$ characterizes the Lie algebra. Again, this also happens to be the adjoint representation, so it is not clear which is important.

Our next goal is to generalize the construction of the third order intrinsic Casimir operators for $so(3)$ to $so(2,1)$. We would like for an identity like 23 to hold. Again, we calculate using the Jacobi identity:

$$[Y_1,[Y_2,[Y_3,A]]] = [[Y_1,Y_2],[Y_3,A]] + [Y_2,[Y_1,[Y_3,A]]]$$

$$= -[Y_3,[Y_3,A]] + [Y_2,[Y_1,[Y_3,A]]]$$

Likewise

$$[Y_2,[Y_3,[Y_1,A]]] = [[Y_2,Y_3],[Y_1,A]] + [Y_3,[Y_2,[Y_1,A]]]$$
$$= [Y_1,[Y_1,A]] + [Y_3,[Y_2,[Y_1,A]]]$$

and

$$[Y_3,[Y_1,[Y_2,A]]] = [[Y_3,Y_1],[Y_2,A]] + [Y_1,[Y_3,[Y_2,A]]]$$
$$= -[Y_2,[Y_2,A]] + [Y_1,[Y_3,[Y_2,A]]]$$

Recall that

$$2C^2A = -[Y_1,[Y_1,A]] + [Y_2,[Y_2,A]] + [Y_3,[Y_3,A]]$$

If we define for $A \in so(2,1)$:

$$C_-^{3'}A = [Y_1,[Y_2,[Y_3,A]]] + [Y_3,[Y_1,[Y_2,A]]] + [Y_2,[Y_3,[Y_1,A]]]$$

and

$$C_+^{3'}A = [Y_3,[Y_2,[Y_1,A]]] + [Y_1,[Y_3,[Y_2,A]]] + [Y_2,[Y_1,[Y_3,A]]]$$

Then adding the three above terms we have

$$C_-^{3'}A = -2C_2A + C_+^{3'}A$$

And we calculate:

$$C_-^{3'}Y_1 = [Y_1,[Y_2,[Y_3,Y_1]]] + [Y_3,[Y_1,[Y_2,Y_1]]] + [Y_2,[Y_3,[Y_1,Y_1]]]$$
$$= 0 + [Y_3,[Y_1,Y_3]] + 0$$
$$= [Y_3,Y_2] = -Y_1$$

$$C_-^{3'}Y_2 = [Y_1,[Y_2,[Y_3,Y_2]]] + [Y_3,[Y_1,[Y_2,Y_2]]] + [Y_2,[Y_3,[Y_1,Y_2]]]$$
$$= +[Y_1,[Y_2,-Y_1]] + 0 + 0$$
$$= -[Y_1,Y_3] = -Y_2$$

$$C_-^{3'}Y_3 = [Y_1,[Y_2,[Y_3,Y_3]]] + [Y_3,[Y_1,[Y_2,Y_3]]] + [Y_2,[Y_3,[Y_1,Y_3]]]$$

$$= +0 + 0 + [Y_2, [Y_3, Y_2]]$$
$$= [Y_2, -Y_1] = -Y_3$$

We have $C_-^{3'} A = -A$ for all $A \in so(2,1)$.

$$C_+^{3'} Y_1 = [Y_3, [Y_2, [Y_1, Y_1]]] + [Y_1, [Y_3, [Y_2, Y_1]]] + [Y_2, [Y_1, [Y_3, Y_1]]]$$
$$= [Y_2, [Y_1, -Y_2]]$$
$$= [Y_2, Y_3] = Y_1$$

$$C_+^{3'} Y_2 = [Y_3, [Y_2, [Y_1, Y_2]]] + [Y_1, [Y_3, [Y_2, Y_2]]] + [Y_2, [Y_1, [Y_3, Y_2]]]$$
$$= [Y_3, [Y_2, -Y_3]] = [Y_3, -Y_1]] = Y_2$$

$$C_+^{3'} Y_3 = [Y_3, [Y_2, [Y_1, Y_3]]] + [Y_1, [Y_3, [Y_2, Y_3]]] + [Y_2, [Y_1, [Y_3, Y_3]]]$$
$$= [Y_1, [Y_3, Y_1]] = [Y_1, -Y_2] = Y_3$$

We have $C_+^{3'} A = A$ for all $A \in so(2,1)$.

Thus $C_-^{3'} A = A$ and $C_+^{3'}$ are verified as third order intrinsic Casimir operators of eigenvalue type for $so(2,1)$. But according to the standard treatment of higher order Casimir elements, $so(2,1)$ should not have any third order Casimir operators which are not a multiple of the second order Casimir operator. Thus, again, the standard treatment of Casimir operators is fundamentally flawed. The construction of higher order intrinsic operators follows the pattern of $so(3)$ and is left to the reader.

6 The Intrinsic Casimir Operators of $sl(2,R)$

A basis for the Lie Algebra sl(2, R) consists of the matrices:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The commutation relations are then

$$[H, X^+] = 2X^+$$

$$[H, X^-] = -2X^-$$

$$[X^+, X^-] = H$$

Lang [16] (p. 194) defines the Casimir operator for $sl(2, R)$ as

$$\Omega = H^2 + 2(X^+X^- + X^-X^+)$$

Which we interpret as:

$$\Omega A = [H, [H, A]] + 2[X^+, [X^-, A]] + 2[X^-, [X^+, A]]$$

Calculating via our method we have

$$\Omega X^+ = [H, [H, X^+]] + 2[X^+, [X^-, X^+]]$$

$$= [H, 2X^+] + 2[X^+, -H] = 4X^+ + 4X^+ = 8X^+$$

$$\Omega X^- = [H, [H, X^-]] + 2[X^-, [X^+, X^-]]$$

$$= 4X^- + 4X^- = 8X^-$$

$$\Omega H = 2[X^+, [X^-, H]] + 2[X^-, [X^+, H]]$$

$$= 2[X^+, 2X^-] + 2[X^-, -2X^+] = 8H$$

Thus the eigenvalue of this Intrinsic Casimir operator is 8.
Humphreys [10](pp. 27-28) defines the Casimir operator of $sl(2, R)$ as

$$\phi = \frac{\Omega}{2}$$

and claims that:

$$\phi = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$$

which is clearly not correct since our calculation shows that the eigenvalue of ϕ is 4, not $\frac{3}{2}$. Also, the intrinsic Casimir operator is not merely a multiple of the identity matrix.

Actually, none of the above calculations gives the intrinsic Casimir operator, i.e. the normalized intrinsic Casimir operator. From the commutation relations, we obtain:

$$[H, [H, X^+]] = [H, 2X^+] = 4X^+$$

$$[H, [H, X^-]] = [H, -2X^-] = 4X^-$$

$Trace([H, [H,) = 8$

$$[X^+, [X^-, H]] = [X^+, 2X^-] = 2H$$

$$\begin{aligned} [X^+, [X^-, X^+]] &= [X^+, -H] \\ &= 2X^+ \end{aligned}$$

Thus $Trace([X^+, [X^-,) = 4$

$$[X^-, [X^+, X^-]] = [X^-, H]$$

$$\begin{aligned} &= 2X^-[X^-, [X^+, H]] \\ &= [X^-, -2X^+] = 2H \end{aligned}$$

And so, $Trace([X^-, [X^+,) = 4$

Thus the normalized intrinsic Casimir operator of $sl(2, R)$ is:

$$CW = \frac{1}{8}[H, [H, W]] + \frac{1}{4}[X^+, [X^-, W]] + \frac{1}{4}[X^-, [X^+, W]] = W$$

and the normalized intrinsic Casimir operator of $sl(2, R)$ acting on an element of $sl(2, R)$ is the identity operator.

After another calculation, the reader can show that acting on an arbitrary 2 by 2 matrix

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

$$\Omega M = \begin{pmatrix} 4m_{11} - 4m_{22} & 8m_{12} \\ 8m_{21} & 4m_{22} - 4m_{11} \end{pmatrix}$$

Setting $\Omega M = 8M$, we obtain

$$4m_{11} - 4m_{22} = 8m_{11}$$

$$8m_{12} = 8m_{12}$$

$$8m_{21} = 8m_{21}$$

$$4m_{22} - 4m_{11} = 8m_{22}$$

These equations require that m_{12} and m_{21} be arbitrary while $m_{22} = -m_{11}$.

Thus $\Omega M = 8M$ implies $M \in sl(2, R)$. And again, at least in this representation, the intrinsic Casimir operator characterizes the Lie Algebra. Within the standard representation, $sl(2, R)$ is the eigenspace of the intrinsic Casimir operator.

Now we will investigate the existence of a third order intrinsic Casimir Operator for $sl(2, R)$. We make the same type of substitution into the second order intrinsic Casimir operator of $sl(2, R)$ as we did for $so(3)$.

$$\Omega A = [H, [H, A]] + 2[X_+, [X_-, A]] + 2[X_-, [X_+, A]]$$

$$= [[X_+, X_-], [H, A]] + [[H, X_+], [X_-, A]] + [[X_-, H], [X_+, A]]$$

Then we note that these terms appear in the expansion of

$$C_3 A = [X_+, [X_-, [H, A]]] + [[H, [X_+, [X_-, A]]] + [X_-, [H, [X_+, A]]]$$

And we check to see if we have indeed constructed an intrinsic Casimir operator:

$$C_3 H = [X_+, [X_-, [H, H]]] + [[H, [X_+, [X_-, H]]] + [X_-, [H, [X_+, H]]]$$

$$= [H, [X_+, 2X_-]] + [X_-, [H,_- 2X_+]]$$

$$= 2[H, H] + -2[X_-, 2X_+] = 4H$$

$$C_3 X_- = [X_+, [X_-, [H, X_-]]] + [[H, [X_+, [X_-, X_-]]] + [X_-, [H, [X_+, X_-]]]$$

$$= [X_+, [X_-, -2X_-]] + [X_-, [H, H]] = 0$$

$$C_3 X_+ = [X_+, [X_-, [H, X_+]]] + [[H, [X_+, [X_-, X_+]]] + [X_-, [H, [X_+, X_+]]]$$

$$= [X_+, [X_-, 2X_+]] + [[H, [X_+, -H]]$$

$$= 2[X_+, -H] + [H, 2X_+] = 4X_+$$

Since C_3 is not an intrinsic Casimir operator for $sl(2, R)$, the procedure is not universally applicable. Suppose we experiment with a new basis:

$$J = X_+ + X_-$$

$$K = X_+ - X_-$$

We obtain the new Lie brackets from the old:

$$[H, J] = 2K$$

$$[H, K] = 2J$$

$$[J, K] = [X_+ + X_-, X_+ - X_-]$$

$$= [X_-, X_+] - [X_+ X_-] = -2H$$

The construction of the intrinsic Casimir operator requires the trace of each quadratic term:

$$[H, [H, J] = [H, 2K] = 4J$$

$$[H, [H, K] = [H, 2J] = 4K$$

$$tr([H, [H,) = 8.$$

$$[J, [J, H] = [J, -2K] = 4H$$

$$[J, [J, K]] = [J, -2H] = 4K$$

$$tr([J, [J,) = 8.$$

$$[K, [K, H]] = [K, -2J] = -4H$$

$$[K, [K, J]] = [K, 2H] = -4J$$

$$tr([K, [K,) = -8.$$

Since the absolute value of the traces is the same, we will not include it, but we need to have a negative sign with the [K,[K, term. The intrinsic Casimir operator in the new basis is:

$$CA = [H, [H, A]] + [J, [J, A]] - [K, [K, A]]$$

And we calculate:

$$[J, [J, H] = [J, -2K] = 4H$$

$$-[K, [K, H]] = -[K, -2J] = 4H$$

$$CH = 8H$$

$$[H, [H, J]] = [H, 2K] = 4J$$

$$-[K, [K, J]] = -[K, 2H] = 4J$$

$$CJ = 8J$$

$$[H, [H, K]] = [H, 2J] = 4K$$

$$[J, [J, K]] = [J, -2H] = 4K$$

$$CK = 8K$$

Thus, we do have an intrinsic Casimir operator. Note that after dividing by 8, the normalized intrinsic Casimir operator has eigenvalue 1.

In order to avoid fractions, we will apply the transition to third order trick to a multiple of the intrinsic Casimir operator:

$$2CA = [2H, [H, A]] + [2J, [J, A]] - [2K, [K, A]]$$

Substituting $2H = [K, J]$, $2J = [H, K]$ and $-2K = [J, K]$, we obtain

$$2CA = [2H, [H, A]] + [2J, [J, A]] - [2K, [K, A]]$$

$$= [[K, J], [H, A]] + [[H, K], [J, A]] + [[J, H], [K, A]]$$

This arises in the expansion of:

$$C^3_+ A = [K, [J, [H, A]]] + [H, [K, [J, A]]] + [J, [H, [K, A]]]$$

The candidate third order intrinsic Casimir operator checks out:

$$C^3_+ H = [K, [J, [H, H]]] + [H, [K, [J, H]]] + [J, [H, [K, H]]]$$

$$= 0 + [H, [K, -2K]] + [J, [H, -2J]]$$
$$= -2[J, 2K] = 8H$$
$$C_+^3 J = [K, [J, [H, J]]] + [H, [K, [J, J]]] + [J, [H, [K, J]]]$$
$$= [K, [J, 2K]] + 0 + [J, [H, 2H]]]$$
$$= 2[K, -2H] = 8J$$
$$C_+^3 K = [K, [J, [H, K]]] + [H, [K, [J, K]]] + [J, [H, [K, K]]]$$
$$= [K, [J, 2J]] + [H, [K, -2H]]$$
$$= -2[H, -2J] = 8K$$

As with the third order intrinsic Casimir operator of $so(3)$ we try reversing the order of the operators and define:

$$C_-^3 A = [H, [J, [K, A]]] + [J, [K, [H, A]]] + [K, [H, [J, A]]]$$

$$C_-^3 H = [H, [J, [K, H]]] + [J, [K, [H, H]]] + [K, [H, [J, H]]]$$
$$= [H, [J, -2J]] + 0 + [K, [H, -2K]]$$
$$= -2[K, 2J] = -8H$$

$$C_-^3 J = [H, [J, [K, J]]] + [J, [K, [H, J]]] + [K, [H, [J, J]]]$$

$$= [H, [J, 2H]] + [J, [K, 2K]]] + 0$$
$$= 2[H, -2K] = -8J$$
$$C_-^3 K = [H, [J, [K, K]]] + [J, [K, [H, K]]] + [K, [H, [J, K]]]$$
$$= [J, [K, 2J]] + [K, [H, -2H]]$$
$$= 2[J, 2H] = -8K$$

This shows that C_-^3 is another third order intrinsic Casimir operator.

7 The Intrinsic Casimir Operators for $so(3,1)$

The standard theory of Casimir operators predicts that the Lorentz Lie algebra, $so(3,1)$ possesses two, both of order 2. In this section, we construct several intrinsic Casimir operators of order 2, providing even more evidence that the standard theory is fatally flawed. A standard basis for $so(3,1)$ consists of the six matrices

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$N_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$N_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The X_i generate $so(3)$ and are compact, while the N_i are the noncompact generators. The brackets of $so(3,1)$ are given in Table I.

TABLE I The commutators for $so(3,1)$

$[X_1, X_2] = X_3$	$[X_2, X_3] = X_1$	$[X_3, X_1] = X_2$
$[X_1, N_1] = 0$	$[X_2, N_1] = -N_3$	$[X_3, N_1] = N_2$
$[X_1, N_2] = N_3$	$[X_2, N_2] = 0$	$[X_3, N_2] = -N_1$
$[X_1, N_3] = -N_2$	$[X_2, N_3] = N_1$	$[X_3, N_3] = 0$
$[N_1, N_2] = -X_3$	$[N_3, N_1] = -X_2$	$[N_2, N_3] = -X_1$

The standard Casimir operators of $so(3,1)$ are (a scalar multiple of):

$$C^2 = X_1^2 + X_2^2 + X_3^2 - N_1^2 - N_2^2 - N_3^2$$

and

$$C' = X_1 N_1 + X_2 N_2 + X_3 N_3$$

To construct the normalized intrinsic Casimir operator, we first need to calculate the traces:

$$[X_1, [X_1, X_2]] = [X_1, X_3] = -X_2$$

$$[X_1, [X_1, X_3]] = [X_1, -X_2] = -X_3$$

$$[X_1, [X_1, N_2]] = [X_1, N_3] = -N_2$$

$$[X_1, [X_1, N_3]] = [X_1, -N_2] = -N_3$$

$$Tr([X_1, [X_1,) = -4$$

Likewise:

$$Tr([X_2, [X_2,) = -4$$

$$Tr([X_3, [X_3,) = -4$$

$$[N_1, [N_1, X_2]] = [N_1, N_3] = X_2$$

$$[N_1, [N_1, X_3]] = [N_1, -N_2] = X_3$$

$$[N_1, [N_1, N_2]] = [N_1, -X_3] = N_2$$

$$[N_1, [N_1, N_3]] = [N_1, X_2] = N_3$$

$$Tr([N_1, [N_1,) = 4$$

Likewise:

$$Tr([N_2, [N_2,) = 4$$

$$Tr([N_3, [N_3,) = 4$$

Thus the normalized intrinsic Casimir operator is:

$$C_n^2 Y = -\frac{1}{4}[X_1, [X_1, Y]] - \frac{1}{4}[X_2, [X_2, Y]] - \frac{1}{4}[X_3, [X_3, Y]] + \frac{1}{4}[N_1, [N_1, Y]]$$

$$+\frac{1}{4}[N_2, [N_2, Y]] + \frac{1}{4}[N_3, [N_3, Y]]$$

However, for typographical reasons we will work with

$$C^2 Y = [X_1, [X_1, Y]] + [X_2, [X_2, Y]] + [X_3, [X_3, Y]]$$

$$-[N_1, [N_1, Y]] - [N_2, [N_2, Y]] - [N_3, [N_3, Y]]$$

We will see, by direct computation, that C^2 is actually the sum of two simpler intrinsic Casimir operators while C' is a different type of Casimir operator than those encountered so far.

Just as we did for $so(3)$, we look for the possible coefficients of an intrinsic Casimir operator on $so(3,1)$:

$$CX_1 = a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + a_{14}N_1 + a_{15}N_2 + a_{16}N_3$$

$$CX_2 = a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + a_{24}N_1 + a_{25}N_2 + a_{26}N_3$$

$$CX_3 = a_{31}X_1 + a_{32}X_2 + a_{33}X_3 + a_{34}N_1 + a_{35}N_2 + a_{36}N_3$$

$$CN_1 = a_{41}X_1 + a_{42}X_2 + a_{43}X_3 + a_{44}N_1 + a_{45}N_2 + a_{46}N_3$$

$$CN_2 = a_{51}X_1 + a_{52}X_2 + a_{53}X_3 + a_{54}N_1 + a_{55}N_2 + a_{56}N_3$$

$$CN_3 = a_{61}X_1 + a_{62}X_2 + a_{63}X_3 + a_{64}N_1 + a_{65}N_2 + a_{66}N_3$$

The intrinsic Casimir operator must satisfy the first limitation on intrinsic Casimir operators:

$$CX_1 = a_{11}X_1 + a_{14}N_1$$

$$CX_2 = a_{22}X_2 + a_{25}N_2$$

$$CX_3 = a_{33}X_3 + a_{36}N_3$$

$$CN_1 = a_{41}X_1 + a_{44}N_1$$
$$CN_2 = a_{52}X_2 + a_{55}N_2$$
$$CN_3 = a_{63}X_3 + a_{66}N_3$$

Starting with the first relation:

$$CX_1 = a_{11}X_1 + a_{14}N_1$$

We bracket each term on the right with X_2:

$$C[X_1, X_2] = a_{11}[X_1, X_2] + a_{14}[N_1, X_2]$$

obtaining:

$$CX_3 = a_{11}X_3 + a_{14}N_3$$

thus we conclude that: $a_{33} = a_{11}$ and $a_{14} = a_{36}$
Again, we bracket the first relationship on the left with X_3:

$$C[X_3, X_1] = a_{11}[X_3, X_1] + a_{14}[X_3, N_1]$$

obtaining:

$$CX_2 = a_{11}X_2 + a_{14}N_2$$

Thus we have shown that $a_{22} = a_{11}$ and $a_{14} = a_{26}$.
Beginning again with the first relation, we bracket each term with N_2:

$$C[X_1, N_2] = a_{11}[X_1, N_2] + a_{14}[N_1, N_2]$$
$$CN_3 = a_{11}X_3 + a_{14}N_3$$

From which we conclude that
$a_{63} = a_{11}$ and $a_{14} = a_{66}$.
Applying the same technique again, we bracket the first relationship with N_3:

$$C[X_1, N_3] = a_{11}[X_1, N_3] + a_{14}[N_1, N_3]$$

obtaining:

$$CN_2 = a_{11}N_2 + a_{14}X_2$$

This final relationship we bracket with X_3:

$$C[N_2, X_3] = a_{11}[N_2, X_3] + a_{14}[X_2, X_3]$$

obtaining:

$$CN_1 = a_{11}N_1 + a_{14}X_1$$

The rest of the coefficients are zero.

Let

$$a = a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66}$$

and

$$b = a_{14} = a_{25} = a_{36} = -a_{63} = -a_{52} = -a_{41}$$

We are left with:

$$CX_1 = aX_1 + bN_1 \tag{24}$$
$$CX_2 = aX_2 + bN_2$$
$$CX_3 = aX_3 + bN_3$$
$$CN_1 = -bX_1 + aN_1$$
$$CN_2 = -bX_2 + aN_2$$
$$CN_3 = -bX_3 + aN_3$$

There are then two independent possibilities for the image of C:

$$C_1X_1 = aX_1 \tag{25}$$

$$C_1X_2 = aX_2$$

$$C_1X_3 = aX_3$$
$$C_1N_1 = aN_1$$
$$C_1N_2 = aN_2$$

$$C_1 N_3 = a N_3$$

and:

$$C_2 X_1 = b N_1 \tag{26}$$

$$C_2 X_2 = b N_2$$

$$C_2 X_3 = b N_3$$

$$C_2 N_1 = -b X_1$$

$$C_2 N_2 = -b X_2$$

$$C_2 N_3 = -b X_3$$

The limitations on intrinsic Casimir operators are necessary, but not sufficient. We have not yet constructed an operator like C_2, but assuming that one exists we now show that C_2 satisfies the conditions of 5.

$$C_2[X_1, X_2] = [X_1, C_2 X_2] = [C_2 X_1, X_2]$$

$$C_2 X_3 = [X_1, b N_2] = [b N_1, X_2]$$

$$b N_3 = b N_3 = b N_3$$

A similar calculation may be done for each entry in Table I.

Thus, we have proven that C_2 commutes with the action of the Lie algebra. This new type of intrinsic Casimir operator, C_2, is not "a multiple of the identity", but we still need to construct operators from the generators of so(3,1) with this type of action.

Define:

$$L^2 Y = [X_1, [X_1, Y]] + [X_2, [X_2, Y]] + [X_3, [X_3, Y]]$$

If X_I is an element of the $so(3)$ subalgebra of $so(3,1)$, it follows from previous work that

$$L^2 X_I = -2 X_I$$

Let us compute the action of L^2 on the non-compact generators:

$$L^2 N_1 = [X_1, [X_1, N_1]] + [X_2, [X_2, N_1]] + [X_3, [X_3, N_1]]$$

$$= [X_2, -N_3] + [X_3, N_2] = -N_1 - N_1 = -2N_1$$
$$L^2N_2 = [X_1, [X_1, N_2]] + [X_2, [X_2, N_2]] + [X_3, [X_3, N_2]]$$
$$= [X_1, N_3] + [X_3, -N_1] = -N_2 - N_2 = -2N_2$$
$$L^2N_3 = [X_1, [X_1, N_3]] + [X_2, [X_2, N_3]] + [X_3, [X_3, N_3]]$$
$$= [X_1, -N_2] + [X_2, N_1] = -N_3 - N_3 = -2N_3$$

Thus $L^2Y = -2Y \quad \forall Y \in so(3,1)$. Hence, surprisingly, L^2 is an intrinsic Casimir operator for $so(3,1)$. This was unexpected because the operator L^2 is only half of the standard Casimir operator of $so(3,1)$. This result is physically interesting because it shows that angular momentum is conserved even under the action of $so(3,1)$. What can we say about the other half of the standard Casimir operator? To this end, we define:

$$N^2Y = [N_1, [N_1, Y]] + [N_2, [N_2, Y]] + [N_3, [N_3, Y]]$$

We calculate:

$$N^2X_1 = [N_1, [N_1, X_1]] + [N_2, [N_2, X_1]] + [N_3, [N_3, X_1]]$$
$$= [N_2, -N_3] + [N_3, +N_2] = 2X_1$$
$$N^2X_2 = [N_1, [N_1, X_2]] + [N_2, [N_2, X_2]] + [N_3, [N_3, X_2]]$$
$$= [N_1, N_3] + [N_3, -N_1] = 2X_2$$
$$N^2X_3 = [N_1, [N_1, X_3]] + [N_2, [N_2, X_3]] + [N_3, [N_3, X_3]]$$
$$= [N_1, -N_2] + [N_2, N_1] = 2X_3$$
$$N^2N_1 = [N_1, [N_1, N_1]] + [N_2, [N_2, N_1]] + [N_3, [N_3, N_1]]$$
$$= [N_2, X_3] + [N_3, -X_2] = 2N_1$$
$$N^2N_2 = [N_1, [N_1, N_2]] + [N_2, [N_2, N_2]] + [N_3, [N_3, N_2]]$$
$$= [N_1, -X_3] + [N_3, X_1] = 2N_2$$
$$N^2N_3 = [N_1, [N_1, N_3]] + [N_2, [N_2, N_3]] + [N_3, [N_3, N_3]]$$
$$= [N_1, X_2] + [N_2, -X_1] = 2N_3$$

Thus $N^2Y = 2Y$ for all $Y \in so(3,1)$ Hence, N^2 is an intrinsic Casimir operator for $so(3,1)$. We have seen that L^2 and N^2 are both intrinsic Casimir operators for $so(3,1)$ while the standard approach admits only $L^2 - N^2$.

Neither L^2 nor N^2, considered as in the standard approach, is a multiple of the identity. Hence having a polynomial of matrices which sum to the identity is not necessary to construct an intrinsic Casimir operator.

Acting on an arbitrary 4 by 4 matrix M, neither L^2 nor N^2 is the identity operator. Requiring

$$L^2 M = -2M$$

does not M to be in $so(3,1)$. Requiring

$$N^2 M = 2M$$

does not force M to be in $so(3,1)$. However, requiring both

$$L^2 M = -2M$$

and

$$N^2 M = 2M$$

does force $M \in so(3,1)$. Or, alternatively, requiring that

$$(L^2 - N^2)M = -4M$$

forces $M \in so(3,1)$. These calculations are so similar to those done for other algebras that we leave the details to the interested reader.

Define

$$C_{2n} = ([X_1)^{2n} + ([X_2)^{2n} + ([X_3)^{2n} - ([N_1)^{2n} - ([N_2)^{2n} - ([N_3)^{2n}$$

Another routine calculation shows that we have an infinite number of intrinsic Casimir operators for $so(3,1)$.

The second Casimir operator of $so(3,1)$ in the standard approach translates to

$$C'Y = [X_1, [N_1, Y]] + [X_2, [N_2, Y]] + [X_3, [N_3, Y]]$$

We calculate:

$$C'X_1 = [X_1, [N_1, X_1]] + [X_2, [N_2, X_1]] + [X_3, [N_3, X_1]]$$

$$= [X_2, -N_3] + [X_3, N_2] = -N_1 - N_1 = -2N_1$$

Likewise,

$$C'X_2 = -2N_2$$

$$C'X_3 = -2N_3$$
$$C'N_1 = 2X_1$$
$$C'N_2 = 2X_2$$
$$C'N_3 = 2X_3$$

In short, C' satisfies:

$$C'X_I = -2N_I$$
$$C'N_I = 2X_I$$

And C' is the second sort of intrinsic Casimir operator, i.e. in 26, we have b = -2. Thus, C' satisfies the conditions of 5 for each entry in Table I.

Note that:

$$C'(X_I + iN_I) = 2i(X_I + iN_I)$$
$$C'(X_I - iN_I) = -2i(X_I - iN_I)$$

showing that we can put C' in eigenvalue form, if we leave the Lie algebra for its complexification. However, there are then two different eigenvalues. Thus, although C' commutes with the action of the Lie algebra, it is not in any sense "a multiple of the identity."

Since X_I and N_I are not eigenvectors of C' , C' is not a Casimir operator of the form we are used to seeing. If $CY = \alpha Y$ for all Y in a Lie algebra, we will call C an intrinsic Casimir operator of eigenvalue type. The standard approach does recognize a difference and calls C' a "pseudoscalar." The above calculations show there are other types of intrinsic Casimir operators. We will call C' an intrinsic Casimir operator of complex structure type because if we define $J = \frac{1}{2}C'$ then:

$$JX_I = -N_I$$
$$JN_I = X_I$$

so

$$J^2X_I = -X_I$$
$$J^2N_I = -N_I$$

Thus J is a complex structure, and we have:

$$J(X_I + iN_I) = i(X_I + iN_I)$$

$$J(X_I - iN_I) = -i(X_I - iN_I)$$

This construction works simply because $so(3,1)$ is isomorphic to the complexification of $so(3)$: $so(3) + iso(3)$. Clearly, analogous complex structures can be constructed on the complexification of any simple real Lie algebra.

Using this observation about the complexification allows us to construct three more new Casimir operators for $so(3,1)$. We will show that (no sum):

$$[X_I, [X_I, Y]] + [N_I, [N_I, Y]] = 0$$

for $I = 1, 2, 3$ for all $Y \in so(3,1)$.

The proof is by direct calculation:

$$[X_1, [X_1, X_1]] + [N_1, [N_1, X_1]] = 0$$

$$[X_1, [X_1, X_2]] + [N_1, [N_1, X_2]]$$
$$= [X_1, X_3] + [N_1, N_3] = -X_2 + X_2 = 0$$
$$[X_1, [X_1, X_3]] + [N_1, [N_1, X_3]]$$
$$= [X_1, -X_2] + [N_1, N_2] = X_3 - X_3 = 0$$
$$[X_1, [X_1, N_1]] + [N_1, [N_1, N_1]] = 0$$
$$[X_1, [X_1, N_2]] + [N_1, [N_1, N_2]]$$
$$= [X_1, N_3] + [N_1, -X_3] = -N_2 + N_2 = 0$$
$$[X_1, [X_1, N_3]] + [N_1, [N_1, N_3]]$$
$$= [X_1, N_2] + [N_1, X_2] = N_3 - N_3 = 0$$

$$[X_2, [X_2, X_1]] + [N_2, [N_2, X_1]]$$

$$= [X_2, -X_3] + [N_2, -N_3] = -X_1 + X_1 = 0$$
$$[X_2, [X_2, X_2]] + [N_2, [N_2, X_2]] = 0$$
$$[X_2, [X_2, X_3]] + [N_2, [N_2, X_3]]$$
$$= [X_2, X_1] + [N_2, N_1] = -X_3 + X_3 = 0$$

$$[X_2, [X_2, N_1]] + [N_2, [N_2, N_1]]$$
$$= [X_2, -N_3] + [N_2, X_3] = -N_1 + N_1 = 0$$
$$[X_2, [X_2, N_2]] + [N_2, [N_2, N_2]] = 0$$
$$[X_2, [X_2, N_3]] + [N_2, [N_2, N_3]]$$
$$= [X_2, N_1] + [N_2, -X_1] = -N_3 + N_3 = 0$$
$$[X_3, [X_3, X_1]] + [N_3, [N_3, X_1]]$$
$$= [X_3, X_2] + [N_3, N_2] = -X_1 + X_1 = 0$$
$$[X_3, [X_3, X_2]] + [N_3, [N_3, X_2]]$$
$$= [X_3, -X_1] + [N_3, -N_1] = -X_2 + X_2 = 0$$
$$[X_3, [X_3, X_3]] + [N_3, [N_3, X_3]] = 0$$
$$[X_3, [X_3, N_1]] + [N_3, [N_3, N_1]]$$
$$= [X_3, N_2] + [N_3, -X_2] = -N_1 + N_1 = 0$$
$$[X_3, [X_3, N_2]] + [N_3, [N_3, N_2]]$$
$$= [X_3, -N_1] + [N_3, X_1] = -N_2 + N_2 = 0$$
$$[X_3, [X_3, N_3]] + [N_3, [N_3, N_3]] = 0$$

This yields three intrinsic Casimir operators of degree two for $so(3,1)$ which do not exist in the standard approach:

$$[X_1, [X_1, +[N_1, [N_1,$$

$$[X_2, [X_2, +[N_2, [N_2,$$
$$[X_3, [X_3, +[N_3, [N_3,$$

To these add the previously confirmed intrinsic Casimir operators of degree two:

$$L^2 = [X_1, [X_1, +[X_2, [X_2, +[X_3, [X_3,$$
$$= ([X_1,)^{2n} + ([X_2,)^{2n} + ([X_3,)^{2n}$$
$$N^2 = [N_1, [N_1, +[N_2, [N_2, +[N_3, [N_3,$$
$$= ([N_1,)^{2n} + ([N_2,)^{2n} + ([N_3,)^{2n}$$
$$C' = [X_1, [N_1, +[X_2, [N_2, +[X_3, [N_3,$$

The standard theory predicts that the Lorentz Lie algebra, $so(3,1)$ has only two Casimir operators, both of order 2. In this section, we constructed six intrinsic Casimir operators for $so(3,1)$ of order 2.

Now things explode, we use the above second order operators to construct families of intrinsic Casimir operators:

$$([X_1,[X_1,)^n + ([N_1,[N_1,)^n$$

$$([X_2,[X_2,)^n + ([N_2,[N_2,)^n$$

$$([X_3,[X_3,)^n + ([N_3,[N_3,)^n$$

$$L^{2n} = ([X_1,)^{2n} + ([X_2,)^{2n} + ([X_3,)^{2n}$$

$$N^{2n} = ([N_1,)^{2n} + ([N_2,)^{2n} + ([N_3,)^{2n}$$

$$C'_n = ([X_1,[N_1,)^n + ([X_2,[N_2,)^n + ([X_3,[N_3,)^n$$

We will do one representative calculation for $n = 2$:

$$C'_2 = ([X_1,[N_1,)^2 + ([X_2,[N_2,)^2 + ([X_3,[N_3,)^2$$

$$C'_2Y = [X_1,[N_1,[X_1,[N_1,Y]]]]+[X_2,[N_2,[X_2,[N_2,Y]]]]+[X_3,[N_3,[X_3,[N_3,Y]]]]$$

We calculate

$$C'_2X_1 = [X_1,[N_1,[X_1,[N_1,X_1]]]]+[X_2,[N_2,[X_2,[N_2,X_1]]]]+[X_3,[N_3,[X_3,[N_3,X_1]]]]$$

$$= [X_2,[N_2,-N_1]] + [X_3,[N_3,-N_1]]$$

$$= -X_1 - X_1 = -2X_1$$

$$C'_2X_2 = [X_1,[N_1,[X_1,[N_1,X_2]]]]+[X_2,[N_2,[X_2,[N_2,X_2]]]]+[X_3,[N_3,[X_3,[N_3,X_2]]]]$$

$$= [X_1,[N_1,-N_2]] + [X_3,[N_3,-N_2]]$$

$$= [X_1,X_3] + [X_3,-X_1]$$

$$= -X_2 - X_2 = -2X_2$$

$$C'_2X_3 = [X_1,[N_1,[X_1,[N_1,X_3]]]]+[X_2,[N_2,[X_2,[N_2,X_3]]]]+[X_3,[N_3,[X_3,[N_3,X_3]]]]$$

$$= [X_1,[N_1,-N_3]] + [X_2,[N_2,-N_3]]$$

$$= [X_1,-X_2] + [X_2,X_1]$$

$$= -2X_3$$

$$C_2'N_1 = [X_1,[N_1,[X_1,[N_1,N_1]]]]+[X_2,[N_2,[X_2,[N_2,N_1]]]]+[X_3,[N_3,[X_3,[N_3,N_1]]]]$$
$$= [X_2,[N_2,X_1]] + [X_3,[N_3,X_1]]$$
$$= -N_1 - N_1 = -2N_1$$

$$C_2'N_2 = [X_1,[N_1,[X_1,[N_1,N_2]]]]+[X_2,[N_2,[X_2,[N_2,N_2]]]]+[X_3,[N_3,[X_3,[N_3,N_2]]]]$$
$$= [X_1,[N_1,X_2]] + [X_3,[N_3,X_2]]$$
$$= -N_2 - N_2 = -2N_2$$

$$C_2'N_3 = [X_1,[N_1,[X_1,[N_1,N_3]]]]+[X_2,[N_2,[X_2,[N_2,N_3]]]]+[X_3,[N_3,[X_3,[N_3,N_3]]]]$$
$$= [X_1,[N_1,X_3]] + [X_2[N_2,X_3]]$$
$$= -2N_3$$

These calculations show that $C_2'Y = -2Y \ \forall \ Y \in so(3,1)$. Thus C_2' is confirmed as a Casimir Operator of $so(3,1)$ of eigenvalue type. The Jacobi identity is of no use in reducing this operator, so it is not clear if it is independent of the others.

8 Third Order intrinsic Casimir Operators for $so(3,1)$

In this section, we construct several intrinsic Casimir operators of order 3, providing even more confirmation that the standard theory is defective.

One third order intrinsic Casimir Operator for $so(3)$ is:

$$C_-^3Y = [X_1,[X_2,[X_3,Y]]] + [X_3,[X_1,[X_2,Y]]] + [X_2,[X_3,[X_1,Y]]]$$

Thus, as we calculated before:

$$C_-^3X_1 = -X_1$$
$$C_-^3X_2 = -X_2$$
$$C_-^3X_3 = -X_3$$

And we now calculate the action of C^3_- on the noncompact generators of $so(3,1)$:

$$C^3_- N_1 = [X_1, [X_2, [X_3, N_1]]] + [X_3, [X_1, [X_2, N_1]]] + [X_2, [X_3, [X_1, N_1]]]$$

$$= [X_1, [X_2, N_2]] + [X_3, [X_1, -N_3]]$$
$$= [X_3, N_2] = -N_1$$
$$C^3_- N_2 = [X_1, [X_2, [X_3, N_2]]] + [X_3, [X_1, [X_2, N_2]]] + [X_2, [X_3, [X_1, N_2]]]$$
$$= [X_1, [X_2, -N_1]] = [X_1, N_3] = -N_2$$
$$C^3_- N_3 = [X_1, [X_2, [X_3, N_3]]] + [X_3, [X_1, [X_2, N_3]]] + [X_2, [X_3, [X_1, N_3]]]$$
$$= [X_2, [X_3, -N_2]] = [X_2, N_1] = -N_3$$

Thus $C^3_- Y = -Y$ for all $Y \in so(3,1)$ and C^3_- is confirmed as an intrinsic Casimir operator of $so(3,1)$ of eigenvalue type. Again, it is surprising that an intrinsic Casimir operator of a subalgebra is an intrinsic Casimir operator of the entire algebra. This does not happen in the standard approach. Since we will need to discuss this situation in more detail, we need a term for the situation. If the Casimir operator of a subalgebra is also a Casimir operator for the entire Lie algebra, we will say that the subalgebra is a *replete* subalgebra. Thus, $so(3)$ is a replete subalgebra of $so(3,1)$. Clearly, any real Lie algebra is a replete subalgebra of its complexification.

Now we begin a search for other intrinsic Casimir operators of $so(3,1)$. If an operator is to be an intrinsic Casimir operator of the eigenvalue type for $so(3,1)$, there must be an even number of N_I as factors. If in C^3_-, we replace 2 of the X_I by the corresponding N_I, we might obtain another intrinsic Casimir operator for $so(3,1)$. To this end we define A^3 by:

$$A^3 Y = [N_1, [X_2, [N_3, Y]]] + [N_3, [X_1, [N_2, Y]]] + [N_2, [X_3, [N_1, Y]]]$$

We calculate:

$$A^3 X_2 = [N_1, [X_2, [N_3, X_2]]]$$

$$= [N_1, [X_2, -N_1]] = [N_1, N_3] = X_2$$
$$A^3 X_1 = [N_3, [X_1, [N_2, X_1]]] = [N_3, [X_1, -N_3]] = [N_3, N_2] = X_1$$
$$A^3 X_3 = [N_2, [X_3, [N_1, X_3]]] = [N_2, [X_3, -N_2]] = [N_2, N_1] = X_3$$

$$A^3N_1 = [N_3, [X_1, [N_2, N_1]]] = [N_3, [X_1, X_3]] = [N3, -X_2] = N_1$$
$$A^3N_2 = [N_1, [X_2, [N_3, N_2]]] = [N_1, [X_2, X_1]] = [N_1, -X_3] = N_2$$
$$A^3N_3 = [N_2, [X_3, [N_1, N_3]]] = [N_2, [X_3, X_2]] = [N_2, -X_1] = N_3$$

Thus $A^3Y = Y$ for all $Y \in so(3,1)$ and A^3 is confirmed as yet another intrinsic Casimir operator of $so(3,1)$ of eigenvalue type. When we have several operators, we have to look for relations between them. Again applying the Jacobi identity:

$$A^3Y = [N_1, [X_2, [N_3, Y]]] + [N_3, [X_1, [N_2, Y]]] + [N_2, [X_3, [N_1, Y]]]$$

$$= [[N_1, X_2], [N_3, Y]] + [X_2, [N_1, [N_3, Y]]] + [[N_3, X_1], [N_2, Y]] + [X_1, [N_3, [N_2, Y]]]$$
$$+ [[N_2, X_3], [N_1, Y]] + [X_3, [N_2, [N_1, Y]]]$$
$$= [N_3, [N_3, Y]] + [X_2, [N_1, [N_3, Y]]] + [N_2, [N_2, Y]] + [X_1, [N_3, [N_2, Y]]]$$
$$+ [N_1, [N_1, Y]] + [X_3, [N_2, [N_1, Y]]]$$
$$= C'Y + [X_2, [N_1, [N_3, Y]]] + [X_1, [N_3, [N_2, Y]]] + [X_3, [N_2, [N_1, Y]]]$$

If in A^3, we replace each of the X_I by the corresponding N_I, we might obtain another intrinsic Casimir operator for $so(3,1)$ .To this end we define F_3 by:

$$F^3Y = [N_1, [N_2, [N_3, Y]]] + [N_3, [N_1, [N_2, Y]]] + [N_2, [N_3, [N_1, Y]]]$$

We calculate:

$$F^3N_1 = [N_3, [N_1, [N_2, N_1]]] = [N_3, [N_1, X_3]] = [N_3, -N_2] = -X_1$$

$$F^3N_2 = [N_1, [N_2, [N_3, N_2]]] = [N_1, [N_2, X_1]] = [N_1, -N_3] = -X_2$$
$$F^3N_3 = [N_2, [N_3, [N_1, N_3]]] = [N_2, [N_3, X_2]] = [N_2, -N_1] = -X_3$$
$$F^3X_1 = [N_3, [N_1, [N_2, X_1]]] = [N_3, [N_1, -N_3]] = [N_3, -X_2] = N_1$$
$$F^3X_2 = [N_1, [N_2, [N_3, X_2]]] = [N_1, [N_2, -N_1]] = [N_1, -X_3] = N_2$$
$$F^3X_3 = [N_2, [N_3, [N_1, X_3]]] = [N_2, [N_3, -N_2]] = [N_2, -X_1] = N_3$$

Thus

$$F^3X_I = N_I$$
$$F^3N_I = -X_I$$

and:

$$F^3(X_I + iN_I) = -i(X_I + iN_I)$$
$$F^3(X_I - iN_I) = i(X_I - iN_I)$$

So,F^3 is confirmed as another intrinsic Casimir operator of complex structure type of $so(3,1)$. For F^3, in 26, we take b = -1 and thus F^3 satisfies the conditions of Theorem 5.

Now we have two operators whose action on a basis is the same, are we to conclude that these two intrinsic Casimir operators are identical? Acting on elements of the Lie algebra, the two operators are the same, but in a representation by differential operators, their actions on functions will be different, so we conclude that they are not identical operators. Physically, we hope they will lead to different conserved quantities.

If we expand

$$F^3Y = [N_1, [N_2, [N_3, Y]]] + [N_3, [N_1, [N_2, Y]]] + [N_2, [N_3, [N_1, Y]]]$$

using the Jacobi identity, term by term:

$$[N_1, [N_2, [N_3, Y]]] = [N_1, [[N_2, N_3], Y]]] + [N_1, [N_3, [N_2, Y]]]$$
$$= [N_1, [-X_1, Y]] + [N_1, [N_3, [N_2, Y]]]$$
$$= -[N_1, [X_1, Y]] + [N_1, [N_3, [N_2, Y]]] + [N_2, [N_3, [N_1, Y]]]$$
$$= [N_2, [[N_3, N_1], Y]]] + [N_2, [N_1, [N_3, Y]]]$$
$$= [N_2, [-X_2, Y]] + [N_2, [N_1, [N_3, Y]]]$$
$$= -[N_2, [X_2, Y]] + [N_2, [N_1, [N_3, Y]]] + [N_3, [N_1, [N_2, Y]]]$$
$$= [N_3, [[N_1, N_2], Y]]] + [N_3, [N_2, [N_1, Y]]]$$
$$= [N_3, [-X_3, Y]] + [N_3, [N_2, [N_1, Y]]]$$
$$= -[N_3, [X_3, Y]] + [N_3, [N_2, [N_1, Y]]]$$

Summing, we obtain the identity

$$F^3Y = -[N_1, [X_1, Y]] + [N_1, [N_3, [N_2, Y]]] - [N_2, [X_2, Y]]$$
$$+[N_2, [N_1, [N_3, Y]]] - [N_3, [X_3, Y]] + [N_3, [N_2, [N_1, Y]]]$$
$$= -C'Y + [N_1, [N_3, [N_2, Y]]] + [N_2, [N_1, [N_3, Y]]] + [N_3, [N_2, [N_1, Y]]]$$

9 Replete Subalgebras of $so(3,1)$

As we showed in section II, the second order intrinsic Casimir Operator for $so(3)$ is:

$$L^2Y = [X_1,[X_1,Y]] + [X_2,[X_2,Y]] + [X_3,[X_3,Y]] = -2Y$$

In section III, we showed that the third order intrinsic Casimir Operators for so(3) are:

$$C_-^3Y = [X_1,[X_2,[X_3,Y]]] + [X_3,[X_1,[X_2,Y]]] + [X_2,[X_3,[X_1,Y]]] = -Y$$

$$C_+^3Y = [X_3,[X_2,[X_1,Y]]] + [X_1,[X_3,[X_2,Y]]] + [X_2,[X_1,[X_3,Y]]] = Y$$

The three intrinsic Casimir operators are related by:

$$C_-^3Y = L^2Y + C_+^3Y$$

These relations clearly hold for all $Y \in so(3)$. Somewhat surprisingly, we showed that these relations also hold for all $Y \in so(3,1)$. Hence these intrinsic Casimir operators for the subalgebra $so(3)$ are also intrinsic Casimir operators for the entire algebra $so(3,1)$, in this situation, we called $so(3)$ a replete subalgebra of $so(3,1)$.

It was unanticipated that the intrinsic Casimir operator for a subalgebra would be an intrinsic Casimir operator for the entire algebra. In this section we will investigate other subalgebras to see if their intrinsic Casimir operators are also intrinsic Casimir operators for the entire algebra. Are there other replete subalgebras of $so(3,1)$ or was $so(3)$ a fluke?

The first subalgebra we investigated was that generated by

$$\{X_1, X_2, X_3,\}$$

which is isomorphic to $so(3)$.

The next subalgebra to be investigated will be denoted subalgebra 2 and is generated by: $\{X_1, N_2, N_3\}$. We demonstrate closure:

$$[X_1, N_2] = N_3$$

$$[X_1, N_3] = -N_2$$

$$[N_2, N_3] = -X_1$$

Subalgebra 2 is isomorphic to $so(2,1)$ and its second order intrinsic Casimir operator is:

$$L_2^2 Y = [X_1, [X_1, Y]] - [N_2, [N_2, Y]] - [N_3, [N_3, Y]]$$

We demonstrate that this is an intrinsic Casimir operator of the entire so(3,1):

$$L_2^2 X_1 = [X_1, [X_1, X_1]] - [N_2, [N_2, X_1]] - [N_3, [N_3, X_1]]$$

$$= [N_2, N_3] - [N_3, N_2] = -2X_1$$

Likewise, as similar calculations show, $L_2^2 Y = -2Y$ for all $Y \in so(3,1)$.

One of the third order intrinsic Casimir operators of this $so(2,1)$ subalgebra is:

$$C_{2+}^3 Y = [X_1, [N_2, [N_3, Y]]] + [N_3, [X_1, [N_2, Y]]] + [N_2, [N_3, [X_1, Y]]]$$

For all $Y \in so(3,1), C_{2+}^3 Y = Y$ and so C_{2+}^3 is an intrinsic Casimir operator of the entire so(3,1). We will do just one representative calculation:

$$C_{2+}^3 N_1 = [X_1, [N_2, [N_3, N_1]]] + [N_3, [X_1, [N_2, N_1]]] + [N_2, [N_3, [X_1, N_1]]]$$

$$= [N_3, [X_1, X_3]] = [N_3, -X_2]] = N_1$$

Expanding via the Jacobi identity we obtain:

$$C_{2+}^3 Y = [X_1, [N_2, [N_3, Y]]] + [N_3, [X_1, [N_2, Y]]] + [N_2, [N_3, [X_1, Y]]]$$

$$= [[X_1, N_2], [N_3, Y]]] + [[N_3, X_1], [N_2, Y]]] + [[N_2, N_3], [X_1, Y]]]$$

$$+[N_2, [X_1, [N_3, Y]]] + [X_1, [N_3, [N_2, Y]]] + [N_3, [N_2, [X_1, Y]]]$$

$$= -[X_1, [X_1, Y]] + [N_2, [N_2, Y]] + [N_3, [N_3, Y]] + [N_2, [X_1, [N_3, Y]]]$$

$$+[X_1, [N_3, [N_2, Y]] + [N_3, [N_2, [X_1, Y]]]$$

$$= -L_2^2 Y + C_{2-}^3 Y$$

From the complement of subalgebra 2, we can construct a third order operator which is the difference between the third order intrinsic Casimir operator of the subalgebra and that of the whole Lie algebra.

$$H^3_{2+}Y = [N_1,[X_2,[X_3,Y]]] + [X_3,[N_1,[X_2,Y]]] + [X_2,[X_3,[N_1,Y]]]$$

We calculate its action:

$$H^3_{2+}X_1 = [N_1,[X_2,[X_3,X_1]]] + [X_3,[N_1,[X_2,X_1]]] + [X_2,[X_3,[N_1,X_1]]]$$
$$= [X_3,[N_1,-X_3]] = [X_3,N_2] = -N_1$$

$$H^3_{2+}X_2 = [N_1,[X_2,[X_3,X_2]]] + [X_3,[N_1,[X_2,X_2]]] + [X_2,[X_3,[N_1,X_2]]]$$
$$= [N_1,[X_2,-X_1]] = [N_1,X_3] = -N_2$$

$$H^3_{2+}X_3 = [N_1,[X_2,[X_3,X_3]]] + [X_3,[N_1,[X_2,X_3]]] + [X_2,[X_3,[N_1,X_3]]]$$
$$= [X_2,[X_3,-N_2] = [X_2,N_1] = -N_3$$

$$H^3_{2+}N_1 = [N_1,[X_2,[X_3,N_1]]] + [X_3,[N_1,[X_2,N_1]]] + [X_2,[X_3,[N_1,N_1]]]$$
$$= [X_3,[N_1,-N_3]] = [X_3,-X_2] = X_1$$

$$H^3_{2+}N_2 = [N_1,[X_2,[X_3,N_2]]] + [X_3,[N_1,[X_2,N_2]]] + [X_2,[X_3,[N_1,N_2]]]$$
$$= [N_1,[X_2,-N_1]] = [N_1,N_3] = X_2$$

$$H^3_{2+}N_3 = [N_1,[X_2,[X_3,N_3]]] + [X_3,[N_1,[X_2,N_3]]] + [X_2,[X_3,[N_1,N_3]]]$$
$$= [X_2,[X_3,X_2] = [X_2,-X_1] = X_3$$

For H^3_{2+} take $b = -1$ in 26, thus H^3_{2+} satisfies the conditions of Theorem 5 and is an intrinsic Casimir operator of complex structure type.

Since the subalgebra is the tangent space of the corresponding homogeneous space, we have constructed an invariant operator on the homogeneous space $so(3,1)/so(2,1)$.

Subalgebra 3 of $so(3,1)$ is generated by N_1, X_2, N_3. We demonstrate closure:

$$[X_2,N_1] = -N_3$$

$$[X_2, N_3] = N_1$$

$$[N_3, N_1] = -X_2$$

Again, this subalgebra is isomorphic to $so(2,1)$ and its second order intrinsic Casimir operator is:

$$L_3^2 Y = [X_2, [X_2, Y]] - [N_1, [N_1, Y]] - [N_3, [N_3, Y]]$$

which is easily confirmed as an intrinsic Casimir operator for the entire $so(3,1)$:

$$L_3^2 X_1 = [X_2, [X_2, X_1]] - [N_1, [N_1, X_1]] - [N_3, [N_3, X_1]]$$

$$= [X_2, -X_3] - [N_3, N_2] = -2X_1$$

The other calculations are left to the interested reader.

One of the third order intrinsic Casimir operators of subalgebra 3 is:

$$C_{3+}^3 Y = [N_1, [X_2, [N_3, Y]]] + [N_3, [N_1, [X_2, Y]]] + [X_2, [N_3, [N_1, Y]]]$$

which we confirm is an intrinsic Casimir operator of the entire algebra:

$$C_{3+}^3 X_1 = [N_1, [X_2, [N_3, X_1]]] + [N_3, [N_1, [X_2, X_1]]] + [X_2, [N_3, [N_1, X_1]]]$$

$$= [N_3, [N_1, -X_3]] = [N_3, N_2] = X_1$$

The other five calculations are again left to the reader.

Expanding, using the Jacobi identity, we obtain:

$$C_{3+}^3 Y = [N_1, [X_2, [N_3, Y]]] + [N_3, [N_1, [X_2, Y]]] + [X_2, [N_3, [N_1, Y]]]$$

$$= [[N_1, X_2], [N_3, Y]] + [[N_3, N_1], [X_2, Y]]] + [[X_2, N_3], [N_1, Y]]]$$

$$+[X_2, [N_1, [N_3, Y]]] + [N_1, [N_3, [X_2, Y]]] + [N_3, [X_2, [N_1, Y]]]$$

$$= [N_3, [N_3, Y]] - [X_2, [X_2, Y]] + [N_1, [N_1, Y]] + [X_2, [N_1, [N_3, Y]]]$$

$$+[N_1, [N_3, [X_2, Y]]] + [N_3, [X_2, [N_1, Y]]]$$

$$= -L_3^2 Y + [X_2, [N_1, [N_3, Y]]] + [N_1, [N_3, [X_2, Y]]] + [N_3, [X_2, [N_1, Y]]]$$

From the complement of Subalgebra 3, we construct:

$$H_{3+}^3 Y = [X_1, [N_2, [X_3, Y]]] + [X_3, [X_1, [N_2, Y]]] + [N_2, [X_3, [X_1, Y]]]$$

Which we confirm is an intrinsic Casimir operator for the entire subalgebra:

$$H^3_{3+}X_1 = [X_1, [N_2, [X_3, X_1]]] + [X_3, [X_1, [N_2, X_1]]] + [N_2, [X_3, [X_1, X_1]]]$$

$$= [X_3, [X_1, -N_3]] = [X_3, N_2] = -N_1$$

$$H^3_{3+}N_1 = [X_1, [N_2, [X_3, N_1]]] + [X_3, [X_1, [N_2, N_1]]] + [N_2, [X_3, [X_1, N_1]]]$$

$$= [X_3, [X_1, X_3]] = [X_3, -X_2] = X_1$$

Just like H^3_{2+}, H^3_{3+} is then an intrinsic Casimir operator of Complex structure type. We expand using the Jacobi identity:

$$H^3_{3+}Y = [X_1, [N_2, [X_3, Y]]] + [X_3, [X_1, [N_2, Y]]] + [N_2, [X_3, [X_1, Y]]]$$

$$= [[X_1, N_2], [X_3, Y]] + [[X_3, X1], [N_2, Y]] + [[N_2, X_3], [X_1, Y]] + [N_2, [X_1, [X_3, Y]]]$$

$$+ [X_1, [X_3, [N_2, Y]]] + [X_3, [N_2, [X_1, Y]]]$$

$$= [N_3, [X_3, Y]] + [X_2, [N_2, Y]] + [N_1, [X_1, Y]] + [N_2, [X_1, [X_3, Y]]]$$

$$+ [X_1, [X_3, [N_2, Y]]] + [X_3, [N_2, [X_1, Y]]]$$

$$= C''Y + [N_2, [X_1, [X_3, Y]]] + [X_1, [X_3, [N_2, Y]]] + [X_3, [N_2, [X_1, Y]]]$$

Subalgebra 4 of $so(3, 1)$ is generated by N_1, N_2, and X_3. We demonstrate closure:

$$[X_3, N_1] = N_2$$

$$[N_1, N_2] = -X_3$$

$$[X_3, N_2] = -N_1$$

Again, this subalgebra is isomorphic to $so(2, 1)$ and its second order intrinsic Casimir operator is:

$$L^2_4 Y = [X_3, [X_3, Y]] - [N_1, [N_1, Y]] - [N_2, [N_2, Y]]$$

which is easily confirmed as an intrinsic Casimir operator for the entire so(3,1):

$$L^2_4 X_1 = [X_3, [X_3, X_1]] - [N_1, [N_1, X_1]] - [N_2, [N_2, X_1]]$$

$$= [X_3, X_2] - [N_2, -N_3] = -2X_1$$

The others are left to the reader.

One of the third order intrinsic Casimir operators of the subalgebra is:

$$C^3_{4+}Y = [N_1, [N_2, [X_3, Y]]] + [X_3, [N_1, [N_2, Y]]] + [N_2, [X_3, [N_1, Y]]]$$

We confirm that C^3_{4+} is an intrinsic Casimir operator for the entire Lie algebra:

$$C^3_{4+}X_1 = [N_1, [N_2, [X_3, X_1]]] + [X_3, [N_1, [N_2, X_1]]] + [N_2, [X_3, [N_1, X_1]]]$$

$$= [X_3, [N_1, -N_3]] = [X_3, -X_2] = X_1$$

Thus C^3_{4+} is confirmed as another intrinsic Casimir operator of eigenvalue type. We expand:

$$C^3_{4+}Y = [N_1, [N_2, [X_3, Y]]] + [X_3, [N_1, [N_2, Y]]] + [N_2, [X_3, [N_1, Y]]]$$

$$= [[N_1, N_2], [X_3, Y]] + [[X_3, N_1], [N_2, Y]]] + [[N_2, X_3], [N_1, Y]]]$$

$$+[N_2, [N_1, [X_3, Y]]] + [N_1, [X_3, [N_2, Y]]] + [X_3, [N_2, [N_1, Y]]]$$

$$= -[[X_3, [X_3, Y]] + [[N_2, [N_2, Y]] + [[N_1, [N_1, Y]] + [N_2, [N_1, [X_3, Y]]]$$

$$+[N_1, [X_3, [N_2, Y]]] + [X_3, [N_2, [N_1, Y]]]$$

$$= -L^2_4 Y + [N_2, [N_1, [X_3, Y]]] + [N_1, [X_3, [N_2, Y]]] + [X_3, [N_2, [N_1, Y]]]$$

The complement of Subalgebra 4 is X_1, X_2, and N_3, from which we construct the third order operator:

$$H^3_{4+}Y = [X_1, [X_2, [N_3, Y]]] + [N_3, [X_1, [X_2, Y]]] + [X_2, [N_3, [X_1, Y]]]$$

We confirm that H^3_{4+} is an intrinsic Casimir operator for the entire algebra.

$$H^3_{4+}X_1 = [X_1, [X_2, [N_3, X_1]]] + [N_3, [X_1, [X_2, X_1]]] + [X_2, [N_3, [X_1, X_1]]]$$

$$= [N_3, [X_1, -X_3]] = [N_3, X_2] = -N_1$$

$$H^3_{4+}N_1 = [X_1, [X_2, [N_3, N_1]]] + [N_3, [X_1, [X_2, N_1]]] + [X_2, [N_3, [X_1, N_1]]]$$

$$= [N_3, [X_1, -N_3]] = [N_3, N_2] = X_1$$

This, together with the similar calculations left for the reader show that H^3_{4+} is an intrinsic Casimir operator of complex structure type.

Expanding using the Jacobi identity, as in previous cases:

$$H^3_{4+}Y = C'Y + [X_2, [X_1, [N_3, Y]]] + [X_1, [N_3, [X_2, Y]]] + [N_3, [X_2, [X_1, Y]]]$$

For each replete subalgebra of $so(3,1)$, higher order intrinsic Casimir operators can be constructed in the same manner as $so(3)$. These are all intrinsic Casimir operators of $so(3,1)$.

10 The Intrinsic Quadratic Casimir Operator of $so(n)$

Let e_{IJ} denote the n by n matrix with 1 in the IJ position and zero elsewhere. A basis for $so(n)$ consists of the matrices $X_{IJ} = e_{IJ} - e_{JI}$, $I < J$, with the Lie bracket:

$$[X_{IJ}, X_{KL}] = X_{IJ}X_{KL} - X_{KL}X_{IJ}$$

We will calculate the trace of $[X_{12}, [X_{12}$ in $so(5), so(4)$ and $so(3)$. For $so(5)$:

$$[X_{12}, [X_{12}, X_{12}] = 0$$

$$[X_{12}, [X_{12}, X_{13}] = [X_{12}, -X_{23}] = -X_{13}$$

$$[X_{12}, [X_{12}, X_{14}] = [X_{12}, -X_{24}] = -X_{14}$$

$$[X_{12}, [X_{12}, X_{15}] = [X_{12}, -X_{25}] = -X_{15}$$

$$[X_{12}, [X_{12}, X_{23}] = [X_{12}, X_{13}] = -X_{23}$$

$$[X_{12}, [X_{12}, X_{24}] = [X_{12}, X_{14}] = -X_{24}$$

$$[X_{12}, [X_{12}, X_{25}] = [X_{12}, X_{15}] = -X_{25}$$

$$[X_{12}, [X_{12}, X_{34}] = 0$$

$$[X_{12}, [X_{12}, X_{35}] = 0$$

$$[X_{12}, [X_{12}, X_{45}] = 0$$

Thus $Trace([X_{12}, [X_{12},) = -6$ in $so(5)$

Simultaneously, we will compute the eigenvalue of the unnormalized intrinsic Casimir operators. In each case, we will only do one representative calculation. The intent is to show that in each case, the eigenvalue is equal to the trace. The unnormalized intrinsic Casimir operator of $so(5)$:

$$[X_{12},[X_{12},X_{45}]] = 0$$

$$[X_{13},[X_{13},X_{45}]] = 0$$

$$[X_{14},[X_{14},X_{45}]] = [X_{14},X_{15}] = -X_{45}$$

$$[X_{15},[X_{15},X_{45}]] = [X_{15},-X_{14}] = -X_{45}$$

$$[X_{23},[X_{23},X_{45}]] = 0$$

$$[X_{24},[X_{24},X_{45}]] = [X_{24},X_{25}] = -X_{45}$$

$$[X_{25},[X_{25},X_{45}]] = [X_{25},-X_{24}] = -X_{45}$$

$$[X_{34},[X_{34},X_{45}]] = [X_{34},X_{35}] = -X_{45}$$

$$[X_{35},[X_{35},X_{45}]] = [X_{35},-X_{34}] = -X_{45}$$

$$[X_{45},[X_{45},X_{45}]] = 0$$

Summing the above, we obtain $-6X_{45}$. Thus the eigenvalue of the unnormalized Intrinsic Casimir operator in $so(5)$ is -6.

For so(4):

$$[X_{12},[X_{12},X_{12}]] = 0$$

$$[X_{12},[X_{12},X_{13}]] = [X_{12},-X_{23}] = -X_{13}$$

$$[X_{12},[X_{12},X_{14}]] = [X_{12},-X_{24}] = -X_{14}$$

$$[X_{12},[X_{12},X_{23}]] = [X_{12},X_{13}] = -X_{23}$$

$$[X_{12},[X_{12},X_{24}]] = [X_{12},X_{14}] = -X_{24}$$

$$[X_{12},[X_{12},X_{34}]] = 0$$

$Trace[X_{12},[X_{12}, = -4$ in $so(4)$

$$[X_{12},[X_{12},X_{34}]] = 0$$

$$[X_{13},[X_{13},X_{34}]] = [X_{13},X_{14}] = -X_{34}$$

$$[X_{14},[X_{14},X_{34}]] = [X_{14},-X_{13}] = -X_{34}$$

$$[X_{23},[X_{23},X_{34}]] = [X_{23},X_{24}] = -X_{34}$$

$$[X_{24},[X_{24},X_{34}]] = [X_{24},-X_{23}] = -X_{34}$$

$$[X_{34},[X_{34},X_{34}]]=0$$

Summing the above, we obtain $-4X_{34}$. Thus the eigenvalue of the unnormalized Intrinsic Casimir operator in $so(4)$ is -4.

For $so(3)$:

$$[X_{12},[X_{12},X_{12}]]=0$$

$$[X_{12},[X_{12},X_{13}]]=[X_{12},-X_{23}]=-X_{13}$$

$$[X_{12},[X_{12},X_{23}]]=[X_{12},X_{13}]=-X_{23}$$

Thus, $Trace([X_{12},[X_{12},)=-2$ in $so(3)$.

$$[X_{12},[X_{12},X_{12}]]=0$$

$$[X_{13},[X_{13},X_{12}]]=[X_{13},X_{23}]=-X_{12}$$

$$[X_{23},[X_{23},X_{12}]]=[X_{23},-X_{13}]=-X_{12}$$

Summing the above, we obtain $-2X_{12}$. Thus, the eigenvalue of the unnormalized Intrinsic Casimir operator in $so(3)$ is -2.

Theorem

In $so(n)$, $Trace([X_{IJ},[X_{IJ},)=-2n+4$,which is also the eigenvalue of the unnormalized Intrinsic Casimir operator in $so(n)$. Thus, the eigenvalue of the normalized Intrinsic Casimir operator is 1.

Proof:

There are two statements to prove: in $so(n)$, the trace of $[X_{IJ},[X_{IJ},$ is $-2n+4$ and in $so(n)$, the eigenvalue of the unnormalized Intrinsic Casimir operator in $so(n)$ is $-2n+4$. We will prove the statements by induction. We have already shown them to be true for n= 3,4, and 5. Those calculations reveal the pattern. The dimension of $so(n)$ is $n(n-1)/2$. Thus going from $so(n)$ to $so(n+1)$ we add $\frac{(n+1)n}{2}-\frac{n(n-1)}{2}=n$ generators.

Computing the trace of X_{IJ} , $I<J<n+1$ in $so(n+1)$, we see that all terms are zero except

$$[X_{IJ},[X_{IJ},X_{I(n+1)}]]=-X_{I(n+1)}$$

$$[X_{IJ},[X_{IJ},X_{J(n+1)}]]=-X_{J(n+1)}$$

Thus the trace of $[X_{IJ},[X_{IJ},$ in $so(n+1)$ is the trace of $[X_{IJ},[X_{IJ},$ in $so(n)$ minus 2, which by the induction hypothesis is $-2n+4-2=-2(n+1)+4$ Thus proving the formula for the trace.

In the intrinsic Casimir operator, going from $so(n)$ to $so(n+1)$, there are n new terms and when acting on X_{IJ}, they are all zero except:

$$[X_{I(n+1)}, [X_{I(n+1)}, X_{IJ}]] = -X_{IJ}$$

$$[X_{J(n+1)}, [X_{J(n+1)}, X_{IJ}]] = -X_{IJ}$$

Thus, the eigenvalue of the intrinsic Casimir operator is also decreased by 2 and the same calculation shows the validity of the formula for the eigenvalue of the intrinsic Casimir operator.

Thus the normalized Intrinsic Casimir Operator of $so(n)$, when acting on $so(n)$, is not just a multiple of the identity, it is the identity operator.

The higher order intrinsic Casimir operators of so(n) follow the pattern of $so(3)$.

11 The intrinsic Casimir Operators of $su(2)$ and $su(3)$

The "$so(3)$ basis" for $su(2)$ is:

$$\alpha_1 = \begin{pmatrix} 0 & -i/2 \\ -i/2 & 0 \end{pmatrix}$$

$$\alpha_2 = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}$$

$$\alpha_3 = \begin{pmatrix} -i/2 & 0 \\ 0 & i/2 \end{pmatrix}$$

and the commutation relations are:

$$[\alpha_1, \alpha_2] = \alpha_3$$

$$[\alpha_2, \alpha_3] = \alpha_1$$

$$[\alpha_3, \alpha_1] = \alpha_2$$

In this basis, the commutation relations are identical with those of so(3) and hence the intrinsic Casimir operators are the same:

$$C^2 A = [\alpha_1, [\alpha_1, A]] + [\alpha_2, [\alpha_2, A]] + [\alpha_3, [\alpha_3, A]] = -2A$$

Since $su(2)$ is isomorphic to $so(3)$, it also possesses third order intrinsic Casimir operators. The third order intrinsic Casimir operators of $su(2)$ are:

$$C_-^3 A = [\alpha_1, [\alpha_2, [\alpha_3, A]]] + [\alpha_3, [\alpha_1, [\alpha_2, A]]]$$

$$+[\alpha_2, [\alpha_3, [\alpha_1, A]]] = -A$$

$$C_+^3 A = [\alpha_3, [\alpha_2, [\alpha_1, A]]] + [\alpha_1, [\alpha_3, [\alpha_2, A]]]$$

$$+[\alpha_2, [\alpha_1, [\alpha_3, A]]] = A$$

And we have the identity in $su(2)$:

$$C_-^3 A = C^2 A + C_+^3 A$$

The standard basis for $su(2)$ is:

$$\beta_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$\beta_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\beta_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

and the commutation relations are:
$[\beta_1, \beta_2] = 2\beta_3$, $[\beta_2, \beta_3] = 2\beta_1$, $[\beta_3, \beta_1] = 2\beta_2$
Since $\beta_i = 2\alpha_i$, in this basis, the intrinsic Casimir operator is:

$$CA = [\beta_1, [\beta_1, A]] + [\beta_2, [\beta_2, A]] + [\beta_3, [\beta_3, A]] = -8A$$

The Lie Algebra $u(3)$ consists of the matrices:

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}$$

$$Y_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$Y_3 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}$$

The intrinsic Casimir operator is a linear combination of the basis elements applied twice. The diagonal elements acting on diagonal elements must yield zero. Thus the non-diagonal elements determine the eigenvalue of the intrinsic Casimir operator acting on diagonal elements. Thus, in order to determine the eigenvalue of the intrinsic Casimir Operator for $u(3)$, all we need to do is to calculate the action of the non-diagonal elements on the diagonal elements. The action of the diagonal elements is given by:

$$[D_1, X_1] = 0$$

$$[D_2, X_1] = -Y_1$$

$$[D_3, X_1] = Y_1$$
$$[D_1, X_2] = Y_2$$
$$[D_2, X_2] = 0$$
$$[D_3, X_2] = -Y_2$$
$$[D_1, X_3] = -Y_3$$
$$[D_2, X_3] = Y_3$$
$$[D_3, X_3] = 0$$
$$[D_1, Y_1] = 0$$
$$[D_2, Y_1] = X_1$$
$$[D_3, Y_1] = -X_1$$
$$[D_1, Y_2] = -X_2$$
$$[D_2, Y_2] = 0$$
$$[D_3, Y_2] = X_2$$
$$[D_1, Y_3] = X_3$$
$$[D_2, Y_3] = -X_3$$
$$[D_3, Y_3] = 0$$

Now we proceed to calculate the action of the intrinsic Casimir operator on the diagonal operators.

$$[X_1, [X_1, D_1]] = 0$$

$$[X_2, [X_2, D_1]] = [X_2, -Y_2] = -2D_1 + 2D_3$$
$$[X_3, [X_3, D_1]] = [X_3, Y_3] = -2D_1 + 2D_2$$
$$[Y_1, [Y_1, D_1]] = 0$$
$$[Y_2, [Y_2, D_1]] = [Y_2, X_2] = -2D_1 + 2D_3$$
$$[Y_3, [Y_3, D_1]] = [Y_3, -X_3] = -2D_1 + 2D_2$$

Summing, we obtain: $CD_1 = -8D_1 + 4D_2 + 4D_3$

$$[X_1, [X_1, D_2]] = [X_1, Y_1] = -2D_2 + 2D_3$$

$$[X_2, [X_2, D_2]] = 0$$

$$[X_3, [X_3, D_2]] = [X_3, -Y_3] = 2D_1 - 2D_2$$

$$[Y_1, [Y_1, D_2]] = [Y_1, -X_1] = -2D_2 + 2D_3$$

$$[Y_2, [Y_2, D_2]] = 0$$

$$[Y_3, [Y_3, D_2]] = [Y_3, X_3] = 2D_1 - 2D_2$$

Summing, we obtain: $CD_2 = 4D_1 - 8D_2 + 4D_3$

$$[X_1, [X_1, D_3]] = [X_1, -Y_1] = 2D_2 - 2D_3$$

$$[X_2, [X_2, D_3]] = [X_2, Y_2] = 2D_1 - 2D_3$$

$$[X_3, [X_3, D_3]] = 0$$

$$[Y_1, [Y_1, D_3]] = [Y_1, X_1] = 2D_2 - 2D_3$$

$$[Y_2, [Y_2, D_3]] = [Y_2, -X_2] = 2D_1 - 2D_3$$

$$[Y_3, [Y_3, D_3]] = 0$$

Summing, we obtain: $CD_3 = 4D_1 + 4D_2 - 8D_3$
We have the action of C on the diagonal elements as follows:

$$CD_1 = -8D_1 + 4D_2 + 4D_3$$

$$CD_2 = 4D_1 - 8D_2 + 4D_3$$

$$CD_3 = 4D_1 + 4D_2 - 8D_3$$

When we diagonalize the matrix

$$\begin{pmatrix} -8 & 4 & 4 \\ 4 & -8 & 4 \\ 4 & 4 & -8 \end{pmatrix}$$

we obtain the eigenvectors
$D = D_1 + D_2 + D_3$ with eigenvalue 0
$R = D_1 - D_2$ with eigenvalue -12
$S = D_1 + D_2 - 2D_3$ with eigenvalue -12

We will take D, R and S as our new diagonal elements. This calculation has given us the eigenvalue of -12 for the intrinsic Casimir operator of $su(3)$

when operating on the diagonal operators. The eigenvalue for the intrinsic Casimir operator must still be -12 when operating on the non-diagonal operators.

$$R = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}$$

TABLE II The commutators for $su(3)$

$[X_1, X_2] = X_3$	$[X_1, X_3] = -X_2$	$[X_1, Y_1] = R - S$
$[X_1, Y_2] = -Y_3$	$[X_1, Y_3] = Y_2$	$[X_2, X_3] = X_1$
$[X_2, Y_1] = Y_3$	$[X_2, Y_2] = R + S$	$[X_2, Y_3] = -Y_1$
$[X_3, Y_1] = -Y_2$	$[X_3, Y_2] = Y_1$	$[X_3, Y_3] = -2R$
$[Y_1, Y_2] = -X_3$	$[Y_1, Y_3] = X_2$	$[Y_2, Y_3] = -X_1$
$[R, X_1] = Y_1$	$[R, X_2] = Y_2$	$[R, X_3] = -2Y_3$
$[R, Y_1] = -X_1$	$[R, Y_2] = -X_2$	$[R, Y_3] = 2X_3$
$[S, X_1] = -3Y_1$	$[S, X_2] = 3Y_2$	$[S, X_3] = 0$
$[S, Y_1] = 3X_1$	$[S, Y_2] = -3X_2$	$[S, Y_3] = 0$

We compute the traces necessary to obtain the normalized intrinsic Casimir operator:

$$[X_1, [X_1, X_1]] = 0$$

$$[X_1, [X_1, X_2]] = [X_1, X_3] = -X_2$$

$$[X_1, [X_1, X_3]] = [X_1, -X_2] = -X_3$$

$$[X_1, [X_1, Y_1]] = [X_1, R - S] = -4Y_1$$

$$[X_1, [X_1, Y_2]] = [X_1, -Y_3] = -Y_2$$

$$[X_1, [X_1, Y_3]] = [X_1, Y_2] = -Y_3$$

$$[X_1, [X_1, R]] = [X_1, -Y_1] = S - R$$

$$[X_1, [X_1, S]] = [X_1, 3Y_1] = -3S + 3R$$

Then the trace of $[X_1, [X_1,$ is -12. In the same manner, the trace of $[X_2, [X_2,$ and $[X_3, [X_3,$ are shown to be -12.

$$[Y_1, [Y_1, X_1]] = [Y_1, S - R] = -4X_1$$

$$[Y_1, [Y_1, X_2]] = [Y_1, -Y_3] = -X_2$$
$$[Y_1, [Y_1, X_3]] = [Y_1, Y_2] = -X_3$$
$$[Y_1, [Y_1, Y_1]] = 0$$
$$[Y_1, [Y_1, Y_2]] = [Y_1, -X_3] = -Y_2$$
$$[Y_1, [Y_1, Y_3]] = [Y_1, X_2] = -Y_3$$
$$[Y_1, [Y_1, R]] = [Y_1, X_1] = S - R$$
$$[Y_1, [Y_1, S]] = [Y_1, -3X_1] = -3S + 3R$$

Thus, the trace of $[Y_1, [Y_1,$ is -12.

In the same manner, the traces of $[Y_2, [Y_2,$ and $[Y_3, [Y_3,$ are shown to be -12.

$$[R, [R, X_1]] = [R, Y_1] = -X_1$$

$$[R, [R, X_2]] = [R, Y_2] = -X_2$$
$$[R, [R, X_3]] = [R, -2Y_3] = -4X_3$$
$$[R, [R, Y_1]] = [R, -X_1] = -Y_1$$
$$[R, [R, Y_2]] = [R, -X_2] = -Y_2$$
$$[R, [R, Y_3]] = [R, 2X_3] = -4Y_3$$
$$[R, [R, R]] = 0$$
$$[R, [R, S]] = 0$$

Thus, the trace of $[R, [R,$ is -12.

$$[S, [S, X_1]] = [S, -3Y_1] = -9X_1$$

$$[S, [S, X_2]] = [S, 3Y_2] = -9X_2$$
$$[S, [S, X_3]] = 0$$
$$[S, [S, Y_1]] = [S, 3X_1] = -9Y_1$$

$$[S,[S,Y_2]] = [S,-3X_2] = -9Y_2$$
$$[S,[S,Y_3]] = 0$$
$$[S,[S,R]] = 0$$
$$[S,[S,S]] = 0$$

Showing that the trace of $[S,[S,$ is -36.

The normalized intrinsic Casimir operator for $su(3)$ is:

$$C_N W = \frac{-1}{12}([X_1,[X_1,W]] + [X_2,[X_2,W]] + [X_3,[X_3,W]]$$
$$+[Y_1,[Y_1,W]] + [Y_2,[Y_2,W]] + [Y_3,[Y_3,W]]$$
$$+[R,[R,W]]) + \frac{-1}{36}[S,[S,W]]$$

If we multiply the normalized intrinsic Casimir operator by -12, we obtain the standard intrinsic Casimir operator for $su(3)$:

$$CW = [X_1,[X_1,W]] + [X_2,[X_2,W]] + [X_3,[X_3,W]]$$
$$+[Y_1,[Y_1,W]] + [Y_2,[Y_2,W]]+$$
$$[Y_3,[Y_3,W]] + [R,[R,W]] + \frac{1}{3}[S,[S,W]]$$

Now we calculate the action of the intrinsic Casimir operator on the non-diagonal elements:

$$[X_1,[X_1,X_1]] = 0$$
$$[X_2,[X_2,X_1]] = -X_1$$
$$[X_3,[X_3,X_1]] = -X_1$$
$$[Y_1,[Y_1,X_1]] = -4X_1$$
$$[Y_2,[Y_2,X_1]] = -X_1$$
$$[Y_3,[Y_3,X_1]] = -X_1$$
$$[R,[R,X_1]] = -X_1(1/3)[S,[S,X1]] = -3X_1$$

Summing we obtain: $CX_1 = -12X_1$. Similar calculations show:

$$CX_2 = -12X_2$$

$$CX_3 = -12X_3$$

$$CY_1 = -12Y_1$$

$$CY_2 = -12Y_2$$

$$CY_3 = -12Y_3$$

$$CR = -12R$$

$$CS = -12S$$

Thus the eigenvalue of the standard intrinsic Casimir operator of $su(3)$ is -12 and the eigenvalue of the normalized intrinsic Casimir operator is 1.

12 The Casimir Operator of su(n)

As with $so(n)$, let e_{IJ} denote the n by n matrix with 1 in the IJ position and zero elsewhere. A basis for $u(n)$ consists of the matrices

$$X_{IJ} = e_{IJ} - e_{JI}$$

$$Y_{IJ} = i(e_{IJ} + e_{JI})$$

$$D_I = ie_{II}$$

$I < J$, with the Lie brackets:

$$[X_{IJ}, X_{KL}] = \delta_{JK}X_{IL} + \delta_{JL}X_{KI} + \delta_{IK}X_{LJ} + \delta_{IL}X_{JK}$$

$$[X_{IJ}, Y_{KL}] = \delta_{JK}Y_{IL} + \delta_{JL}Y_{KI} - \delta_{IK}Y_{LJ} - \delta_{IL}Y_{JK}$$

$$[Y_{IJ}, Y_{KL}] = -\delta_{JK}X_{IL} - \delta_{JL}X_{KI} - \delta_{IK}X_{LJ} - \delta_{IL}X_{JK}$$

$$[X_{IJ}, Y_{IJ}] = D_I - D_J$$

$$[D_I, X_{IJ}] = Y_{IJ}$$

$$[D_J, X_{IJ}] = -Y_{IJ}$$

$$[D_I, Y_{IJ}] = -X_{IJ}$$

$$[D_J, Y_{IJ}] = X_{IJ}$$

We first need to compute the trace of the elements of the Lie algebra. We start with the trace of X_{12} in $su(5)$ as a representative calculation:

$$[X_{12}, [X_{12}, X_{12}]] = 0$$

$$[X_{12}, [X_{12}, X_{13}]] = [X_{12}, -X_{23}] = -X_{13}$$
$$[X_{12}, [X_{12}, X_{14}]] = [X_{12}, -X_{24}] = -X_{14}$$
$$[X_{12}, [X_{12}, X_{15}]] = [X_{12}, -X_{25}] = -X_{15}$$
$$[X_{12}, [X_{12}, X_{23}]] = [X_{12}, X_{13}] = -X_{23}$$
$$[X_{12}, [X_{12}, X_{24}]] = [X_{12}, X_{14}] = -X_{24}$$
$$[X_{12}, [X_{12}, X_{25}]] = [X_{12}, X_{15}] = -X_{25}$$
$$[X_{12}, [X_{12}, X_{34}]] = 0$$
$$[X_{12}, [X_{12}, X_{35}]] = 0$$
$$[X_{12}, [X_{12}, X_{45}]] = 0$$
$$[X_{12}, [X_{12}, Y_{12}]] = -4Y_{12}$$
$$[X_{12}, [X_{12}, Y_{13}]] = [X_{12}, -Y_{23}] = -Y_{13}$$
$$[X_{12}, [X_{12}, Y_{14}]] = [X_{12}, -Y_{24}] = -Y_{14}$$
$$[X_{12}, [X_{12}, Y_{15}]] = [X_{12}, -Y_{25}] = -Y_{15}$$
$$[X_{12}, [X_{12}, Y_{23}]] = [X_{12}, Y_{13}] = -Y_{23}$$
$$[X_{12}, [X_{12}, Y_{24}]] = [X_{12}, Y_{14}] = -Y_{24}$$
$$[X_{12}, [X_{12}, Y_{25}]] = [X_{12}, Y_{15}] = -Y_{25}$$
$$[X_{12}, [X_{12}, Y_{34}]] = 0$$
$$[X_{12}, [X_{12}, Y_{35}]] = 0$$
$$[X_{12}, [X_{12}, Y_{45}]] = 0$$

Trace of $[X_{12}, [X_{12}, = -20$ in $su(5)$.
We proceed with the terms of the Casimir operator of $su(5)$:

$$[X_{12}, [X_{12}, X_{45}]] = 0$$

$$[X_{13},[X_{13},X_{45}]] = 0$$
$$[X_{14},[X_{14},X_{45}]] = [X_{14},X_{15}] = -X_{45}$$
$$[X_{15},[X_{15},X_{45}]] = [X_{15},-X_{14}] = -X_{45}$$
$$[X_{23},[X_{23},X_{45}]] = 0$$
$$[X_{24},[X_{24},X_{45}]] = [X_{24},X_{25}] = -X_{45}$$
$$[X_{25},[X_{25},X_{45}]] = [X_{25},-X_{24}] = -X_{45}$$
$$[X_{34},[X_{34},X_{45}]] = [X_{34},X_{35}] = -X_{45}$$
$$[X_{35},[X_{35},X_{45}]] = [X_{35},-X_{34}] = -X_{45}$$
$$[X_{45},[X_{45},X_{45}]] = 0$$
$$[Y_{12},[Y_{12},X_{45}]] = 0$$
$$[Y_{13},[Y_{13},X_{45}]] = 0$$
$$[Y_{14},[Y_{14},X_{45}]] = [Y_{14},Y_{15}] = -X_{45}$$
$$[Y_{15},[Y_{15},X_{45}]] = [Y_{15},-Y_{14}] = -X_{45}$$
$$[Y_{23},[Y_{23},X_{45}]] = 0$$
$$[Y_{24},[Y_{24},X_{45}]] = [Y_{24},Y_{25}] = -X_{45}$$
$$[Y_{25},[Y_{25},X_{45}]] = [Y_{25},-Y_{24}] = -X_{45}$$
$$[Y_{34},[Y_{34},X_{45}]] = [Y_{34},Y_{35}] = -X_{45}$$
$$[Y_{35},[Y_{35},X_{45}]] = [Y_{35},-Y_{34}] = -X_{45}$$
$$[Y_{45},[Y_{45},X_{45}]] = -4X_{45}$$

Summing the above, we obtain $CX_{45} = -20X_{45}$. Thus the eigenvalue of the unnormalized Intrinsic Casimir operator in $su(5)$ is -20.

We continue with the trace of X_{12} in $su(4)$:

$$[X_{12},[X_{12},X_{12}]] = 0$$

$$[X_{12},[X_{12},X_{13}]] = [X_{12},-X_{23}] = -X_{13}$$
$$[X_{12},[X_{12},X_{14}]] = [X_{12},-X_{24}] = -X_{14}$$
$$[X_{12},[X_{12},X_{23}]] = [X_{12},X_{13}] = -X_{23}$$

$$[X_{12}, [X_{12}, X_{24}]] = [X_{12}, X_{14}] = -X_{24}$$
$$[X_{12}, [X_{12}, X_{34}]] = 0$$
$$[X_{12}, [X_{12}, Y_{12}]] = -4Y_{12}$$
$$[X_{12}, [X_{12}, Y_{13}]] = [X_{12}, -Y_{23}] = -Y_{13}$$
$$[X_{12}, [X_{12}, Y_{14}]] = [X_{12}, -Y_{24}] = -Y_{14}$$
$$[X_{12}, [X_{12}, Y_{23}]] = [X_{12}, Y_{13}] = -Y_{23}$$
$$[X_{12}, [X_{12}, Y_{24}]] = [X_{12}, Y_{14}] = -Y_{24}$$
$$[X_{12}, [X_{12}, Y_{34}]] = 0$$

Trace of $[X_{12}, [X_{12}, = -16$ in $su(4)$.

We proceed with the terms of the Casimir operator of $su(4)$:

$$[X_{12}, [X_{12}, X_{34}]] = 0$$

$$[X_{13}, [X_{13}, X_{34}]] = [X_{13}, X_{14}] = -X_{34}$$
$$[X_{14}, [X_{14}, X_{34}]] = [X_{14}, -X_{13}] = -X_{34}$$
$$[X_{23}, [X_{23}, X_{34}]] = [X_{23}, X_{24}] = -X_{34}$$
$$[X_{24}, [X_{24}, X_{34}]] = [X_{24}, -X_{23}] = -X_{34}$$
$$[X_{34}, [X_{34}, X_{34}]] = 0$$
$$[Y_{12}, [Y_{12}, X_{34}]] = 0$$
$$[Y_{13}, [Y_{13}, X_{34}]] = [Y_{13}, X_{14}] = -X_{34}$$
$$[Y_{14}, [Y_{14}, X_{34}]] = [Y_{14}, -X_{13}] = -X_{34}$$
$$[Y_{23}, [Y_{23}, X_{34}]] = [Y_{23}, X_{24}] = -X_{34}$$
$$[Y_{24}, [Y_{24}, X_{34}]] = [Y_{24}, -X_{23}] = -X_{34}$$
$$[Y_{34}, [Y_{34}, X_{34}]] = -4X_{34}$$

Summing the above, we obtain $CX_{34} = -16X_{34}$. Thus the eigenvalue of the unnormalized Intrinsic Casimir operator in $su(4)$ is -16.

We continue with the trace of X_{12} in $su(3)$:

$$[X_{12}, [X_{12}, X_{12}]] = 0$$

$$[X_{12},[X_{12},X_{13}]] = [X_{12},-X_{23}] = -X_{13}$$
$$[X_{12},[X_{12},X_{23}]] = [X_{12},X_{13}] = -X_{23}$$
$$[X_{12},[X_{12},Y_{12}]] = -4Y_{12}$$
$$[X_{12},[X_{12},Y_{13}]] = [X_{12},-X_{23}] = -X_{13}$$
$$[X_{12},[X_{12},Y_{23}]] = [X_{12},Y_{13}] = -X_{23}$$

Trace of $[X_{12},[X_{12}, = -12$ in $su(3)$.

We proceed with the terms of the Casimir operator of $su(3)$:

$$[X_{12},[X_{12},X_{12}]] = 0$$

$$[X_{13},[X_{13},X_{12}]] = [X_{13},X_{23}] = -X_{12}$$
$$[X_{23},[X_{23},X_{12}]] = [X_{23},-X_{13}] = -X_{12}$$
$$[Y_{12},[Y_{12},X_{12}]] = 0$$
$$[Y_{13},[Y_{13},X_{12}]] = [Y_{13},Y_{23}] = -X_{12}$$
$$[Y_{23},[Y_{23},X_{12}]] = [Y_{23},-Y_{13}] = -X_{12}$$

Summing the above, we obtain $CX_{34} = -12X_{34}$. Thus, the eigenvalue of the unnormalized Intrinsic Casimir operator in $su(3)$ is -12.

Theorem In $su(n)$, the trace of $[X_{IJ},[X_{IJ},$ is $-4n$ which is also the eigenvalue of the unnormalized Intrinsic Casimir operator in $su(n)$. Thus, the eigenvalue of the normalized Intrinsic Casimir operator is 1.

Proof There are two statements to prove: in $su(n)$, the trace of $[X_{IJ},[X_{IJ},$ is $-4n$ and in $su(n)$, the eigenvalue of the unnormalized Intrinsic Casimir operator in $su(n)$ is $-4n$. We will prove the statements by induction. We have already shown them to be true for $n = 3, 4,$ and 5. Those calculations reveal the patterns. The dimension of $su(n)$ is n^2-1. Thus going from $su(n)$ to $su(n+1)$ we add $(n+1)^2-1-(n^2-1) = 2n+1$ generators. Computing the trace of X_{IJ}, $I < J < n+1$. we see that all terms are zero except

$$[X_{IJ},[X_{IJ},X_{I(n+1)}]] = -X_{I(n+1)}$$

$$[X_{IJ},[X_{IJ},X_{J(n+1)}]] = -X_{J(n+1)}$$
$$[X_{IJ},[X_{IJ},Y_{I(n+1)}]] = -Y_{I(n+1)}$$
$$[X_{IJ},[X_{IJ},Y_{J(n+1)}]] = -Y_{J(n+1)}$$

Thus the trace of $[X_{IJ}, [X_{IJ},$ in $su(n+1)$ = the trace of $[X_{IJ}, [X_{IJ},$ in $su(n)$ minus 4, which by the induction hypothesis is $-4n - 4 = -4(n+1)$ Thus proving the formula for the trace.

In the intrinsic Casimir operator, there are n new terms and when acting on X_{IJ}, they are all zero except:

$$[X_{I(n+1)}, [X_{I(n+1)}, X_{IJ}]] = -X_{IJ}$$

$$[X_{J(n+1)}, [X_{J(n+1)}, X_{IJ}]] = -X_{IJ}$$

$$[Y_{I(n+1)}, [Y_{I(n+1)}, X_{IJ}]] = -X_{IJ}$$

$$[Y_{J(n+1)}, [Y_{J(n+1)}, X_{IJ}]] = -X_{IJ}$$

Thus, the eigenvalue of the intrinsic Casimir operator is also decreased by 4 and the same calculation show the validity of the formula for the eigenvalue of the intrinsic Casimir operator. Thus the normalized Intrinsic intrinsic Casimir Operator of $su(n)$ is not just a multiple of the identity, it is the identity operator, but only when acting on elements of su(n)!

13 An Attempt at Reconciliation

Perhaps we can reconcile the two different ways of looking at the Casimir operator if we expand the action of the intrinsic Casimir operator of $so(3)$ in terms of matrix multiplication:

$$CA = [X_1, [X_1, A]] + [X_2, [X_2, A]] + [X_3, [X_3, A]] \tag{27}$$

$$= [X_1, X_1A - AX_1] + [X_2, X_2A - AX_2]] + [X_3, X_3A - X_3]$$

$$= X_1X_1A - X_1AX_1 + X_2X_2A - X_2AX_2 + X_3X_3A - X_3AX_3$$

$$-(X_1AX_1 - AX_1X_1 + X_2AX_2 - AX_2X_2 + X_3AX_3 - AX_3X_3)$$

$$= X_1X_1A + X_2X_2A + X_3X_3A - 2X_1AX_1 - 2X_2AX_2 - 2X_3AX_3$$

$$+AX_1X_1 + AX_2X_2 + AX_3X_3$$

$$= (X_1X_1 + X_2X_2 + X_3X_3)A - 2X_1AX_1 - 2X_2AX_2 - 2X_3AX_3$$

$$+A(X_1X_1 + X_2X_2 + X_3X_3)$$

$$= (X_1X_1 + X_2X_2 + X_3X_3)A \quad (28)$$

(a)

$$-2(X_1AX_1 + X_2AX_2 + X_3AX_3)$$

(b)

$$+A(X_1X_1 + X_2X_2 + X_3X_3)$$

(c)

The terms in (28) have been labeled in order to facilitate discussion. The expressions in parentheses in (28) (a and c), i.e. the sum of the squares

$$(X_1X_1 + X_2X_2 + X_3X_3)$$

is the standard way of interpreting the intrinsic Casimir operator. Thus, the sum of the squares of the matrices is a multiple of the identity matrix, but this is not an invariant operator since the eigenvalue varies with the representation. The invariant operator is obtained as the sum of the matrix terms and its eigenvalue does not depend on the representation.

There is another point of confusion in Quantum Mechanics: the dual role of the Casimir operator as a matrix operator and as an invariant differential operator. Acting on elements of the Lie algebra the intrinsic Casimir operator has only one eigenvalue. Acting on functions, the intrinsic Casimir operator has an infinite range of eigenvalues. Schiff [26](p. 82) has the Casimir operator (as differential operator) acting on the spherical harmonics:

$$L^2Y_{lm}(\theta, \phi) = l(l+1)Y_{lm}(\theta, \phi)$$

This action of L^2 is as a differential operator acting on functions defined on the manifold $SO(3)$. Because Schiff follows the physics tradition and multiplies each generator by a factor of i in our notation we would have instead:

$$CY_{lm}(\theta, \phi) = -l(l+1)Y_{lm}(\theta, \phi)$$

The standard approach to quantum theory of angular momentum attempts to force the matrix representations do the job the eigenfunctions should be doing.

In order to obtain a spectrum from the Casimir operator using matrices, the standard approach requires an new representation. In order to obtain a spectrum from a differential operator representation of the intrinsic Casimir operator, a new representation is not necessary, rather using a representation by differential operators, a family of eigenfunctions of the same differential operator is constructed (i.e. the spherical harmonics). Then in any representation the eigenvalues are the same, otherwise the differential operator is not well defined.

In one approach applying group theoretical ideas to quantum theory, according to Salam [25]:

> The wave functions are classified in terms of representations of the group...

Unfortunately, that program degenerated into finding matrix representations. That program presumes that quantum numbers vary from one matrix representation to another matrix representation. In light of the results presented here, that program must be abandoned and replaced by the geometric program in which the only numbers which matter are those which are independent of the representation or coordinate system. These numbers are not just obtained from matrices, but from eigenvalue equations involving the intrinsic Casimir operators (i.e. from a maximal set of commuting differential operators). In the standard approach, the quantum numbers are eigenvalues of the intrinsic Casimir operators operating on matrices. In the author's approach to quantum theory, some of those quantum numbers come from the eigenvalues of the intrinsic Casimir operators, as differential operators, acting on appropriate functions. Since there is a relation between eigenfunctions and representations, the two approaches are not as disparate as they sound.

We can go one step further and replace the Lie bracket by the Lie derivative. Recall that the Lie derivative satisfies $L_X Y = [X, Y]$. Then the intrinsic Casimir operator is:

$$C = \sum_{ij} g^{ij} L_{X_i} L_{X_j}$$

and for a vector field Y,

$$CY = \sum_{ij} g^{ij} L_{X_i} L_{X_j} Y = \sum_{ij} g^{ij} [X_i, [X_j, Y]$$

This new interpretation of the Casimir operator now allows us to investigate the action of C on other geometric objects.

14 The Enveloping Algebra

Let us examine the definition of a Lie algebra:

A real (complex) Lie algebra is a real (complex) vector space with a rule of composition denoted

$[X, Y]$ satisfying:

(i) $[,]$ is R (C) bilinear;

(ii) $[X, Y] = -[Y, X]$;

(iii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

This is standard mathematical fare. The algebraic rules for the Lie algebras follow directly from the algebraic rules for Lie groups in much the same way the rules for logarithms follow from the rules for exponents.

Compare the above rules with the four principle rules for commutator algebras as given by Messiah [21]:

$[A, B] = -[B, A]$ (V.63)

$[A, B + C] = [A, B] + [A, C]$ (V.64)

$[A, BC] = [A, B]C + B[A, C]$ (V.65)

$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$ (V.66)

Clearly, Messiah's (V.63) is the same as (i); (V.66) is the same as (iii); (V.64) is part of (i), but Messiah's (V.65) has no counterpart in the definition of a Lie algebra.

Barry G. Adams [1] discusses "Commutator Gymnastics":

> The following two identities are useful for moving operators in products outside the commutator brackets:
>
> $[A, BC] = [A, B]C + B[A, C]$
>
> $[AB, C] = A[B, C] + [A, C]B$
>
> These rules are of fundamental importance and are used in virtually all calculations involving the simplification of commutators. They have the same structure and importance as the product rule for differentiation does in the calculus: in fact, defining $D_A(B) = [A, B]$, the first rule becomes
>
> $$D_A(BC) = BD_A(C) + D_A(B)C.$$

Beginning with the Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Take two terms to the right hand side of the equation:

$$[X,[Y,Z]] = -[Y,[Z,X]] - [Z,[X,Y]]$$

Apply the anti-symmetry property to the two terms on the right hand side:

(iv) $[X,[Y,Z]] = [Y,[X,Z]] + [[X,Y],Z]$

Comparing (iv) to the second of Adams' relations and Messiah's (v.65) we see a structural similarity. And, indeed, if we define $D_X(Y) = [X,Y]$ then we have the identity:

$$D_X([Y,Z]) = [Y,D_X(Z)] + [D_X(Y),Z].$$

This derivative is well known and is called the Lie derivative.

Thus it seems that the derivative defined by Adams is a cheap parody, a grotesque caricature of the Lie derivative.

The rules of "commutator gymnastics" are not the rules for Lie algebras and chaos follows if the two are confused.

Revisit the definition: a Lie algebra is a vector space equipped with an operation denoted by $[A,B]$. There is no mention of the Lie algebra being equipped with a product AB. Indeed, it is *not* part of the definition. Once two matrices are multiplied together, the product is outside the structure of Lie Algebras.

Let us digress and discuss a more elementary example. The allowed operations on a logarithm are:

$$log(xy) = logx + logy$$

$$log(x/y) = logx - logy$$

$$logy^r = rlogy$$

Now, $logx$ and $logy$ are real numbers and real numbers can be multiplied together: $(logx)(logy)$. But the product has no significance as a logarithm, unless you are changing bases.

The transition from Lie Algebra to Lie Group is via the exponential map. Thus the map from Lie Group to Lie Algebra is the logarithm. In this case, we are dealing with matrices, and matrices (even those which represent Lie algebras) can be multiplied together, but once you have multiplied them together, they have ceased to have significance as logarithms, they have ceased to represent the Lie algebra. Multiplication of matrices is meaningless in the context of Lie algebras (except as a tool for computing the Lie bracket within a given representation).

For a Lie algebra $\mathcal{L}$, the universal enveloping algebra of $\mathcal{L}$, $U(\mathcal{L})$ is the "free algebra" with the elements of $\mathcal{L}$ as generators modulo the relation $XY - YX - [X, Y] = 0$ [9] (pp. 72-73).

Alternately, [5] the universal enveloping algebra of $\mathcal{L}$ can be defined in terms of the tensor algebra of $\mathcal{L}$ which is a vector space over the field k. If we define

$$T_n = \mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L} \ldots \otimes \mathcal{L}$$

(n factors of $\mathcal{L}$), with $T_0 = k$

Define

$$T = T_0 \otimes T_1 \otimes T_2 \otimes \ldots T_n \otimes \ldots$$

The product in T is just tensor multiplication.

Let J be the two-sided ideal of T generated by all objects

$$X \otimes Y - Y \otimes X - [X, Y]$$

for all X,Y in $\mathcal{L}$. In this setting, $U(\mathcal{L}) = T/J$. The two definitions are equivalent but the second is more elegant.

In the enveloping algebra approach, the Casimir operators are the elements of the center of $U(\mathcal{L})$, which is denoted $Z(\mathcal{L})$. If we are to keep this relationship with our reinterpretation of the Casimir operator, we must also reinterpret the universal enveloping algebra formalism.

The relation $X \otimes Y - Y \otimes X - [X, Y] = 0$ holds in $U(\mathcal{L})$ and is supposed to mimic the relation $XY - YX = [X, Y]$, which as Hermann [9] points out is an attempt to force a meaning to the symbol XY. If this is the case, exactly what meaning does the theory of the enveloping algebra force upon XY? The symbol XY can be given a meaning without such elaborate underpinnings. As we did with the intrinsic Casimir operator, we can interpret XY as an operator: XYZ becomes $[X, [Y, Z]]$. To emphasize this interpretation, we will write [X,[Y instead of just XY.

From the Jacobi identity, for $X, Y, Z \in \mathcal{L}$,

$$[X, [Y, Z]] - [Y, [X, Z]] = [[X, Y], Z]$$

Dropping the Z], we obtain an operator equation:

$$[X, [Y, -[Y, [X, = [[X, Y],$$

With this interpretation, the relation $X \otimes Y - Y \otimes X - [X, Y]$ in the definition of $U(\mathcal{L})$ is shown to be the Jacobi identity, *not the commutator* (or perhaps we should say that the commutator and the Jacobi identity are the same). Geometrically, we are replacing the element X of the Lie algebra by the Lie derivative with respect to X. The Lie derivative satisfies:

$$\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X = \mathcal{L}_{[X,Y]}$$

Indeed, since $\mathcal{L}_X Z = [X, Z]$, the rearranged Jacobi identity:

$$[X, [Y, Z]] - [Y, [X, Z]] = [[X, Y], Z]$$

is a Lie derivative equation:

$$\mathcal{L}_X \mathcal{L}_Y(Z) - \mathcal{L}_Y \mathcal{L}_X(Z) = \mathcal{L}_{[X,Y]} Z$$

So it seems that we are led to interpret the universal enveloping algebra of $\mathcal{L}$, $U(\mathcal{L})$ as the "free algebra" with generators the Lie derivatives with respect to the elements of $\mathcal{L}$. The equivalence relation "modulo the relation $XY - YX - [X, Y]$" is not necessary since it is built into the Lie derivative.

To avoid confusion, we need a name for the new formalism and I suggest it be called the "geometric enveloping algebra."

In the Enveloping algebra approach to the Casimir operator, the Casimir operator is taken to be an element of the Enveloping Algebra in which case $C^2 Y = \sum_{ij} g^{ij} X_i \otimes X_j \otimes Y$, with the tensor product as the multiplication.

In the geometric enveloping algebra, this translates as:

$$CY = \sum_{ij} g^{ij} [X_i, [X_j, [Y \tag{29}$$

This provides an alternative way of interpreting the symbols, but it is not a definition of the Casimir operator C, instead, (29) is a definition of the operator CY. Still, some expressions in the standard theory of enveloping algebras must be interpreted in terms of (29) instead of (8). The difference between (29) and (8) comes down to the difference in roles of X and Y in the expression $[X, Y]$ interpreted as the operator, $[X,$ acting on the element of the vector space, $Y]$. This is the same problem in mathematical semantics addressed by Dirac in his treatment of Bras and Kets. If Y is being acted upon (a ket), then (8) holds and is viewed as

$$C^2(Y]) = \sum_{ij} g^{ij}[X_i, [X_j, Y]] \tag{30}$$

If instead, Y is viewed as an operator (a bra), then (29) holds and is viewed as:

$$C([Y) = \sum_{ij} g^{ij}[X_i, [X_j, [Y, \tag{31}$$

The geometric enveloping algebra is associative because left bracketing is a function from $\mathcal{L}$ to $\mathcal{L}$ and composition of functions is associative. The enveloping algebra has only bras, no kets. Thus, we could say that the Lie Bracket is "left associative" and becomes non-associative only when left and right brackets are mixed. A free algebra of only kets would also be associative. The Lie bracket is also right associative. It is only when bras and kets are mixed that the algebra becomes nonassociative.

15 Intrinsic Casimir operators with eigenvalue Zero

Examining the commutators for $so(3,1)$, we observe that there are precisely two ways to obtain each X_I:

$$[X_2, X_3] = X_1$$

$$[N_2, N_3] = -X_1$$
$$[X_3, X_1] = X_2$$
$$[N_3, N_1] = -X_2$$
$$[X_1, X_2] = X_3$$
$$[N_1, N_2] = -X_3$$

Thus we can write the Zero operator as:

$$0 = X_1 - X_1 + X_2 - X_2 + X_3 - X_3$$

$$= [X_2, X_3] + [N_2, N_3] + [X_3, X_1] + [N_3, N_1] + [X_1, X_2] + [N_1, N_2]$$

Which arises in the expansion of

$$\sigma^2 Y = [X_1, [X_2, Y]] + [X_2, [X_3, Y]] + [X_3, [X_1, Y]]$$
$$+[N_1, [N_2, Y]] + [N_2, [N_3, Y]] + [N_3, [N_1, Y]]$$

and we check the eigenvalues:

$$\sigma^2 X_1 = [X_1, [X_2, X_1]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_1]]$$
$$+[N_1, [N_2, X_1]] + [N_2, [N_3, X_1]] + [N_3, [N_1, X_1]]$$
$$= [X_1, -X_3] + [N_1, -N_3] = X_2 - X_2 = 0$$

$$\sigma^2 X_2 = [X_1, [X_2, X_2]] + [X_2, [X_3, X_2]] + [X_3, [X_1, X_2]]$$
$$+[N_1, [N_2, X_2]] + [N_2, [N_3, X_2]] + [N_3, [N_1, X_2]]$$
$$= [X_2, -X_1] + [N_2, -N_1] = X_3 - X_3 = 0$$

$$\sigma^2 X_3 = [X_1, [X_2, X_3]] + [X_2, [X_3, X_3]] + [X_3, [X_1, X_3]]$$
$$+[N_1, [N_2, X_3]] + [N_2, [N_3, X_3]] + [N_3, [N_1, X_3]]$$
$$= [X_3, -X_2] + [N_3, -N_2] = X_1 - X_1 = 0$$

$$\sigma^2 N_1 = [X_1, [X_2, N_1]] + [X_2, [X_3, N_1]] + [X_3, [X_1, N_1]]$$
$$+[N_1, [N_2, N_1]] + [N_2, [N_3, N_1]] + [N_3, [N_1, N_1]]$$
$$= [X_1, -N_3] + [N_1, X_3] = N_2 - N_2 = 0$$

$$\sigma^2 N_2 = [X_1, [X_2, N_2]] + [X_2, [X_3, N_2]] + [X_3, [X_1, N_2]]$$
$$+[N_1, [N_2, N_2]] + [N_2, [N_3, N_2]] + [N_3, [N_1, N_2]]$$
$$= [X_2, -N_1] + [N_2, X_1] = N_3 - N_3 = 0$$

$$\sigma^2 N_3 = [X_1, [X_2, N_3]] + [X_2, [X_3, N_3]] + [X_3, [X_1, N_3]]$$
$$+[N_1, [N_2, N_3]] + [N_2, [N_3, N_3]] + [N_3, [N_1, N_3]]$$

$$= [X_3, -N_2] + [N_3, X_2] = -N_1 + N_1 = 0$$

Thus σ^2 is a Casimir operator with eigenvalue zero.

We can repeat the process by noting that there are two ways to obtain each N_I.

Starting with the relations:

$$[X_2, N_3] = N_1$$

$$-[N_2, X_3] = -N_1$$

$$[X_3, N_1] = N_2$$

$$-[N_3, X_1] = -N_2$$

$$[X_1, N_2] = N_3$$

$$-[N_1, X_2] = -N_3$$

we add to obtain:

$$[X_2, N_3] - [N_2, X_3] + [X_3, N_1] - [N_3, X_1] + [X_1, N_2] - [N_1, X_2] = 0$$

Which arises in the expansion of:

$$\kappa Y = [X_2, [N_3, Y]] - [N_2, [X_3, Y]] + [X_3, [N_1, Y]]$$

$$-[N_3, [X_1, Y]] + [X_1, [N_2, Y]] - [N_1, [X_2, Y]]$$

Computing the action of κ, we obtain:

$$\kappa X_1 = [X_2, [N_3, X_1]] - [N_2, [X_3, X_1]] + [X_3, [N_1, X_1]]$$

$$-[N_3, [X_1, X_1]] + [X_1, [N_2, X_1]] - [N_1, [X_2, X_1]]$$

$$= [X_1, -N_3] - [N_1, X_3] = N_2 - N_2 = 0$$

$$\kappa X_2 = [X_2, [N_3, X_2]] - [N_2, [X_3, X_2]] + [X_3, [N_1, X_2]]$$

$$-[N_3, [X_1, X_2]] + [X_1, [N_2, X_2]] - [N_1, [X_2, X_2]]$$

$$= [X_2, -N_1] - [N_2, -X_1] = N_3 - N_3 = 0$$

$$\kappa X_3 = [X_2, [N_3, X_3]] - [N_2, [X_3, X_3]] + [X_3, [N_1, X_3]]$$

$$-[N_3,[X_1,X_3]]+[X_1,[N_2,X_3]]-[N_1,[X_2,X_3]]$$
$$=[X_3,-N_2]-[N_3,-X_2]=N_1-N_1=0$$

$$\kappa N_1=[X_2,[N_3,N_1]]-[N_2,[X_3,N_1]]+[X_3,[N_1,N_1]]$$
$$-[N_3,[X_1,N_1]]+[X_1,[N_2,N_1]]-[N_1,[X_2,N_1]]$$
$$=[X_1,X_3]-[N_1,-N_3]=-X_2+X_2=0$$

$$\kappa N_2=[X_2,[N_3,N_2]]-[N_2,[X_3,N_2]]+[X_3,[N_1,N_2]]$$
$$-[N_3,[X_1,N_2]]+[X_1,[N_2,N_2]]-[N_1,[X_2,N_2]]$$
$$=[X_2,X_1]-[N_2,-N_1]=-X_3+X_3=0$$

$$\kappa N_3=[X_2,[N_3,N_3]]-[N_2,[X_3,N_3]]+[X_3,[N_1,N_3]]$$
$$-[N_3,[X_1,N_3]]+[X_1,[N_2,N_3]]-[N_1,[X_2,N_3]]$$
$$=[X_3,X_2]-[N_3,-N_2]=-X_1+X_1=0$$

Thus we have another unexpected Casimir operator of eigenvalue type.

The Casimir operator

$$\sigma^2 Y=[X_1,[X_2,Y]]+[X_2,[X_3,Y]]+[X_3,[X_1,Y]]$$
$$+[N_1,[N_2,Y]]+[N_2,[N_3,Y]]+[N_3,[N_1,Y]]$$

may be viewed as being associated with the decomposition of $so(3,1)$ into the $so(3)$ subalgebra generated by the X_I, and its complement, consisting of the N_I. The corresponding operator with respect to the subalgebra 2 decomposition is:

$$\sigma_2^2 Y=[X_1,[N_2,Y]]+[N_2,[N_3,Y]]+[N_3,[X_1,Y]]$$
$$-[N_1,[X_2,Y]]+[X_2,[X_3,Y]]-[X_3,[N_1,Y]]$$

We calculate:

$$\sigma_2^2 X_1=[X_1,[N_2,X_1]]+[N_2,[N_3,X_1]]+[N_3,[X_1,X_1]]$$
$$-[N_1,[X_2,X_1]]+[X_2,[X_3,X_1]]-[X_3,[N_1,X_1]]$$

$$= [X_1, -N_3] - [N_1, -X_3] = N_2 - N_2 = 0$$

$$\sigma_2^2 X_2 = [X_1, [N_2, X_2]] + [N_2, [N_3, X_2]] + [N_3, [X_1, X_2]]$$

$$-[N_1, [X_2, X_2]] + [X_2, [X_3, X_2]] - [X_3, [N_1, X_2]]$$

$$= [N_2, -N_1] + [X_2, -X_1] = -X_3 + X_3 = 0$$

$$\sigma_2^2 X_3 = [X_1, [N_2, X_3]] + [N_2, [N_3, X_3]] + [N_3, [X_1, X_3]]$$

$$-[N_1, [X_2, X_3]] + [X_2, [X_3, X_3]] - [X_3, [N_1, X_3]]$$

$$= [N_3, -X_2] - [X_3, -N_2] = N_1 - N_1 = 0$$

$$\sigma_2^2 N_1 = [X_1, [N_2, N_1]] + [N_2, [N_3, N_1]] + [N_3, [X_1, N_1]]$$

$$-[N_1, [X_2, N_1]] + [X_2, [X_3, N_1]] - [X_3, [N_1, N_1]]$$

$$= [X_1, X_3] - [N_1, -N_3] = -X_2 + X_2 = 0$$

$$\sigma_2^2 N_2 = [X_1, [N_2, N_2]] + [N_2, [N_3, N_2]] + [N_3, [X_1, N_2]]$$

$$-[N_1, [X_2, N_2]] + [X_2, [X_3, N_2]] - [X_3, [N_1, N_2]]$$

$$= [N_2, X_1] + [X_2, -N_1] = -N_3 + N_3 = 0$$

$$\sigma_2^2 N_3 = [X_1, [N_2, N_3]] + [N_2, [N_3, N_3]] + [N_3, [X_1, N_3]]$$

$$-[N_1, [X_2, N_3]] + [X_2, [X_3, N_3]] - [X_3, [N_1, N_3]]$$

$$= [N_3, -N_2] - [X_3, X_2] = -X_1 + X_1 = 0$$

Thus σ_2^2 is a Casimir operator with eigenvalue 0.
The operator constructed from subalgebra 3, corresponding to σ^2 is:

$$\sigma_3^2 Y = [N_1, [X_2, Y]] + [X_2, [N_3, Y]] + [N_3, [N_1, Y]]$$

$$-[X_1, [N_2, Y]] - [N_2, [X_3, Y]] + [X_3, [X_1, Y]]$$

We calculate the action on a basis:

$$\sigma_3^2 X_1 = [N_1, [X_2, X_1]] + [X_2, [N_3, X_1]] + [N_3, [N_1, X_1]]$$

$$-[X_1, [N_2, X_1]] - [N_2, [X_3, X_1]] + [X_3, [X_1, X_1]]$$

$$= [N_1, -X_3] - [X_1, -N_3] = N_2 - N_2 = 0$$

$$\sigma_3^2 X_2 = [N_1, [X_2, X_2]] + [X_2, [N_3, X_2]] + [N_3, [N_1, X_2]]$$
$$-[X_1, [N_2, X_2]] - [N_2, [X_3, X_2]] + [X_3, [X_1, X_2]]$$
$$= [X_2, -N_1] + [N_2, -X_1] = N_3 - N_3 = 0$$

$$\sigma_3^2 X_3 = [N_1, [X_2, X_3]] + [X_2, [N_3, X_3]] + [N_3, [N_1, X_3]]$$
$$-[X_1, [N_2, X_3]] - [N_2, [X_3, X_3]] + [X_3, [X_1, X_3]]$$
$$= [N_3, -N_2] + [X_3, -X_2]] = -X_1 + X_1 = 0$$

$$\sigma_3^2 N_1 = [N_1, [X_2, N_1]] + [X_2, [N_3, N_1]] + [N_3, [N_1, N_1]]$$
$$-[X_1, [N_2, N_1]] - [N_2, [X_3, N_1]] + [X_3, [X_1, N_1]]$$
$$= [N_1, -N_3] - [X_1, X_3] = -X_2 + X_2 = 0$$

$$\sigma_3^2 N_2 = [N_1, [X_2, N_2]] + [X_2, [N_3, N_2]] + [N_3, [N_1, N_2]]$$
$$-[X_1, [N_2, N_2]] - [N_2, [X_3, N_2]] + [X_3, [X_1, N_2]]$$
$$= [X_2, X_1] - [N_2, -N_1] = -X_3 + X_3 = 0$$

$$\sigma_3^2 N_3 = [N_1, [X_2, N_3]] + [X_2, [N_3, N_3]] + [N_3, [N_1, N_3]]$$
$$-[X_1, [N_2, N_3]] - [N_2, [X_3, N_3]] + [X_3, [X_1, N_3]]$$
$$= [N_3, X_2] + [X_3, -N_2]] = -N_1 + N_1 = 0$$

This confirms that σ_3^2 is a Casimir operator with eigenvalue 0.
The σ-like operator constructed for subalgebra 4 is:

$$\sigma_4^2 Y = [N_1, [N_2, Y]] + [N_2, [X_3, Y]] + [X_3, [N_1, Y]]$$
$$+[X_1, [X_2, Y]] - [X_2, [N_3, Y]] - [N_3, [X_1, Y]]$$

The calculations proceed as above:

$$\sigma_4^2 X_1 = [N_1, [N_2, X_1]] + [N_2, [X_3, X_1]] + [X_3, [N_1, X_1]]$$
$$+[X_1, [X_2, X_1]] - [X_2, [N_3, X_1]] - [N_3, [X_1, X_1]]$$

$$= [N_1, -N_3] + [X_1, -X_3] = -X_2 + X_2 = 0$$

$$\sigma_4^2 X_2 = [N_1, [N_2, X_2]] + [N_2, [X_3, X_2]] + [X_3, [N_1, X_2]]$$
$$+[X_1, [X_2, X_2]] - [X_2, [N_3, X_2]] - [N_3, [X_1, X_2]]$$
$$= [N_2, -X_1] - [X_2, -N_1] = N_3 - N_3 = 0$$

$$\sigma_4^2 X_3 = [N_1, [N_2, X_3]] + [N_2, [X_3, X_3]] + [X_3, [N_1, X_3]]$$
$$+[X_1, [X_2, X_3]] - [X_2, [N_3, X_3]] - [N_3, [X_1, X_3]]$$
$$= [X_3, -N_2] - [N_3, -X_2] = N_1 - N_1 = 0$$

$$\sigma_4^2 N_1 = [N_1, [N_2, N_1]] + [N_2, [X_3, N_1]] + [X_3, [N_1, N_1]]$$
$$+[X_1, [X_2, N_1]] - [X_2, [N_3, N_1]] - [N_3, [X_1, N_1]]$$
$$= [N_1, X_3] + [X_1, -N_3] = -N_2 + N_2 = 0$$

$$\sigma_4^2 N_2 = [N_1, [N_2, N_2]] + [N_2, [X_3, N_2]] + [X_3, [N_1, N_2]]$$
$$+[X_1, [X_2, N_2]] - [X_2, [N_3, N_2]] - [N_3, [X_1, N_2]]$$
$$= [N_2, -N_1] - [X_2, X_1] = -X_3 + X_3 = 0$$

$$\sigma_4^2 N_3 = [N_1, [N_2, N_3]] + [N_2, [X_3, N_3]] + [X_3, [N_1, N_3]]$$
$$+[X_1, [X_2, N_3]] - [X_2, [N_3, N_3]] - [N_3, [X_1, N_3]]$$
$$= [X_3, X_2] - [N_3, -N_2] = -X_1 + X_1 = 0$$

Thus σ_4^2 is a Casimir operator with eigenvalue 0.

16 Solo Casimir Operators

Surprisingly, there is yet another type of intrinsic Casimir operator.

The characteristic equation of the adjoint representation of $X_1 \in so(3)$ is the determinant of

$$det \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{pmatrix} = -\lambda^3 - \lambda$$

Since a matrix satisfies its own characteristic equation, we have:

$$-(adX_1)^3 - adX_1 = 0$$

We use the property of the adjoint representation to write the operator equation:

$$S_{X_1} = [X_1, [X_1, [X_1 + [X_1 = 0$$

and we compute, for verification:

$$S_{X_1} X_2 = [X_1, [X_1, [X_1, X_2]]] + [X_1, X_2]$$

$$= [X_1, [X_1, X_3]] + X_3 = [X_1, -X_2] + X_3$$

$$= -X_3 + X_3 = 0$$

$$S_{X_1} X_3 = [X_1, [X_1, [X_1, X_3]]] + [X_1, X_3] = 0$$

$$= [X_1, [X_1, -X_2]] - X_2$$

$$= [X_1, -X_3] - X_2 = X_2 - X_2 = 0$$

Likewise, from the adjoint representation of X_2 we obtain:

$$S_{X_2} = [X_2, [X_2, [X_2 + [X_2 = 0$$

From the adjoint representation of X_3 we obtain:

$$S_{X_3} = [X_3, [X_3, [X_3 + [X_3 = 0$$

Thus, we have three more third order Casimir operators for $so(3)$. Since the definition of S_{X_i} involves only X_i, these are a novel sort of Casimir operator which we will call solo-Casimir operators. We state the obvious:

Theorem on Solo-Casimir Operators

Every element of a simple Lie algebra $\mathcal{L}$ generates a solo Casimir operator.

Proof: Let $P(\lambda)$ be the characteristic equation of the matrix adX. Since a matrix satisfies its own characteristic equation, we have: $P(adX) = 0$ Thus, we have the operator equation $P([X) = 0$ which defines a solo-Casimir operator.

Observe that although the characteristic equation of adX has degree equal to the dimension of the Lie Algebra, there may be a polynomial of lower degree which adX may satisfy. The lowest degree polynomial which adX satisfies will be called the degree of X.

We leave it to the interested reader to confirm that for a generic matrix M,

$$S_{X_1}M = [X_1, [X_1, [X_1, M]]] + [X_1, M] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3m_{23} + 3m_{32} & -3m_{22} + 3m_{33} \\ 0 & -3m_{22} + 3m_{33} & -3m_{23} - 3m_{32} \end{pmatrix}$$

And furthermore, if a three by three matrix M is an eigenvector of all three solo operators:

$$S_{X_1}M = [X_1, [X_1, [X_1, M]]] + [X_1, M] = 0$$

$$S_{X_2}M = [X_2, [X_2, [X_2, M]]] + [X_2, M] = 0$$

$$S_{X_3}M = [X_3, [X_3, [X_3, M]]] + [X_3, M] = 0$$

then $M \in so(3)$.

The $su(2)$ solo operators follow the pattern for $so(3)$:

$$S_1 = [\alpha_1, [\alpha_1, [\alpha_1 + [\alpha_1$$

$$S_2 = [\alpha_2, [\alpha_2, [\alpha_2 + [\alpha_2$$

$$S_3 = [\alpha_3, [\alpha_3, [\alpha_3 + [\alpha_3$$

This construction of the third order solo-Casimir operators and the calculations done for $so(3)$ needs to be done for $so(2,1)$. The characteristic equation for the matrix of Y_1 in the adjoint representation is $\lambda_3 + \lambda = 0$ which leads us to consider the operator:

$$S_{Y_1} = [Y_1, [Y_1, [Y_1 + [Y_1$$

and we compute, for verification:

$$S_{Y_1}Y_2 = [Y_1, [Y_1, [Y_1, Y_2]]] + [Y_1, Y_2]$$

$$= [Y_1, [Y_1, -Y_3]] - Y_3$$

$$= [Y_1, -Y_2] - Y_3 = Y_3 - Y_3 = 0$$

The other computations are left to the reader.

Likewise, from the matrix of Y_2 in the adjoint representation, we obtain the characteristic equation $\lambda^3 - \lambda = 0$ and hence, the operator:

$$S_{Y_2} = [Y_2, [Y_2, [Y_2 - [Y_2 = 0$$

and we compute, for verification:

$$S_{Y_2}Y_1 = [Y_2, [Y_2, [Y_2, Y_1]]] - [Y_2, Y_1]$$

$$= [Y_2, [Y_2, Y_3]] + Y_3$$

$$= [Y_2, Y_1] - Y_3 = Y_3 - Y_3 = 0$$

Again, from the adjoint representation of Y_3 we obtain:

$$S_{Y_3} = [Y_3, [Y_3, [Y_3 - [Y_3 = 0$$

We compute the action of the solo Casimir operator of $so(3), S_{X_1}$, on the remaining elements of $so(3,1)$:

$$S_{X_1}N_1 = [X_1, [X_1, [X_1, N_1]]] + [X_1, N_1] = 0$$

$$S_{X_1}N_2 = [X_1, [X_1, [X_1, N_2]]] + [X_1, N_2]$$

$$= [X_1, [X_1, N_3]] + N_3$$

$$= [X_1, -N_2] + N_3 = -N_3 + N_3 = 0$$

$$S_{X_1}N_3 = [X_1, [X_1, [X_1, N_3]]] + [X_1, N_3]$$

$$= [X_1, [X_1, -N_2]] - N_2 = [X_1, -N_3] - N_2 = N_2 - N_2 = 0$$

Likewise, for the action of S_{X_2} we obtain:

$$S_{X_2}N_1 = [X_2, [X_2, [X_2, N_1]]] + [X_2, N_1]$$

$$= [X_2, [X_2, -N_3]] - N_3$$

$$= [X_2, -N_1] - N_3 = N_3 - N_3 = 0$$

$$S_{X_2}N_2 = [X_2, [X_2, [X_2, N_2]]] + [X_2, N_2] = 0$$

$$S_{X_2}N_3 = [X_2, [X_2, [X_2, N_3]]] + [X_2, N_3]$$

$$= [X_2, [X_2, N_1]] + N_1 = [X_2, -N_3] + N_1 = -N_1 + N_1 = 0$$

In the same way, for the action of S_{X_3} we obtain:

$$S_{X_3}N_1 = [X_3, [X_3, [X_3, N_1]]] + [X_3, N_1] = 0$$

$$= [X_3, [X_3, N_2]] + N_2$$

$$= [X_3, -N_1] + N_2 = -N_2 + N_2 = 0$$

$$S_{X_3}N_2 = [X_3, [X_3, [X_3, N_2]]] + [X_3, N_2]$$

$$= [X_3, [X_3, -N_1]] - N_1 = [X_3, -N_2] - N_1 = N_1 - N_1 = 0$$

$$S_{X_3}N_3 = [X_3, [X_3, [X_3, N_3]]] + [X_3, N_3] = 0$$

Thus, three more third order Casimir operators of $so(3)$ prove also to be Casimir operators for $so(3,1)$. These are solo-Casimir operators of degree three while the Theorem on solo-Casimir operators would only guarantee degree six.

We imitate the action of S_{X_I} by replacing the X_I with the corresponding N_I and defining S_{N_I} (we also replace the + by -, since the N are non-compact):

$$S_{N_I} = [N_I, [N_I, [N_I - [N_I$$

and we compute the action of the S_{N_I} on so(3,1):

$$S_{N_1}X_1 = [N_1, [N_1, [N_1, X_1]]] - [N_1, X_1] = 0$$

$$S_{N_1}X_2 = [N_1, [N_1, [N_1, X_2]]] - [N_1, X_2]$$

$$= [N_1, [N_1, N_3]] - N_3 = [N_1, X_2] - N_3 = N_3 - N_3 = 0$$

$$S_{N_1}X_3 = [N_1, [N_1, [N_1, X_3]]] - [N_1, X_3]$$

$$= [N_1, [N_1, -N_2]] + N_2 = [N_1, -X_3] + N_2 = -N_2 + N_2 = 0$$

$$S_{N_1}N_1 = [N_1, [N_1, [N_1, N_1]]] - [N_1, N_1] = 0$$

$$S_{N_1}N_2 = [N_1, [N_1, [N_1, N_2]]] - [N_1, N_2]$$

$$= [N_1, [N_1, -X_3]] + X_3 = [N_1, N_2] + X_3 = -X_3 + X_3 = 0$$

$$S_{N_1}N_3 = [N_1, [N_1, [N_1, N_3]]] - [N_1, N_3]$$

$$= [N_1, [N_1, X_2]] - X_2 = [N_1, N_3] - X_2 = X_2 - X_2 = 0$$

Likewise, for the action of S_{N_2} we obtain:

$$S_{N_2}X_1 = [N_2, [N_2, [N_2, X_1]]] - [N_2, X_1]$$

$$= [N_2, [N_2, -N_3]] + N_3 = [N_2, X_1] + N_3 = -N_3 + N_3 = 0$$

$$S_{N_2}X_2 = [N_2, [N_2, [N_2, X_2]]] - [N_2, X_2] = 0$$

$$S_{N_2}X_3 = [N_2, [N_2, [N_2, X_3]]] - [N_2, X_3]$$

$$= [N_2, [N_2, N_1]] - N_1 = [N_2, X_3] - N_1 = N_1 - N_1 = 0$$

$$S_{N_2}N_1 = [N_2, [N_2, [N_2, N_1]]] - [N_2, N_1]$$

$$= [N_2, [N_2, X_3]] - X_3 = [N_2, N_1] - X_3 = X_3 - X_3 = 0$$

$$S_{N_2}N_2 = [N_2, [N_2, [N_2, N_2]]] - [N_2, N_2] = 0$$

$$S_{N_2}N_3 = [N_2,[N_2,[N_2,N_3]]] - [N_2,N_3] = [N_2,[N_2,-X_1]] + X_1$$

$$= [N_2,N_3] + X_1 = -X_1 + X_1 = 0$$

For the action of S_{N_3} we obtain:

$$S_{N_3}N_1 = [N_3,[N_3,[N_3,N_1]]] - [N_3,N_1] = [N_3,[N_3,-X_2]] + X_2$$

$$= [N_3,N_1] + X_2 = -X_2 + X_2 = 0$$

$$S_{N_3}N_2 = [N_3,[N_3,[N_3,N_2]]] - [N_3,N_2] = [N_3,[N_3,X_1]] - X_1$$

$$= [N_3,N_2] - X_1 = X_1 - X_1 = 0$$

$$S_{N_3}N_3 = [N_3,[N_3,[N_3,N_3]]] - [N_3,N_3] = 0$$

$$S_{N_3}N_1 = [N_3,[N_3,[N_3,N_1]]] - [N_3,N_1] = [N_3,[N_3,-X_2]] + X_2$$

$$= [N_3,N_1] + X_2 = -X_2 + X_2 = 0$$

$$S_{N_3}N_2 = [N_3,[N_3,[N_3,N_2]]] - [N_3,N_2]$$

$$= [N_3,[N_3,X_1]] - X_1$$

$$= [N_3,N_2] - X_1 = X_1 - X_1 = 0$$

$$S_{N_3}N_3 = [N_3,[N_3,[N_3,N_3]]] - [N_3,N_3] = 0$$

So now we have three more third order Casimir operators for $so(3,1)$ for a total of six solo-Casimir operators all of degree three.

The construction of solo-Casimir operators remains to be done for $su(3)$. We begin by checking to see if:

$$S_{X_1} = [X_1,[X_1,[X_1 + [X_1 = 0$$

acting on $su(3)$.

From our calculations for $so(3)$ we have:

$$S_{X_1} X_1 = 0$$

$$S_{X_1} X_2 = 0$$

$$S_{X_1} X_3 = 0$$

and we compute the remaining actions:

$$S_{X_1} Y_1 = [X_1, [X_1, [X_1, Y_1]]] + [X_1, Y_1] = 0$$

$$= [X_1, [X_1, R - S]] + R - S$$

$$= [X_1, -4Y_1] + R - S = -3R + 3S$$

At this point we could stop with the observation that it fails. However, we will continue to see exactly by how much it fails.

$$S_{X_1} Y_2 = [X_1, [X_1, [X_1, Y_2]]] + [X_1, Y_2]$$

$$= [X_1, [X_1, -Y_3]] - Y_3$$

$$= [X_1, -Y_2] - Y_3 = Y_3 - Y_3 = 0$$

$$S_{X_1} Y_3 = [X_1, [X_1, [X_1, Y_3]]] + [X_1, Y_3]$$

$$= [X_1, [X_1, -Y_2]] - Y_2$$

$$= [X_1, -Y_3] - Y_2 = Y_2 - Y_2 = 0$$

$$S_{X_1} R = [X_1, [X_1, [X_1, R]]] + [X_1, R] = 0$$

$$S_{X_1} S = [X_1, [X_1, [X_1, S]]] + [X_1, S] = 0$$

Thus S_{X_1} fails as a solo-Casimir operator for $su(3)$ only in its action on Y_1. To find out why it fails we look at the eigenvalues of $[X_1, [X_1,$

$$[X_1, [X_1, X_1]] = 0$$

$$[X_1, [X_1, X_2]] = -X_2$$

$$[X_1,[X_1,X_3]] = -X_3$$
$$[X_1,[X_1,Y_1]] = -4Y_1$$
$$[X_1,[X_1,Y_2]] = -Y_2$$
$$[X_1,[X_1,Y_3]] = -Y3$$
$$[X_1,[X_1,R]] = [X_1,-Y_1] = S-R$$
$$[X_1,[X_1,S]] = [X_1,3Y_1] = 3R-3S$$

We can find an eigen-basis: $3R+S$ with eigenvalue 0 and $S-R$ with eigenvalue -4.

For each eigenvalue there is a factor in the minimal equation. The eigenvalue 0 leads to the factor $[X_1,$, while the eigenvalue -1 leads to the factor $[X_1,[X_1,+1$ and the eigenvalue -4 leads to the factor $[X_1,[X_1,+4$. Note that the eigenvalue -4 is associated only with the eigenvector $Y_1]$.

We multiply to obtain the minimal polynomial:

$$([X_1,)([X_1,[X_1,+1)([X_1,[X_1,+4) = ([X_1,)^5 + 5([X_1,)^3 + 4[X_1,$$

This is degree five, less than the degree eight guaranteed by the theorem on solo Casimir operators.

The eigenvalues of $[X_2,[X_2,$ lead to the identical conclusion:

$$[X_2,[X_2,X_1]] = -X_1$$
$$[X_2,[X_2,X_2]] = 0$$
$$[X_2,[X_2,X_3]] = -X_3$$
$$[X_2,[X_2,Y_1]] = -Y_1$$
$$[X_2,[X_2,Y_2]] = -4Y_2$$
$$[X_2,[X_2,Y_3]] = -Y_3$$
$$[X_2,[X_2,R]] = [X_2,-Y_2] = -S-R$$
$$[X_2,[X_2,S]] = [X_2,-3Y_2] = -3R-3S$$

Again we can diagonalize these last two to obtain an eigen-basis: $3R-S$ with eigenvalue 0 and $R+S$ with eigenvalue -4.

The eigenvalues again lead to factors of the minimal equation: the eigenvalue 0 implies the factor $([X_2, [X_2,$; the eigenvalue -1 implies the factor $([X_2, [X_2, +1$ while the eigenvalue -4 implies the factor $([X_2, [X_2, +4)$

Again, we multiply to obtain the minimal polynomial:

$$([X_2,)([X_2, [X_2, +1)([X_2, [X_2, +4) = ([X_2)^5 + 5([X_2)^3 + 4[X_2$$

In like manner, we observe that the corresponding eigenvalues all of the other generators of $su(3)$ are 0, -1 and -4 and hence all share the same minimal equation and each one generates the same sort of solo-Casimir operator except the generator S. The eigenvalues of $[S, [S,$ are -9 and 0, hence the minimal equation of S is

$$[S, [S, [S, +9[S = 0.$$

Each of the elements of $u(3)$ has an associated solo-Casimir operator which leads to a linear differential operator and hence to eigenvalue equations for functions. In the author's geometric model of elementary particles, each generator of $u(3,1)$ corresponds to a family of elementary particles. These elementary particles are modeled as fX, where X is an element of $u(3,1)$ and f is a function satisfying certain differential equations, which depend on X. These solo-operators them seem to offer the possibility of fulfilling that program.

17 Conclusion

According to the standard wisdom on Casimir operators, the number of Casimir Operators is equal to the rank of the Lie algebra. The rank of $so(3,1)$ is two, but we have constructed an infinite number of intrinsic Casimir operators with no indication that this exhausts the list (but it has exhausted the author). The existence of these new intrinsic Casimir operators for $so(3,1)$ shows that the standard approach to Casimir operators is fatally flawed, but for now, we have no program to replace it.

Perhaps the most surprising result presented here is the existence of intrinsic Casimir operators of complex structure type. Their existence is not even hinted at in the standard approach which recognizes only Casimir operators of eigenvalue type (Multiples of the identity). Clearly, one task for future study is to characterize those Lie algebras which possess each type of intrinsic Casimir operator and to look for other types.

Wigner [33] classified elementary particles on the basis of the two invariants (eigenvalues of the Casimir operators) of the inhomogeneous Lorentz Group (the Poincaré Group). His classification was based on the assumption that these eigenvalues change value with the representation. The work presented here shows clearly that this program is in error since the eigenvalues of the Casimir operators are true invariants and do not change with the representation. The modern theory based on this assumption (the theory of quarks) is likewise based on the same faulty mathematics.

Weyl [32] stated the program for the application of group theory to quantum mechanics as it was understood in 1930:

> All quantum numbers, with the exception of the so-called principal quantum number, are indices characterizing representations of groups.

This program operated under the assumption that the eigenvalues of the Casimir operators vary from representation to representation. That assumption has been shown to be false and hence the program as stated by Weyl is mathematically unsound. Indeed, according to Borel [2], Casimir's work was a generalization of Weyl's work on $so(3)$. The Casimir operator is a generalization to an arbitrary Lie algebra of the "square of the magnitude of the moment of momentum" which Weyl [32] introduced in his book *The Theory of Groups and Quantum Mechanics*. Weyl's operator is defined in terms of matrix multiplication, which is not defined in a Lie algebra.

The geometry of a four dimensional space-time is required to describe gravitation. Consequently, the inclusion of additional forces in a geometric model requires the use of higher dimensions The idea of obtaining additional forces by embedding the observed space-time in a larger manifold is due to Kaluza [11]. Kaluza added one extra dimension to Minkowski space in an attempt to incorporate electromagnetism into the geometric framework of Einstein's general theory of relativity. Klein [12] showed that Kaluza's extra dimension must be compact in order to account for quantization. In the early 1960's, this program experienced a great revival. Most of the early attempts to unify gravitation with the other forces were by means of combining space-time symmetries with the internal symmetries of compact gauge groups. Ne'eman [22] discusses the history of these early attempts.

The interest in extensions of the Poincaré group waned after O'Raifeartaigh [23] 'proved' his infamous "no-go theorem" showing that such an extension led to an unphysical mass spectrum. This theorem and its 'improvements' by others showed that extension of the Poincaré group was not the correct idea. The article by Coleman [4] was an attempt to put such attempts to rest.

The no-go theorems of O'Raifeartaigh, I.E. Segal [29] and others relied heavily on the belief that the eigenvalues of the Casimir operators of the Poincaré algebra vary from representation to representation. This belief has been shown to be erroneous and so the "no-go" theorems are not valid. The theories of supersymmetry, super-gravity and super-strings were attempts to circumvent these no-go theorems. Since the no-go theorems are not valid, there no longer exists any reason to pursue supersymmetry.

In the final analysis, as far as physics is concerned, we want to be working with the invariant differential operators, which leads to invariant differential equations and we want differential operators which commute with the group action and hence with every element of the Lie algebra in a representation as differential operators. To say that a Casimir operator is a multiple of the identity operator is meaningless in this context, but the same calculations in terms of Lie Bracket are valid in any representation. Thus, these calculations in the context of matrix representations of the Lie Algebras are just preliminaries to a treatment of representations in terms of differential operators. These are also the representations of interest in harmonic analysis.

Helgason [8] defines $D(G/K)$ as the algebra of all differential operators on G/K which are invariant under the action of G. Harmonic analysis then is the study of the joint eigenfunctions of $D(G/K)$, i.e. functions which are simultaneously eigenfunctions of each operator in $D(G/K)$. In order to do harmonic analysis then, the invariant operators must be properly identified. The intrinsic Casimir operators are just $D(G)$ and as we showed above, they have not been correctly identified. Given the importance of the Casimir operators, a program of finding the true Casimir (invariant) differential operators of the Lie algebras important in physics seems essential.

It has evidently not been realized before that this program of "conserved quantities equals eigenvalues of Casimir operators" is inconsistent with the Lagrangian formalism. According to Noether's theorem, in a Lagrangian theory, there is a one to one correspondence between conserved quantities and elements of the Lie Algebra. For each conserved quantity there is a basis element of the Lie Algebra and vice-versa. The Lagrangian program is then inconsistent with the Casimir operator program where the eigenvalues of the Casimir operators and of the Cartan subalgebra are the only invariants.

But the Lagrangian approach to conserved quantities is internally inconsistent. Suppose a Lagrangian system has a maximum of n conserved quantities. By Noether's theorem the corresponding Lie Algebra is then n dimensional. If this Lie Algebra has a Casimir operator, the eigenvalue of the Casimir operator is also a conserved quantity, resulting in $n+1$ conserved quantities, a contradiction.

The Casimir operator approach to conserved quantities is automatically built into the Lie Algebra formalism, leading to invariant differential equations and conserved quantities and quantization via Schrödinger's [27] approach to "Quantization as an Eigenvalue Problem". Fortunately, the Lie algebras of physical interest have more generalized Casimir operators than generators.

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