Elementary Particles as Oscillations in Anti-de Sitter Space-Time

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Abstract

Using the spinor differential operator representation of $U(3,2)$ to explore the hidden symmetries of the complex space-time $U(3,2)/U(3,1)\times U(1)$ leads to an interpretation of this complex space-time as excited states of Anti-de Sitter space-time. This in turn leads to new Lie Algebraic Quantum Field Theory and a mathematical model of the internal structure of elementary particles as oscillations of complex space-time. This is a quantum theory of gravity which satisfies Einstein's criteria for a unified field theory.

PACS: 12.60.-i Models of particles and fields beyond the standard model

1 Introduction

John Wheeler [57] claimed that:

> Physical law expresses itself through group theory and symmetry, but group theory and symmetry hide the machinery beneath that physical law.

In this section, a model of particle interactions is analyzed using the Lie Algebra $u(3,2)$ to expose the machinery beneath.

Most physicists would probably agree with this statement by A. Salam [49]:

> Throughout the history of quantum theory, a battle has raged between the amateurs and professional group theorists. The amateurs have maintained that everything one needs in the theory of groups can be discovered by the light of nature provided one knows how to multiply two matrices. In support of this claim, they of course, justifiably, point to the successes of that prince of amateurs in this field, Dirac, particularly with the spinor representations of the Lorentz group.
>
> As an amateur myself, I strongly believe in the truth of the non-professionalist creed. I think perhaps there is not much one has to learn in the way of methodology from the group theorists except caution. But this does not mean one should not be aware of the riches which have been amassed over the course of years particularly in that most highly developed of all mathematical disciplines - the theory of Lie groups.

Since Salam refered to Dirac as the "prince of amateurs" it seems fitting to start with a very amateurish paper by Dirac. Dirac [6] attempted to obtain "the de-Sitter analogues of some of the important equations of physics." Unfortunately, Dirac made so many mathematical mistakes that the equations he "derived" are devoid of meaning. One would expect that an error of this magnitude made so long ago would have been exposed and so would be irrelevant today. Unfortunately this is not the case.

I was motivated to understand what Dirac had done on anti-de Sitter space, $SO(3,2)/SO(3,1)$ since I had suggested using the complex space time $SU(3,2)/SU(3,1)\times U(1)$ in a unification program [33].

2 Dirac's Mistakes

De Sitter space is the submanifold of R^5 defined by

(1) $x_1^2 + x_2^2 + x_3^2 - x_4^2 + x_5^2 = R^2$

(Equation numbers in parenthesis follow Dirac's numbering)

The manifold now known as Anti-de Sitter space, which Dirac called "a rather similar space" is:

(2) $x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 = -R^2$

And Dirac states:

> ... most of our work will apply equally well to either space. We can take this into account by working with five coordinates satisfying the symmetrical equation
>
> (3) $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = R^2$
>
> and supposing x_4 to be pure imaginary when we want to have space (1), and x_4, x_5 and R to be pure imaginary when we want to have space (2). We shall write (3) in the contracted form
>
> (4) $x_\mu x_\mu = R^2$
>
> the suffix μ running from 1 to 5.

Now while these substitutions work perfectly well in (3), they do not work in most of the other formulas which Dirac considers. For clarity, I will not attempt to treat both cases simultaneously and will only treat case (2), thus "Dirac's substitution" will refer to replacing x_4 by ix_4 and x_5 by ix_5. Similar comments would apply to case (1).

Dirac correctly points out that

> The only processes of differentiation which it will have a meaning to apply to a physical function will be those referring to differentiations along directions in the de-Sitter space. The operators expressing such differentiations will be of the form
>
> (5) $a_\mu \frac{\partial}{\partial x_\mu}$

So far, so good. But then Dirac claims that

> ... the a_μ are functions of the coordinates x of the point where the differentiation is performed, satisfying
>
> (6) $a_\mu x_\mu = 0.$

This is Dirac's first mistake. Dirac goes on to say that:

> These operators may be characterized by the condition that they commute with the left-hand side of (4). The most fundamental of them are
>
> (7) $x_\mu \partial_\nu - x_\nu \partial_\mu$
>
> corresponding to the infinitesimal rotations of the de-Sitter space.

What he should have said was the operators (7) acting on (4) yield zero.

The "infinitesimal rotations of R^5" form the Lie algebra $so(5)$, a basis of which is:

$$x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}$$

$$x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1}$$

$$x_1 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_1}$$

$$x_1 \frac{\partial}{\partial x_5} - x_5 \frac{\partial}{\partial x_1}$$

$$x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}$$

$$x_2 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_2}$$

$$x_2 \frac{\partial}{\partial x_5} - x_5 \frac{\partial}{\partial x_2}$$

$$x_3 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_3}$$

$$x_3 \frac{\partial}{\partial x_5} - x_5 \frac{\partial}{\partial x_3}$$

$$x_4 \frac{\partial}{\partial x_5} - x_5 \frac{\partial}{\partial x_4}$$

Since each of these operators acting on $(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2)$ yields zero the flows generated by these vector fields remain on the sphere they started on.

The "infinitesimal rotations of anti-deSitter space" are the Lie algebra $so(3,2)$, a basis of which is:

$$X_{12} = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}$$
$$X_{13} = x_1\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_1}$$
$$X_{14} = x_1\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_1}$$
$$X_{15} = x_1\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_1}$$
$$X_{23} = x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}$$
$$X_{24} = x_2\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_2}$$
$$X_{25} = x_2\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_2}$$
$$X_{34} = x_3\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_3}$$
$$X_{35} = x_3\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_3}$$
$$X_{45} = x_4\frac{\partial}{\partial x_5} - x_5\frac{\partial}{\partial x_4}$$

Note that each of these operators acting on $(x_1^2+x_2^2+x_3^2-x_4^2-x_5^2)$ yields zero.

Applying Dirac's substitution to the basis of $so(5)$ we obtain:

$$\begin{array}{lcc}
x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} & \Longrightarrow & unchanged \\
x_1\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_1} & \Longrightarrow & unchanged \\
x_1\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial x_1} & \Longrightarrow & -i(x_1\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_1}) \\
x_1\frac{\partial}{\partial x_5} - x_5\frac{\partial}{\partial x_1} & \Longrightarrow & -i(x_1\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_1}) \\
x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2} & \Longrightarrow & unchanged \\
x_2\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial x_2} & \Longrightarrow & -i(x_2\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_2}) \\
x_2\frac{\partial}{\partial x_5} - x_5\frac{\partial}{\partial x_2} & \Longrightarrow & -i(x_2\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_2}) \\
x_3\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial x_3} & \Longrightarrow & -i(x_3\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_3}) \\
x_3\frac{\partial}{\partial x_5} - x_5\frac{\partial}{\partial x_3} & \Longrightarrow & -i(x_3\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_3}) \\
x_4\frac{\partial}{\partial x_5} - x_5\frac{\partial}{\partial x_4} & \Longrightarrow & unchanged
\end{array}$$

We put in the details for one of the calculations:

$$x_3 \frac{\partial}{\partial x_5} - x_5 \frac{\partial}{\partial x_3} \Longrightarrow x_3 \frac{\partial}{\partial i x_5} - i x_5 \frac{\partial}{\partial x_3} = -i(x_3 \frac{\partial}{\partial x_5} + x_5 \frac{\partial}{\partial x_3})$$

Thus, the Dirac substitution has only four of the ten generators correct, the others are the non-compact generators of $so(3,2)$ multiplied by -i. Recall that multiplication by i changes a compact generator into a noncompact generator and vice-versa [25]. Again, multiplication by i takes us out of the tangent space, which is a real vector space.

The tangent space of anti-de Sitter space is spanned by:

$$x_1 \frac{\partial}{\partial x_5} + x_5 \frac{\partial}{\partial x_1}$$

$$x_2 \frac{\partial}{\partial x_5} + x_5 \frac{\partial}{\partial x_2}$$

$$x_3 \frac{\partial}{\partial x_5} + x_5 \frac{\partial}{\partial x_3}$$

$$x_4 \frac{\partial}{\partial x_5} - x_5 \frac{\partial}{\partial x_4}$$

In Dirac's version, the corresponding vectors are

$$-i(x_1 \frac{\partial}{\partial x_5} + x_5 \frac{\partial}{\partial x_1})$$

$$-i(x_2 \frac{\partial}{\partial x_5} + x_5 \frac{\partial}{\partial x_2})$$

$$-i(x_3 \frac{\partial}{\partial x_5} + x_5 \frac{\partial}{\partial x_3})$$

$$x_4 \frac{\partial}{\partial x_5} - x_5 \frac{\partial}{\partial x_4}$$

Only the last of these is actually in the tangent space.

Dirac's equation (9) is the same as (6) with the same mistake. In order to obtain his equation (17) Dirac differentiates (9) , which is wrong to begin with, with respect to x_ν which is not allowed (since ∂_ν is not in the tangent space), thus compounding his errors.

Dirac's equation (12) is the quadratic Casimir operator of $so(3,2)$ acting on ρ^2. This is one equation which is correct with Dirac's substitution.

Dirac's equations (15), (16) and (22) involve differentiation by $\frac{\partial}{\partial x_\mu}$ which is not in the tangent space of anti-de Sitter space and the equations are thus not valid. In his equation (24), Dirac repeats the same sign errors he made in (7).

The remainder of Dirac's paper builds on the errors pointed out above. Even amateurs need to be careful. Unfortunately the effects of Dirac's mistake still resonates through the Physics literature, Fronsdal [16](equation 4.2) misidentified the basis of $so(5)$ as the basis for $so(3,2)$. Halpern [21] used several of Dirac's erroneous equations.

The basic problem is as Salam points out:

> I shall state theorems; and with a physicist's typical unconcern rarely, if ever, shall I prove these.

When generations of articles pass without proofs, errors will inevitably appear. Without proofs, then, the erroneous statements become accepted as true, finally choking further progress.

3 The Preliminary Problem

An attempt to correct Dirac's mistakes and find the correct equations on $QAdS$ led to the following.

The Lie Group $U(2)$ is the group of 2×2 complex matrices which preserve the form

$$f = z_1\bar{z}_1 + z_2\bar{z}_2$$

The dimension of $U(2)$ is 4.

Set

$$z_I = x_I + iy_I$$

then

$$z_I\bar{z}_I = (x_I + iy_I)(x_I - iy_I) = x_I^2 + y_I^2$$

so,

$$f = x_1^2 + y_1^2 + x_2^2 + y_2^2$$

which is the form preserved by $SO(4)$ with dimension 6. What happened to the 2 missing generators in the Lie algebra $u(2)$? Our first goal is to find them. The investigation will take us on a detour into some unfamiliar territory which has some landmarks which are interesting in their own right. We will pause along the way to make some observations. This trip takes us to the author's $U(3,2)$ theory of matter, exposing the here-to-fore hidden substructure of elementary particles. (The answer, as the expert will know is that the four generators are holomorphic and the other two are not. When we get to the complex space-time $U(3,2)/U(3,1)\times U(1)$ the question will become relevant: do we want to restrict ourselves to the holomorphic generators or allow the other symmetries?)

As will be shown in detail later, $QAdS$ is a complex spacetime. We want to deal with the tangent spaces $T^{1,0}$ and $T^{0,1}$. Since we want to use the Lie algebra to construct differential equations, we need the representation of $u(3,2)$ in terms of differential operators.

4 Nonclassical Lie Algebras

Let $f : R^n \to R$ be a C^∞ function. Define

$$Z_f = \{X \in T(R^n) | Xf = 0\}$$

This is the idea behind the Killing fields in General Relativity, where f would be a space-time metric. These vector fields were called *kinematical operators* by Fubini, Hanson and Jackiw [17].

Theorem: Z_f is a Lie algebra.[32]

Proof: If $X, Y \in Z_f$, so $Xf = 0$ and $Yf = 0$ then

$$[X, Y]f = XYf - YXf = 0.$$

Thus Z_f is closed as a subalgebra of $T(R^n)$.

Corollary: Z_f is a module over $C^\infty(R^n)$

Proof: If $X \in Z_f$ and $\psi \in C^\infty(R^n)$ then $\psi Xf = 0$ thus $\psi X \in Z_f$

Given an arbitrary f, we would like to have a method of finding Z_f. That turns out to be rather easy.

Theorem (The Recipe): If $f \in C^\infty(R^n)$ then

$$\left(\frac{\partial f}{\partial x_I}\right)\frac{\partial}{\partial x_J} - \left(\frac{\partial f}{\partial x_J}\right)\frac{\partial}{\partial x_I} \in Z_f \tag{1}$$

Proof: Direct calculation.

Collary: There are $\binom{n}{2}$ such operators.

Let $f : R^n \to R$ be a C^∞ function. Define

$$S_f = \{X \in T(R^n) | \qquad Xf = \alpha f \; for \; some \; \alpha \in C\}$$

Theorem: S_f is a Lie algebra.

Proof: If $X, Y \in Z_f$, so $Xf = \alpha f$ and $Yf = \beta f$ then

$$[X, Y]f = XYf - YXf = X\beta f - Y\alpha f = \alpha\beta f - \beta\alpha f = 0$$

Corollary: Z_f is an ideal in S_f.

Example:

In classical Hamiltonian mechanics, the Hamiltonian vector field generated by H:

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right)$$

is the sum of operators of this form [1, 2].

Example: In R^3, let $f = x_1^3 + x_2^3 + x_3^3$.

Then using the Recipe we obtain a basis for Z_f:

$$\{x_2^2 \frac{\partial}{\partial x_1} - x_1^2 \frac{\partial}{\partial x_2}, x_1^2 \frac{\partial}{\partial x_3} - x_3^2 \frac{\partial}{\partial x_1}, x_3^2 \frac{\partial}{\partial x_2} - x_2^2 \frac{\partial}{\partial x_3}\}$$

Let

$$A = x_2^2 \frac{\partial}{\partial x_1} - x_1^2 \frac{\partial}{\partial x_2}$$

$$B = x_1^2 \frac{\partial}{\partial x_3} - x_3^2 \frac{\partial}{\partial x_1}$$

$$C = x_3^2 \frac{\partial}{\partial x_2} - x_2^2 \frac{\partial}{\partial x_3}$$

then

$$[A, B] = 2x_2 C \quad [B, C] = 2x_3 A \quad [C, A] = 2x_1 B$$

In order for a Lie algebra to integrate into a Lie group, the bracket of two elements must be a linear combination of the basis elements, one must have structure *constants*, not structure *functions*. This is a Lie algebra which does not integrate into a Lie group.

Example: In R^3, let $f = x_1 x_2 x_3$.

Then using the Recipe we obtain a basis for Z_f:

$$\{x_1x_2\frac{\partial}{\partial x_1} - x_2x_3\frac{\partial}{\partial x_3}, x_1x_3\frac{\partial}{\partial x_3} - x_1x_2\frac{\partial}{\partial x_2}, x_3x_2\frac{\partial}{\partial x_2} - x_1x_3\frac{\partial}{\partial x_1}\}$$

We compute one bracket:

$$\begin{aligned}
&[x_1x_2\frac{\partial}{\partial x_1} - x_2x_3\frac{\partial}{\partial x_3}, x_1x_3\frac{\partial}{\partial x_3} - x_1x_2\frac{\partial}{\partial x_2}] \\
&= x_1x_2x_1\frac{\partial}{\partial x_1} - x_1x_2x_2\frac{\partial}{\partial x_2} \\
&= x_1x_2\left(x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2}\right)
\end{aligned}$$

Beginning with these operators, we obtain a rather complicated Lie algebraic structure with structure functions.

In order to simplify the brackets, it seems reasonable to divide the first generator by x_2, the second by x_1 and the third by x_3. We then obtain as a basis for Z_f:

$$\{x_1\frac{\partial}{\partial x_1} - x_3\frac{\partial}{\partial x_3}, x_3\frac{\partial}{\partial x_3} - x_2\frac{\partial}{\partial x_2}, x_2\frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_1}\}$$

These operators commute, showing that the Lie algebra is Abelian, the structure functions were not essential. This case shows that while the recipe gives us a start at finding an appropriate basis for the Lie algebra, suitable multiplications or factorizations may greatly simplify the situation.

Example: In R^3, take

$$f = \left(x_1^4 + x_2^4 + x_3^4\right)$$

The recipe for Z_f yields:

$$\begin{array}{c}
x_2^3\frac{\partial}{\partial x_1} - x_1^3\frac{\partial}{\partial x_2} \\
x_3^3\frac{\partial}{\partial x_1} - x_1^3\frac{\partial}{\partial x_3} \\
x_3^3\frac{\partial}{\partial x_2} - x_2^3\frac{\partial}{\partial x_3}
\end{array}$$

We will compute one commutator as a sample:

$$[x_2^3\frac{\partial}{\partial x_1} - x_1^3\frac{\partial}{\partial x_2}, x_3^3\frac{\partial}{\partial x_1} - x_1^3\frac{\partial}{\partial x_3}] = 3x_1^2\left(x_3^3\frac{\partial}{\partial x_2} - x_2^3\frac{\partial}{\partial x_3}\right)$$

Again, this is an example of a Lie algebra which does not integrate into a Lie group.

5 The Classical Lie algebras

The method developed in the previous section can be applied to obtain the classical Lie algebras as well.

Example: In R^3, let $f = x_1^2 + x_2^2 + x_3^2$. The recipe yields:

$$x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}$$

$$x_1\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_1}$$

$$x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}$$

Which we recognize as the standard basis of the Lie algebra $so(3)$. The Lie algebra $so(n)$ is obtained in the same way.

Example: In R^3, let $f = x_1^2 + x_2^2 + x_3^2 - c^2t^2$. The recipe yields the generators of $so(3)$ plus:

$$x_1\frac{\partial}{\partial t} - c^2t\frac{\partial}{\partial x_1}$$

$$x_2\frac{\partial}{\partial t} - c^2t\frac{\partial}{\partial x_2}$$

$$x_3\frac{\partial}{\partial t} - c^2t\frac{\partial}{\partial x_3}$$

This is a basis for the Lorentz algebra.

In order to obtain physics, we need to scale the Lie algebra. This example shows that scaling the function automatically scales the Lie algebra.

Also relevant are the examples of the rotations of R^5 and the symmetries of de Sitter space and "anti-de Sitter space" discussed above. The operators representing the Lie algebras of $so(n)$ can be obtained in the same way. The operator representation of $su(n)$ is the topic of the next few sections.

6 Complex Symmetries

If $f = x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2$, then Z_f is isomorphic to the anti-de Sitter Lie algebra $so(3,2)$. In [33], I suggested that the group underlying nature was $SU(3,2)$ and that $SU(3,2)/SU(3,1) \times U(1)$ is a complex spacetime (dubbed

Quantum Anti-deSitter space, $QAdS$) which in some sense is a complexification of Anti-de Sitter space $SO(3,2)/SO(3,1)$. Define

$$\rho^2 = z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 - z_4\bar{z}_4 - z_5\bar{z}_5$$

A major purpose of this section is to study the relation between Z_{ρ^2}, the tangent space of $QAdS$ and the Lie Algebra $u(3,2)$. The relation is not as straight forward as with AdS since the recipe does not distinguish between the tangent spaces $T^{1,0}$ and $T^{0,1}$.

We begin with the preliminary problem: In C^2 define

$$f = z_1\bar{z}_1 + z_2\bar{z}_2$$

Apply the recipe (1) to obtain:

$$\bar{z}_1\frac{\partial}{\partial\bar{z}_1} - z_1\frac{\partial}{\partial z_1}$$

$$\bar{z}_2\frac{\partial}{\partial z_1} - \bar{z}_1\frac{\partial}{\partial z_2}$$

$$z_2\frac{\partial}{\partial\bar{z}_1} - z_1\frac{\partial}{\partial\bar{z}_2}$$

$$\bar{z}_1\frac{\partial}{\partial\bar{z}_2} - z_2\frac{\partial}{\partial z_1}$$

$$\bar{z}_2\frac{\partial}{\partial\bar{z}_1} - z_1\frac{\partial}{\partial z_2}$$

$$\bar{z}_2\frac{\partial}{\partial\bar{z}_2} - z_2\frac{\partial}{\partial z_2}$$

This gives us the 6 operators we were looking for. The problem is that the recipe does not distinguish between $T^{1,0}$ and $T^{0,1}$. Instead we must test each of these operators to see which are in $u(2)$. Recall that $u(2)$ is closed under the following operation:

1. Take the transpose (interchange the indices i and j)
2. Take the complex conjugate
3. Multiply by -1.

Let us apply this sequence of steps to

$$\bar{z}_2\frac{\partial}{\partial\bar{z}_1} - z_1\frac{\partial}{\partial z_2}$$

1.

$$\bar{z}_1 \frac{\partial}{\partial \bar{z}_2} - z_2 \frac{\partial}{\partial z_1}$$

2.

$$z_1 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_1}$$

3.

$$-z_1 \frac{\partial}{\partial z_2} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_1}$$

Which is what we started with, so this operator is in $u(2)$.

Now we apply the same sequence of steps to:

$$\bar{z}_1 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial z_1}$$

1.

$$\bar{z}_2 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial z_2}$$

2.

$$z_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial \bar{z}_2}$$

3.

$$-z_2 \frac{\partial}{\partial \bar{z}_1} + z_1 \frac{\partial}{\partial \bar{z}_2}$$

The final result is not what we started with, so this operator is not in $u(2)$, but it is on our list.

We conclude that the operators in $u(2)$ are those of the form

$$\bar{z}_J \frac{\partial}{\partial \bar{z}_I} - z_I \frac{\partial}{\partial z_J}$$

Since there are two choices for each of I and J, this yields all 4 of the basis vectors of $u(2)$. The operators in $u(2)$ segregate the z_I from the $\bar{z}_I$. The operators not in $u(2)$ mix the z_I with the $\bar{z}_I$.

Since these operators are close in form to the operator B_3 which Naimark [39] (equation 9.4(8)) presents as an "infinitesimal operator of a spinor representation", it seems that we have arrived at a representation of $u(2)$ in terms of spinor derivatives. This is not surprising since Nash [40] showed "There exists an exceptional equivalence of a complex Dirac spinor and a complex Minkowski space-time vector."

7 A Deeper Level

Since the standard representation of $u(2)$ is in terms of x_I and y_I instead of z_I, we make the change of variables. For the time being, we will ignore the tangent vectors not in $u(2)$.

The operators in $u(2)$:

$$u_{21} = \bar{z}_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial z_2}$$

$$= (x_2 - iy_2)\left(\frac{1}{2}\right)\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial y_1}\right) - (x_1 + iy_1)\left(\frac{1}{2}\right)\left(\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial y_2}\right)$$

$$= \left(\frac{1}{2}\right)\left(x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}\right) + \left(\frac{1}{2}\right)\left(y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2}\right)$$

$$+ \left(\frac{i}{2}\right)\left[\left(x_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_2}\right) + \left(x_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_1}\right)\right]$$

$$u_{12} = \bar{z}_1 \frac{\partial}{\partial \bar{z}_2} - z_2 \frac{\partial}{\partial z_1}$$

$$= (x_1 - iy_1)\left(\frac{1}{2}\right)\left(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial y_2}\right) - (x_2 + iy_2)\left(\frac{1}{2}\right)\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial y_1}\right)$$

$$= \left(\frac{1}{2}\right)\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right) + \left(\frac{1}{2}\right)\left(y_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_1}\right)$$

$$+ \left(\frac{i}{2}\right)\left[\left(x_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_1}\right) + \left(x_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_2}\right)\right]$$

The following operator acting on ρ^2 yields zero, but there are problems with considering it as a basis element:

$$\bar{z}_1 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial z_1}$$

$$= (x_1 - iy_1)\left(\frac{1}{2}\right)\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial y_1}\right) - (x_1 + iy_1)\left(\frac{1}{2}\right)\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial y_1}\right)$$

$$= \left(\frac{i}{2}\right)\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right)$$

The problem is that the operator should be compact while the factor of i makes it noncompact. We will define:

$$u_{11} = \left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right)$$

The same comments hold for:

$$\bar{z}_2\frac{\partial}{\partial \bar{z}_2} - z_2\frac{\partial}{\partial z_2}$$

$$= (x_2 - iy_2)\left(\frac{1}{2}\right)\left(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial y_2}\right) - (x_2 + iy_2)\left(\frac{1}{2}\right)\left(\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial y_2}\right)$$

$$= \left(\frac{i}{2}\right)\left(x_2\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_2}\right)$$

We define:

$$u_{22} = \left(x_2\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_2}\right)$$

8 The matrix generators of u(3,2)

We list the matrix generators of $u(3,2)$:

$$\gamma_1 = \begin{pmatrix} i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \end{pmatrix}$$

$$X_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Y_{12} = \begin{pmatrix} 0 & i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Y_{13} = \begin{pmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{14} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Y_{14} = \begin{pmatrix} 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Y_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Y_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Y_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{25} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$Y_{25} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \end{pmatrix}$$

$$X_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Y_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{35} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$Y_{35} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \end{pmatrix}$$

$$X_{45} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$Y_{45} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & i & 0 \end{pmatrix}$$

9 The brackets of u(3,2)

The matrices of $u(3,2)$ satisfy the following commutation relations.

$$[X_{12}, X_{13}] = -X_{23}$$

$$[X_{12}, X_{14}] = -X_{24}$$
$$[X_{12}, X_{15}] = -X_{25}$$
$$[X_{12}, X_{23}] = X_{13}$$
$$[X_{12}, X_{24}] = X_{14}$$
$$[X_{12}, X_{25}] = X_{15}$$
$$[X_{12}, X_{34}] = 0$$
$$[X_{12}, X_{35}] = 0$$
$$[X_{12}, X_{45}] = 0$$

$$[X_{13}, X_{14}] = -X_{34}$$
$$[X_{13}, X_{15}] = -X_{35}$$
$$[X_{13}, X_{23}] = -X_{12}$$
$$[X_{13}, X_{24}] = 0$$
$$[X_{13}, X_{25}] = 0$$
$$[X_{13}, X_{34}] = X_{14}$$
$$[X_{13}, X_{35}] = X_{15}$$
$$[X_{13}, X_{45}] = 0$$

$$[X_{14}, X_{15}] = X_{45}$$
$$[X_{14}, X_{23}] = 0$$
$$[X_{14}, X_{24}] = X_{12}$$
$$[X_{14}, X_{25}] = 0$$
$$[X_{14}, X_{34}] = X_{13}$$
$$[X_{14}, X_{35}] = 0$$
$$[X_{14}, X_{45}] = X_{15}$$

$$[X_{15}, X_{23}] = 0$$

$$[X_{15}, X_{24}] = 0$$

$$[X_{15}, X_{25}] = X_{12}$$

$$[X_{15}, X_{34}] = 0$$

$$[X_{15}, X_{35}] = X_{13}$$

$$[X_{15}, X_{45}] = -X_{14}$$

$$[X_{23}, X_{24}] = -X_{34}$$

$$[X_{23}, X_{25}] = -X_{35}$$

$$[X_{23}, X_{34}] = X_{24}$$

$$[X_{23}, X_{35}] = X_{25}$$

$$[X_{23}, X_{45}] = 0$$

$$[X_{24}, X_{25}] = X_{45}$$

$$[X_{24}, X_{34}] = X_{23}$$

$$[X_{24}, X_{35}] = 0$$

$$[X_{24}, X_{45}] = X_{25}$$

$$[X_{25}, X_{34}] = 0$$

$$[X_{25}, X_{35}] = X_{23}$$

$$[X_{25}, X_{45}] = -X_{24}$$

$$[X_{34}, X_{35}] = X_{45}$$

$$[X_{34}, X_{45}] = X_{35}$$

$$[X_{35}, X_{45}] = -X_{34}$$

$$[Y_{12}, Y_{13}] = -X_{23}$$

$$[Y_{12}, Y_{14}] = -X_{24}$$

$$[Y_{12}, Y_{15}] = -X_{25}$$
$$[Y_{12}, Y_{23}] = -X_{13}$$
$$[Y_{12}, Y_{24}] = -X_{14}$$
$$[Y_{12}, Y_{25}] = -X_{15}$$
$$[Y_{12}, Y_{34}] = 0$$
$$[Y_{12}, Y_{35}] = 0$$
$$[Y_{12}, Y_{45}] = 0$$

$$[Y_{13}, Y_{14}] = -X_{34}$$

$$[Y_{13}, Y_{15}] = -X_{35}$$
$$[Y_{13}, Y_{23}] = -X_{12}$$
$$[Y_{13}, Y_{24}] = 0$$
$$[Y_{13}, Y_{25}] = 0$$
$$[Y_{13}, Y_{34}] = -X_{14}$$
$$[Y_{13}, Y_{35}] = -X_{15}$$
$$[Y_{13}, Y_{45}] = 0$$
$$[Y_{14}, Y_{15}] = X_{45}$$
$$[Y_{14}, Y_{23}] = 0$$
$$[Y_{14}, Y_{24}] = X_{12}$$
$$[Y_{14}, Y_{25}] = 0$$
$$[Y_{14}, Y_{34}] = X_{13}$$
$$[Y_{14}, Y_{35}] = 0$$
$$[Y_{14}, Y_{45}] = -X_{15}$$

$$[Y_{15}, Y_{23}] = 0$$

$$[Y_{15}, Y_{24}] = 0$$
$$[Y_{15}, Y_{25}] = X_{12}$$

$$[Y_{15}, Y_{34}] = 0$$

$$[Y_{15}, Y_{35}] = X_{13}$$

$$[Y_{15}, Y_{45}] = -X_{14}$$

$$[Y_{23}, Y_{24}] = -X_{34}$$

$$[Y_{23}, Y_{25}] = -X_{35}$$

$$[Y_{23}, Y_{34}] = -X_{24}$$

$$[Y_{23}, Y_{35}] = -X_{25}$$

$$[Y_{23}, Y_{45}] = 0$$

$$[Y_{24}, Y_{25}] = X_{45}$$

$$[Y_{24}, Y_{34}] = X_{23}$$

$$[Y_{24}, Y_{35}] = 0$$

$$[Y_{24}, Y_{45}] = -X_{25}$$

$$[Y_{25}, Y_{34}] = 0$$

$$[Y_{25}, Y_{35}] = X_{23}$$

$$[Y_{25}, Y_{45}] = -X_{24}$$

$$[Y_{34}, Y_{35}] = X_{45}$$

$$[Y_{34}, Y_{45}] = -X_{35}$$

$$[Y_{35}, Y_{45}] = -X_{34}$$

$$[X_{12}, Y_{12}] = 2\gamma_1 - 2\gamma_2$$

$$[X_{12}, Y_{13}] = -Y_{23}$$

$$[X_{12}, Y_{14}] = -Y_{24}$$

$$[X_{12}, Y_{15}] = -Y_{25}$$

$$[X_{12}, Y_{23}] = Y_{13}$$

$$[X_{12}, Y_{24}] = Y_{14}$$

$$[X_{12}, Y_{25}] = Y_{15}$$

$$[X_{12}, Y_{34}] = 0$$
$$[X_{12}, Y_{35}] = 0$$
$$[X_{12}, Y_{45}] = 0$$

$$[X_{13}, Y_{12}] = -Y_{23}$$
$$[X_{13}, Y_{13}] = 2\gamma_1 - 2\gamma_3$$
$$[X_{13}, Y_{14}] = -Y_{34}$$
$$[X_{13}, Y_{15}] = -Y_{35}$$
$$[X_{13}, Y_{23}] = Y_{12}$$
$$[X_{13}, Y_{24}] = 0$$
$$[X_{13}, Y_{25}] = 0$$
$$[X_{13}, Y_{34}] = Y_{14}$$
$$[X_{13}, Y_{35}] = Y_{15}$$
$$[X_{13}, Y_{45}] = 0$$

$$[X_{14}, Y_{12}] = -Y_{24}$$
$$[X_{14}, Y_{13}] = -Y_{34}$$
$$[X_{14}, Y_{14}] = -2\gamma_1 + 2\gamma_4$$
$$[X_{14}, Y_{15}] = Y_{45}$$
$$[X_{14}, Y_{23}] = 0$$
$$[X_{14}, Y_{24}] = -Y_{12}$$
$$[X_{14}, Y_{25}] = 0$$
$$[X_{14}, Y_{34}] = -Y_{13}$$
$$[X_{14}, Y_{35}] = 0$$
$$[X_{14}, Y_{45}] = Y_{15}$$
$$[X_{15}, Y_{12}] = -Y_{25}$$
$$[X_{15}, Y_{13}] = -Y_{35}$$

$$[X_{15}, Y_{14}] = Y_{45}$$
$$[X_{15}, Y_{15}] = -2\gamma_1 + 2\gamma_5$$
$$[X_{15}, Y_{23}] = 0$$
$$[X_{15}, Y_{24}] = 0$$
$$[X_{15}, Y_{25}] = -Y_{12}$$
$$[X_{15}, Y_{34}] = 0$$
$$[X_{15}, Y_{35}] = -Y_{13}$$
$$[X_{15}, Y_{45}] = Y_{14}$$

$$[X_{23}, Y_{12}] = -Y_{13}$$
$$[X_{23}, Y_{13}] = Y_{12}$$
$$[X_{23}, Y_{14}] = 0$$
$$[X_{23}, Y_{15}] = 0$$
$$[X_{23}, Y_{23}] = 2\gamma_2 - 2\gamma_3$$
$$[X_{23}, Y_{24}] = -Y_{34}$$
$$[X_{23}, Y_{25}] = -Y_{35}$$
$$[X_{23}, Y_{34}] = Y_{24}$$
$$[X_{23}, Y_{35}] = Y_{25}$$
$$[X_{23}, Y_{45}] = 0$$

$$[X_{24}, Y_{12}] = -Y_{14}$$
$$[X_{24}, Y_{13}] = 0$$
$$[X_{24}, Y_{14}] = -Y_{12}$$
$$[X_{24}, Y_{15}] = 0$$
$$[X_{24}, Y_{23}] = -Y_{34}$$
$$[X_{24}, Y_{24}] = -2\gamma_2 + 2\gamma_4$$
$$[X_{24}, Y_{25}] = Y_{45}$$

$$[X_{24}, Y_{34}] = -Y_{23}$$
$$[X_{24}, Y_{35}] = 0$$
$$[X_{24}, Y_{45}] = Y_{25}$$

$$[X_{25}, Y_{12}] = -Y_{15}$$
$$[X_{25}, Y_{13}] = 0$$
$$[X_{25}, Y_{14}] = 0$$
$$[X_{25}, Y_{15}] = -Y_{12}$$
$$[X_{25}, Y_{23}] = -Y_{35}$$
$$[X_{25}, Y_{24}] = Y_{45}$$
$$[X_{25}, Y_{25}] = -2\gamma_2 + 2\gamma_5$$
$$[X_{25}, Y_{34}] = 0$$
$$[X_{25}, Y_{35}] = -Y_{23}$$
$$[X_{25}, Y_{45}] = Y_{24}$$

$$[X_{34}, Y_{12}] = 0$$
$$[X_{34}, Y_{13}] = -Y_{14}$$
$$[X_{34}, Y_{14}] = -Y_{13}$$
$$[X_{34}, Y_{15}] = 0$$
$$[X_{34}, Y_{23}] = -Y_{24}$$
$$[X_{34}, Y_{24}] = -Y_{23}$$
$$[X_{34}, Y_{25}] = 0$$
$$[X_{34}, Y_{34}] = -2\gamma_3 + 2\gamma_4$$
$$[X_{34}, Y_{35}] = Y_{45}$$
$$[X_{34}, Y_{45}] = Y_{35}$$
$$[X_{35}, Y_{12}] = 0$$
$$[X_{35}, Y_{13}] = -Y_{15}$$

$$[X_{35}, Y_{14}] = 0$$
$$[X_{35}, Y_{15}] = -Y_{13}$$
$$[X_{35}, Y_{23}] = -Y_{25}$$
$$[X_{35}, Y_{24}] = 0$$
$$[X_{35}, Y_{25}] = -Y_{23}$$
$$[X_{35}, Y_{34]} = Y_{45}$$
$$[X_{35}, Y_{35}] = -2\gamma_3 + 2\gamma_5$$
$$[X_{35}, Y_{45}] = Y_{34}$$
$$[X_{45}, Y_{12}] = 0$$
$$[X_{45}, Y_{13}] = 0$$
$$[X_{45}, Y_{14}] = -Y_{15}$$
$$[X_{45}, Y_{15}] = Y_{14}$$
$$[X_{45}, Y_{23}] = 0$$
$$[X_{45}, Y_{24}] = -Y_{25}$$
$$[X_{45}, Y_{25}] = Y_{24}$$
$$[X_{45}, Y_{34}] = -Y_{35}$$
$$[X_{45}, Y_{35}] = Y_{34}$$
$$[X_{45}, Y_{45}] = -2\gamma_5 + 2\gamma_4$$

$$[\gamma_1, X_{12}] = Y_{12}$$
$$[\gamma_1, X_{13}] = Y_{13}$$
$$[\gamma_1, X_{14}] = Y_{14}$$
$$[\gamma_1, X_{15}] = Y_{15}$$
$$[\gamma_1, X_{23}] = 0$$
$$[\gamma_1, X_{24}] = 0$$
$$[\gamma_1, X_{25}] = 0$$
$$[\gamma_1, X_{34}] = 0$$

$$[\gamma_1, X_{35}] = 0$$
$$[\gamma_1, X_{45}] = 0$$

$$[\gamma_1, Y_{12}] = -X_{12}$$
$$[\gamma_1, Y_{13}] = -X_{13}$$
$$[\gamma_1, Y_{14}] = -X_{14}$$
$$[\gamma_1, Y_{15}] = -X_{15}$$
$$[\gamma_1, Y_{23}] = 0$$
$$[\gamma_1, Y_{24}] = 0$$
$$[\gamma_1, Y_{25}] = 0$$
$$[\gamma_1, Y_{34}] = 0$$
$$[\gamma_1, Y_{35}] = 0$$
$$[\gamma_1, Y_{45}] = 0$$
$$[\gamma_2, X_{12}] = -Y_{12}$$
$$[\gamma_2, X_{13}] = 0$$
$$[\gamma_2, X_{14}] = 0$$
$$[\gamma_2, X_{15}] = 0$$
$$[\gamma_2, X_{23}] = Y_{23}$$
$$[\gamma_2, X_{24}] = Y_{24}$$
$$[\gamma_2, X_{25}] = Y_{25}$$
$$[\gamma_2, X_{34}] = 0$$
$$[\gamma_2, X_{35}] = 0$$
$$[\gamma_2, X_{45}] = 0$$
$$[\gamma_2, Y_{12}] = X_{12}$$
$$[\gamma_2, Y_{13}] = 0$$
$$[\gamma_2, Y_{14}] = 0$$
$$[\gamma_2, Y_{15}] = 0$$

$$[\gamma_2, Y_{23}] = -X_{23}$$
$$[\gamma_2, Y_{24}] = -X_{24}$$
$$[\gamma_2, Y_{25}] = -X_{25}$$
$$[\gamma_2, Y_{34}] = 0$$
$$[\gamma_2, Y_{35}] = 0$$
$$[\gamma_2, Y_{45}] = 0$$

$$[\gamma_3, X_{12}] = 0$$
$$[\gamma_3, X_{13}] = -Y_{13}$$
$$[\gamma_3, X_{14}] = 0$$
$$[\gamma_3, X_{15}] = 0$$
$$[\gamma_3, X_{23}] = -Y_{23}$$
$$[\gamma_3, X_{24}] = 0$$
$$[\gamma_3, X_{25}] = 0$$
$$[\gamma_3, X_{34}] = Y_{34}$$
$$[\gamma_3, X_{35}] = Y_{35}$$
$$[\gamma_3, X_{45}] = 0$$
$$[\gamma_3, Y_{12}] = 0$$
$$[\gamma_3, Y_{13}] = X_{13}$$
$$[\gamma_3, Y_{14}] = 0$$
$$[\gamma_3, Y_{15]} = 0$$
$$[\gamma_3, Y_{23}] = X_{23}$$
$$[\gamma_3, Y_{24}] = 0$$
$$[\gamma_3, Y_{25}] = 0$$
$$[\gamma_3, Y_{34}] = -X_{34}$$
$$[\gamma_3, Y_{35]} = -X_{35}$$
$$[\gamma_3, Y_{45}] = 0$$

$$[\gamma_4, X_{12}] = 0$$

$$[\gamma_4, X_{13}] = 0$$

$$[\gamma_4, X_{14}] = -Y_{14}$$

$$[\gamma_4, X_{15}] = 0$$

$$[\gamma_4, X_{23}] = 0$$

$$[\gamma_4, X_{24}] = -Y_{24}$$

$$[\gamma_4, X_{25}] = 0$$

$$[\gamma_4, X_{34}] = -Y_{34}$$

$$[\gamma_4, X_{35}] = 0$$

$$[\gamma_4, X_{45}] = Y_{45}$$

$$[\gamma_4, Y_{12}] = 0$$

$$[\gamma_4, Y_{13}] = 0$$

$$[\gamma_4, Y_{14}] = X_{14}$$

$$[\gamma_4, Y_{15}] = 0$$

$$[\gamma_4, Y_{23}] = 0$$

$$[\gamma_4, Y_{24}] = X_{24}$$

$$[\gamma_4, Y_{25}] = 0$$

$$[\gamma_4, Y_{34}] = X_{34}$$

$$[\gamma_4, Y_{35}] = 0$$

$$[\gamma_4, Y_{45}] = -X_{45}$$

$$[\gamma_5, X_{12}] = 0$$

$$[\gamma_5, X_{13}] = 0$$

$$[\gamma_5, X_{14}] = 0$$

$$[\gamma_5, X_{15}] = -Y_{15}$$

$$[\gamma_5, X_{23}] = 0$$

$$[\gamma_5, X_{24}] = 0$$
$$[\gamma_5, X_{25}] = -Y_{25}$$
$$[\gamma_5, X_{34}] = 0$$
$$[\gamma_5, X_{35}] = -Y_{35}$$
$$[\gamma_5, X_{45}] = -Y_{45}$$
$$[\gamma_5, Y_{12}] = 0$$
$$[\gamma_5, Y_{13}] = 0$$
$$[\gamma_5, Y_{14}] = 0$$
$$[\gamma_5, Y_{15}] = X_{15}$$
$$[\gamma_5, Y_{23}] = 0$$
$$[\gamma_5, Y_{24}] = 0$$
$$[\gamma_5, Y_{25}] = X_{25}$$
$$[\gamma_5, Y_{34}] = 0$$
$$[\gamma_5, Y_{35}] = X_{35}$$
$$[\gamma_5, Y_{45}] = X_{45}$$

10 The Differential Operator-Matrix Correspondence

In order to determine the correspondence between the matrix representation of $u(3,2)$ and the Differential Operator representation, we need to compute the spectrum of the u_{ii} acting on the u_{ij}.

Since u_{12} is the eigenvector of the first diagonal operator u_{11} with eigenvalue i in the differential operator representation, the corresponding matrix representation is (a multiple of) the eigenvector of the first diagonal operator with eigenvalue i.

Working with the u(2) Lie sub-algebra :

$$\gamma_1 = \begin{pmatrix} i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Y_{12} = \begin{pmatrix} 0 & i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[X_{12}, Y_{12}] = 2\gamma_1 - 2\gamma_2$$

$$[\gamma_1, X_{12}] = Y_{12}$$

$$[\gamma_2, X_{12}] = -Y_{12}$$

$$[\gamma_1, Y_{12}] = -X_{12}$$

$$[\gamma_2, Y_{12}] = X_{12}$$

$$[\gamma_1, X_{12} - iY_{12}] = Y_{12} + iX_{12} = i\,(X_{12} - iY_{12})$$

$$[\gamma_2, X_{12} - iY_{12}] = -Y_{12} - iX_{12} = -i\,(X_{12} - iY_{12})$$

$$[\gamma_1, X_{12} + iY_{12}] = Y_{12} - iX_{12} = i\,(X_{12} + iY_{12})$$

$$[\gamma_2, X_{12} + iY_{12}] = -Y_{12} + iX_{12} = -i\,(X_{12} + iY_{12})$$

Now we look at the differential operator brackets:

$$[u_{11}, u_{12}]$$

$$= [\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), \left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right) + \left(y_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_1}\right)$$

$$+i\left(x_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_2}\right) + i\left(x_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_1}\right)]$$

We do the calculation term by term:

$$[\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), \left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right)] = (-i)\,i\left(x_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_2}\right)$$

$$[\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), \left(y_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_1}\right)] = (-i)\,i\left(x_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_1}\right)$$

$$[\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), i\left(x_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_2}\right)] =$$

$$(-i)\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right)$$

$$[\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), i\left(x_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_1}\right)] =$$

$$(-i)\left(y_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_1}\right)$$

Summing, we obtain

$$[u_{11}, u_{12}] = -iu_{12}$$

Computing the interaction term by term using the differential operator representation:

$$[u_{22}, u_{12}] =$$

$$[\left(x_2\frac{\partial}{\partial y_2}-y_2\frac{\partial}{\partial x_2}\right),\left(x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1}\right)]=\left(x_1\frac{\partial}{\partial y_2}-y_2\frac{\partial}{\partial x_1}\right)$$

$$[\left(x_2\frac{\partial}{\partial y_2}-y_2\frac{\partial}{\partial x_2}\right),\left(y_1\frac{\partial}{\partial y_2}-y_2\frac{\partial}{\partial y_1}\right)]=\left(x_2\frac{\partial}{\partial y_1}-y_1\frac{\partial}{\partial x_2}\right)$$

$$[\left(x_2\frac{\partial}{\partial y_2}-y_2\frac{\partial}{\partial x_2}\right),i\left(x_2\frac{\partial}{\partial y_1}-y_1\frac{\partial}{\partial x_2}\right)]=\left(y_2\frac{\partial}{\partial y_1}-y_1\frac{\partial}{\partial y_2}\right)$$

$$[\left(x_2\frac{\partial}{\partial y_2}-y_2\frac{\partial}{\partial x_2}\right),i\left(x_1\frac{\partial}{\partial y_2}-y_2\frac{\partial}{\partial x_1}\right)]=-i\left(x_2\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_2}\right)$$

Adding, we obtain

$$[u_{22},u_{12}]=iu_{12}$$

Comparing the matrix brackets with the differential operator brackets, we obtain a correspondence:

$$u_{11}\rightarrow\gamma_1$$

$$u_{22}\rightarrow\gamma_2$$

$$u_{12}\rightarrow X_{12}+iY_{12}$$

$$u_{21}\rightarrow X_{12}-iY_{12}$$

11 Operator Representation

A few more calculations show that the following vector fields on $QAdS$ satisfy the same relations as do the corresponding matrices. The same symbols are used for the abstract elements of the Lie algebra, the defining matrix representation and the operator representation. Since these are the only representations we use, there should be no confusion.

Our work on $u(2)$ showed that we should add a 'y' copy to each 'x' generator. We do that for the generators of $so(3,2)$ from section 1.

$$X_{12}=x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1}+y_1\frac{\partial}{\partial y_2}-y_2\frac{\partial}{\partial y_1}$$

$$X_{13}=x_1\frac{\partial}{\partial x_3}-x_3\frac{\partial}{\partial x_1}+y_1\frac{\partial}{\partial y_3}-y_3\frac{\partial}{\partial y_1}$$

$$X_{14}=x_1\frac{\partial}{\partial x_4}+x_4\frac{\partial}{\partial x_1}+y_1\frac{\partial}{\partial y_4}+y_4\frac{\partial}{\partial y_1}$$

$$X_{15} = x_1 \frac{\partial}{\partial x_5} + x_5 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_5} + y_5 \frac{\partial}{\partial y_1}$$
$$X_{23} = x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_3} - y_3 \frac{\partial}{\partial y_2}$$
$$X_{24} = x_2 \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_4} + y_4 \frac{\partial}{\partial y_2}$$
$$X_{25} = x_2 \frac{\partial}{\partial x_5} + x_5 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_5} + y_5 \frac{\partial}{\partial y_2}$$
$$X_{34} = x_3 \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_3} + y_3 \frac{\partial}{\partial y_4} + y_4 \frac{\partial}{\partial y_3}$$
$$X_{35} = x_3 \frac{\partial}{\partial x_5} + x_5 \frac{\partial}{\partial x_3} + y_3 \frac{\partial}{\partial y_5} + y_5 \frac{\partial}{\partial y_3}$$
$$X_{45} = x_4 \frac{\partial}{\partial x_5} - x_5 \frac{\partial}{\partial x_4} + y_4 \frac{\partial}{\partial y_5} - y_5 \frac{\partial}{\partial y_4}$$

Then we add the Cartan subalgebra (spectrum generating operators) to the basis:

$$\gamma_1 = \left(x_1 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_1} \right)$$
$$\gamma_2 = \left(x_2 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial x_2} \right)$$
$$\gamma_3 = \left(x_3 \frac{\partial}{\partial y_3} - y_3 \frac{\partial}{\partial x_3} \right)$$
$$\gamma_4 = \left(x_4 \frac{\partial}{\partial y_4} - y_4 \frac{\partial}{\partial x_4} \right)$$
$$\gamma_5 = \left(x_5 \frac{\partial}{\partial y_5} - y_5 \frac{\partial}{\partial x_5} \right)$$

The previous foray into $u(2)$ was necessary in order find these operators.

In order to determine the operator representation of the Y_{IJ} we compute the interaction of the diagonal operators with our hybrid generators.

$$[\gamma_1, X_{12}] = [\left(x_1 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_1} \right), x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial y_1}]$$

$$= x_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_1} = Y_{12}$$

$$[\gamma_2, X_{12}] = [\left(x_2\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_2}\right), x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_1}]$$

$$= -x_2\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial y_2} + y_2\frac{\partial}{\partial x_1} = -Y_{12}$$

$$[\gamma_1, X_{13}] = [\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), x_1\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial y_1}]$$

$$= x_3\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_1} = Y_{13}$$

$$[\gamma_3, X_{13}] = [\left(x_3\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_3}\right), x_1\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial y_1}]$$

$$= -x_3\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial x_3} - x_1\frac{\partial}{\partial y_3} + y_3\frac{\partial}{\partial x_1} = -Y_{13}$$

$$[\gamma_1, X_{14}] = [\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), x_1\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_4} + y_4\frac{\partial}{\partial y_1}]$$

$$= x_1\frac{\partial}{\partial y_4} + y_4\frac{\partial}{\partial x_1} - x_4\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_4} = Y_{14}$$

$$[\gamma_4, X_{14}] = [\left(x_4\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_4}\right), x_1\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_4} + y_4\frac{\partial}{\partial y_1}]$$

$$= -x_1\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial x_4} = -Y_{14}$$

$$[\gamma_1, X_{15}] = [\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), x_1\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial y_1}]$$

$$= x_1\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial x_1} - x_5\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_5} = Y_{15}$$

$$[\gamma_5, X_{15}] = [\left(x_5\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_5}\right), x_1\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial y_1}]$$

$$= -x_1\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_1} + x_5\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial x_5} = -Y_{15}$$

$$[\gamma_2, X_{23}] = [\left(x_2\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_2}\right), x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial y_2}]$$

$$= x_2\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_3} = Y_{23}$$

$$[\gamma_3, X_{23}] = [\left(x_3\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_3}\right), x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial y_2}]$$

$$= -x_2\frac{\partial}{\partial y_3} + y_3\frac{\partial}{\partial x_2} - x_3\frac{\partial}{\partial y_2} + y_2\frac{\partial}{\partial x_3} = -Y_{23}$$

$$[\gamma_2, X_{24}] = [\left(x_2\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_2}\right), x_2\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_4} + y_4\frac{\partial}{\partial y_2}]$$

$$= x_2\frac{\partial}{\partial y_4} + y_4\frac{\partial}{\partial x_2} - x_4\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_4} = Y_{24}$$

$$[\gamma_4, X_{24}] = [\left(x_4\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_4}\right), x_2\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_4} + y_4\frac{\partial}{\partial y_2}]$$

$$= -x_2\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial y_2} + y_2\frac{\partial}{\partial x_4} = -Y_{24}$$

$$[\gamma_2, X_{25}] = [\left(x_2\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_2}\right), x_2\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial y_2}]$$

$$= x_2\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial x_2} - x_5\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_5} = Y_{25}$$

$$[\gamma_5, X_{25}] = [\left(x_5\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_5}\right), x_2\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial y_2}]$$

$$= -x_2\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_2} + x_5\frac{\partial}{\partial y_2} + y_2\frac{\partial}{\partial x_5} = -Y_{25}$$

$$[\gamma_3, X_{34}] = [\left(x_3\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_3}\right), x_3\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial y_4} + y_4\frac{\partial}{\partial y_3}]$$

$$= x_3\frac{\partial}{\partial y_4} + y_4\frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_4} = Y_{34}$$

$$[\gamma_4, X_{34}] = [\left(x_4\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_4}\right), x_3\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial y_4} + y_4\frac{\partial}{\partial y_3}]$$

$$= -x_3\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial y_3} + y_3\frac{\partial}{\partial x_4} = -Y_{34}$$

$$[\gamma_3, X_{35}] = [\left(x_3\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_3}\right), x_3\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial y_3}]$$

$$= x_3\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial x_3} - x_5\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_5} = Y_{35}$$

$$[\gamma_5, X_{35}] = [\left(x_5\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_5}\right), x_3\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial y_3}]$$

$$= -x_3\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_3} + x_5\frac{\partial}{\partial y_3} + y_3\frac{\partial}{\partial x_5} = -Y_{35}$$

$$[\gamma_4, X_{45}] = [\left(x_4\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_4}\right), x_4\frac{\partial}{\partial x_5} - x_5\frac{\partial}{\partial x_4} + y_4\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial y_4}]$$

$$= x_4\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_4} + x_5\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_5} = Y_{45}$$

$$[\gamma_5, X_{45}] = [\left(x_5\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_5}\right), x_4\frac{\partial}{\partial x_5} - x_5\frac{\partial}{\partial x_4} + y_4\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial y_4}]$$

$$= -x_4\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial x_4} - x_5\frac{\partial}{\partial y_4} + y_4\frac{\partial}{\partial x_5} = -Y_{45}$$

Just as Penrose's [42] "Twistors... are the spinors for the conformal group" ($SU(2,2)$), the operators defined above are the spinors for $U(3,2)$.

Normally, the coordinates are given as functions and the momenta are operators. Snyder [51] reversed the role to obtain a set of operators representing the coordinates and then used functions to represent the corresponding momenta. To totally utilize the Lie algebra approach, we should work in the space of frames or, equivalently the entire Lie algebra. Thus both the coordinate and the momenta must be represented by first order differential operators. Snyder wrote down some of the above operators with certain combinations occuring with only a plus and others with only a minus. The above operators then include Snyder's with all possible combinations of plus and minus and in addition the diagonal operators to close the algebra. The complex combinations $X+iY$ are then the combinations $x+ip$ as Rosen [45] advocated, albeit with the commutation relations of the Lie algebra $u(3,2)$ instead of the canonical commutation relations.

12 Ignorable symmetries?

Looking at the function

$$\rho^2 = z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 - z_4\bar{z}_4 - z_5\bar{z}_5,$$

in the form

$$\rho_R^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 + y_3^2 - x_4^2 - y_4^2 - x_5^2 - y_5^2,$$

the recipe allows for operators not in $u(3,2)$. Their role (if any) in the theory is not clear at this point. However, we need to determine the consequences of including them in our Lie algebra before we can decide whether to admit them or not. They may be considered to be hidden symmetries. Including them would lead to new conservation laws.

$$v_{12} = \bar{z}_1\frac{\partial}{\partial z_2} - \bar{z}_2\frac{\partial}{\partial z_1}$$

$$= (x_1 - iy_1)\left(\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial y_2}\right) - (x_2 - iy_2)\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial y_1}\right)$$

$$\begin{aligned} &= \left(x_1\tfrac{\partial}{\partial x_2} - x_2\tfrac{\partial}{\partial x_1}\right) + \left(y_2\tfrac{\partial}{\partial y_1} - y_1\tfrac{\partial}{\partial y_2}\right) \\ &+i\left[\left(x_2\tfrac{\partial}{\partial y_1} - y_1\tfrac{\partial}{\partial x_2}\right) + \left(y_2\tfrac{\partial}{\partial x_1} - x_1\tfrac{\partial}{\partial y_2}\right)\right] \end{aligned}$$

$$w_{12} = \bar{z}_1 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial z_1}$$

$$= \left(x_1 \tfrac{\partial}{\partial x_2} - x_2 \tfrac{\partial}{\partial x_1}\right) + \left(y_2 \tfrac{\partial}{\partial y_1} - y_1 \tfrac{\partial}{\partial y_2}\right)$$
$$-i\left[\left(x_2 \tfrac{\partial}{\partial y_1} - y_1 \tfrac{\partial}{\partial x_2}\right) + \left(y_2 \tfrac{\partial}{\partial x_1} - x_1 \tfrac{\partial}{\partial y_2}\right)\right]$$

Then

$$\frac{1}{2}\left(v_{12} + w_{12}\right) = \left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right) + \left(y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2}\right)$$

and

$$\frac{1}{2i}\left(v_{12} - w_{12}\right) = \left[\left(x_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_2}\right) + \left(y_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial y_2}\right)\right]$$

This leads us to define the V_{ij} as the X_{ij} with the sign of the y terms changed:

$$V_{12} = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} - y_1 \frac{\partial}{\partial y_2} + y_2 \frac{\partial}{\partial y_1}$$

$$V_{13} = x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1} - y_1 \frac{\partial}{\partial y_3} + y_3 \frac{\partial}{\partial y_1}$$

$$V_{14} = x_1 \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_1} - y_1 \frac{\partial}{\partial y_4} - y_4 \frac{\partial}{\partial y_1}$$

$$V_{15} = x_1 \frac{\partial}{\partial x_5} + x_5 \frac{\partial}{\partial x_1} - y_1 \frac{\partial}{\partial y_5} - y_5 \frac{\partial}{\partial y_1}$$

$$V_{23} = x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} - y_2 \frac{\partial}{\partial y_3} + y_3 \frac{\partial}{\partial y_2}$$

$$V_{24} = x_2 \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_2} - y_2 \frac{\partial}{\partial y_4} - y_4 \frac{\partial}{\partial y_2}$$

$$V_{25} = x_2 \frac{\partial}{\partial x_5} + x_5 \frac{\partial}{\partial x_2} - y_2 \frac{\partial}{\partial y_5} - y_5 \frac{\partial}{\partial y_2}$$

$$V_{34} = x_3 \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_3} - y_3 \frac{\partial}{\partial y_4} - y_4 \frac{\partial}{\partial y_3}$$

$$V_{35} = x_3\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_3} - y_3\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial y_3}$$

$$V_{45} = x_4\frac{\partial}{\partial x_5} - x_5\frac{\partial}{\partial x_4} - y_4\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial y_4}$$

Then we compute the action of the Cartan subalgebra on the V_{ij} :

$$[\gamma_1, V_{12}] = [\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} - y_1\frac{\partial}{\partial y_2} + y_2\frac{\partial}{\partial y_1}]$$

$$= x_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial y_2} + y_2\frac{\partial}{\partial x_1} = -W_{12}$$

The reason for the negative will be explained in a moment.

$$[\gamma_2, V_{12}] = [\left(x_2\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_2}\right), x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} - y_1\frac{\partial}{\partial y_2} + y_2\frac{\partial}{\partial y_1}]$$

$$= x_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial y_2} + y_2\frac{\partial}{\partial x_1} = -W_{12}$$

Recall that that $[\gamma_i, X_{ij}] = -Y_{ij}$ and $[\gamma_i, Y_{ij}] = X_{ij}$ so the signs were different. Thus

$$[\gamma_i, [\gamma_i, X_{ij}]] = -X_{ij}$$

and

$$[\gamma_i, [\gamma_i, Y_{ij}]] = -Y_{ij}.$$

The V_{ij} satisfy:

$$[\gamma_i, [\gamma_i, V_{ij}]] = V_{ij}$$

$$[\gamma_1, V_{13}] = [\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), x_1\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_1} - y_1\frac{\partial}{\partial y_3} + y_3\frac{\partial}{\partial y_1}]$$

$$= x_3\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_3} - x_1\frac{\partial}{\partial y_3} + y_3\frac{\partial}{\partial x_1} = -W_{13}$$

$$[\gamma_3, V_{13}] = [\left(x_3\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_3}\right), x_1\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_1} - y_1\frac{\partial}{\partial y_3} + y_3\frac{\partial}{\partial y_1}]$$

$$= x_3\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_3} - x_1\frac{\partial}{\partial y_3} + y_3\frac{\partial}{\partial x_1} = -W_{13}$$

In contrast to the relations we had before:

$$[\gamma_i, X_{ij}] = -Y_{ij}$$

and

$$[\gamma_j, X_{ij}] = Y_{ij}$$

these new vectors satisfy:

$$[\gamma_i, V_{ij}] = -W_{ij}$$

and

$$[\gamma_j, V_{ij}] = -W_{ij}$$

$$[\gamma_1, V_{14}] = [\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), x_1\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_1} - y_1\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial y_1}]$$

$$= -x_1\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_1} - x_4\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_4} = -W_{14}$$

If we had the positive sign for W_{14}, then each term would be negative, hence the negatives were put in only for esthetic reasons.

$$[\gamma_4, V_{14}] = [\left(x_4\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_4}\right), x_1\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_1} - y_1\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial y_1}]$$

$$= -x_1\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_1} - x_4\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_4} = -W_{14}$$

$$[\gamma_1, V_{15}] = [\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), x_1\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_1} - y_1\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial y_1}]$$

$$= -x_1\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_1} - x_5\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_5} = -W_{15}$$

$$[\gamma_5, V_{15}] = [\left(x_5\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_5}\right), x_1\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_1} - y_1\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial y_1}]$$

$$= -x_1\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_1} - x_5\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_5} = -W_{15}$$

$$[\gamma_2, V_{23}] = [\left(x_2\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_2}\right), x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2} - y_2\frac{\partial}{\partial y_3} + y_3\frac{\partial}{\partial y_2}]$$

$$= -x_2\frac{\partial}{\partial y_3} + y_3\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_3} = -W_{23}$$

$$[\gamma_3, V_{23}] = [\left(x_3\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_3}\right), x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2} - y_2\frac{\partial}{\partial y_3} + y_3\frac{\partial}{\partial y_2}]$$

$$= -x_2\frac{\partial}{\partial y_3} + y_3\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_3} = -W_{23}$$

$$[\gamma_2, V_{24}] = [\left(x_2\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_2}\right), x_2\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_2} - y_2\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial y_2}]$$

$$= -x_2\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_2} - x_4\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_4} = -W_{24}$$

$$[\gamma_4, V_{24}] = [\left(x_4\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_4}\right), x_2\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_2} - y_2\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial y_2}]$$

$$= -x_2\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_2} - x_4\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_4} = -Y_{24}$$

$$[\gamma_2, V_{25}] = [\left(x_2\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_2}\right), x_2\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_2} - y_2\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial y_2}]$$

$$= -x_2\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_2} - x_5\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_5} = -W_{25}$$

$$[\gamma_5, V_{25}] = [\left(x_5\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_5}\right), x_2\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_2} - y_2\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial y_2}]$$

$$= -x_2\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_2} - x_5\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_5} = -W_{25}$$

$$[\gamma_3, V_{34}] = [\left(x_3\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_3}\right), x_3\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_3} - y_3\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial y_3}]$$

$$= -x_3\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_4} = -W_{34}$$

$$[\gamma_4, V_{34}] = [\left(x_4\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_4}\right), x_3\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_3} - y_3\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial y_3}]$$

$$= -x_3\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_4} = -W_{34}$$

$$[\gamma_3, V_{35}] = [\left(x_3\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_3}\right), x_3\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_3} - y_3\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial y_3}]$$

$$= -x_3\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_3} - x_5\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_5} = -W_{35}$$

$$[\gamma_5, V_{35}] = [\left(x_5\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_5}\right), x_3\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_3} - y_3\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial y_3}]$$

$$= -x_3\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_3} - x_5\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_5} = -W_{35}$$

$$[\gamma_4, V_{45}] = [\left(x_4\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_4}\right), x_4\frac{\partial}{\partial x_5} - x_5\frac{\partial}{\partial x_4} - y_4\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial y_4}]$$

$$= -x_4\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial x_4} + x_5\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_5} = -W_{45}$$

$$[\gamma_5, V_{45}] = [\left(x_5\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_5}\right), x_4\frac{\partial}{\partial x_5} - x_5\frac{\partial}{\partial x_4} - y_4\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial y_4}]$$

$$= -x_4\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial x_4} + x_5\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_5} = -W_{45}$$

13 Particle-Operator Correspondence

In order to obtain the operator representation of the elementary particles, we first need to calculate the eigenvalues (quantum numbers).

$$[\gamma_1, X_{12} + iY_{12}] = -i\,(X_{12} + iY_{12})$$

$$[\gamma_2, X_{12} + iY_{12}] = i\,(X_{12} + iY_{12})$$

$$[\gamma_1, X_{12} - iY_{12}] = i\,(X_{12} - iY_{12})$$

$$[\gamma_2, X_{12} - iY_{12}] = -i\,(X_{12} + iY_{12})$$

$$[\gamma_1, X_{13} + iY_{13}] = -i\,(X_{13} + iY_{13})$$

$$[\gamma_3, X_{13} + iY_{13}] = i\,(X_{13} + iY_{13})$$

$$[\gamma_1, X_{13} - iY_{13}] = i\,(X_{13} - iY_{13})$$

$$[\gamma_3, X_{13} - iY_{13}] = -i\,(X_{13} - iY_{13})$$

$$[\gamma_1, X_{14} + iY_{14}] = -i\,(X_{14} + iY_{14})$$

$$[\gamma_4, X_{14} + iY_{14}] = i\,(X_{14} + iY_{14})$$

$$[\gamma_1, X_{14} - iY_{14}] = -i\,(X_{14} - iY_{14})$$

$$[\gamma_4, X_{14} - iY_{14}] = -i\,(X_{14} - iY_{14})$$

$$[\gamma_2, X_{23} + iY_{23}] = -i\,(X_{23} + iY_{23})$$

$$[\gamma_3, X_{23} + iY_{23}] = i\,(X_{23} + iY_{23})$$

$$[\gamma_2, X_{23} - iY_{23}] = i\,(X_{23} - iY_{23})$$

$$[\gamma_3, X_{23} - iY_{23}] = -i\,(X_{23} - iY_{23})$$

$$[\gamma_2, X_{24} + iY_{24}] = -i\,(X_{24} + iY_{24})$$

$$[\gamma_4, X_{24} + iY_{24}] = i\,(X_{24} + iY_{24})$$

$$[\gamma_2, X_{24} - iY_{24}] = i\,(X_{24} - iY_{24})$$

$$[\gamma_4, X_{24} - iY_{24}] = -i\,(X_{24} - iY_{24})$$

$$[\gamma_3, X_{34} + iY_{34}] = -i\,(X_{34} + iY_{34})$$

$$[\gamma_4, X_{34} + iY_{34}] = i\,(X_{24} + iY_{24})$$

$$[\gamma_3, X_{34} - iY_{34}] = i\,(X_{34} - iY_{34})$$

$$[\gamma_4, X_{34} - iY_{34}] = -i\,(X_{34} - iY_{34})$$

Now we have the information necessary to determine the operator corresponding to each particle.

	γ_1	γ_2	γ_3	γ_4	
ν	i	$-i$	0	0	$X_{12} - iY_{12}$
$\bar{\nu}$	$-i$	i	0	0	$X_{12} + iY_{12}$
H	i	0	$-i$	0	$X_{13} - iY_{13}$
$\bar{H}$	$-i$	0	i	0	$X_{13} + iY_{13}$
e^-	i	0	0	$-i$	$X_{14} - iY_{14}$
e^+	$-i$	0	0	i	$X_{14} + iY_{14}$
n	0	i	$-i$	0	$X_{23} - iY_{23}$
$\bar{n}$	0	$-i$	i	0	$X_{23} + iY_{23}$
π^-	0	i	0	$-i$	$X_{24} - iY_{24}$
π^+	0	$-i$	0	i	$X_{24} + iY_{24}$
p^+	0	0	$-i$	i	$X_{34} + iY_{34}$
p^-	0	0	i	$-i$	$X_{34} - iY_{34}$

14 Field-Particle Interactions

> The connection between the field and its source has always been and still is the most difficult problem in classical and quantum electrodynamics.—D.K. Sen [50]

The formalism developed in the previous sections allows us to reveal "the connection between the field and its source." We will see how the fields act on the particle as well as how the particle generates its fields, which is a problem in all field theories. Using the differential operator representation, we compute the interaction of the γ_1 field with the antineutrino term by term:

$$[\gamma_1, \bar{\nu}] =$$

$$[\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), \left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right) + \left(y_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_1}\right)$$

$$+i\left(x_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_2}\right) + i\left(x_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_1}\right)]$$

We will do the calculation term by term:

$$[\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), \left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right)]$$

$$= \left(x_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_2}\right)$$

$$= -(i)(i)\left(x_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_2}\right)$$

$$[\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), \left(y_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_1}\right)]$$

$$= \left(x_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_1}\right)$$

$$= -(i)(i)\left(x_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_1}\right)$$

$$[\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), i\left(x_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_2}\right)]$$

$$= -i\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right)$$

$$[\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), i\left(x_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_1}\right)]$$

$$= -i\left(y_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_1}\right)$$

Summing, we obtain $-i$ times what we started with. But there is more going on, the field permutes the four components.

It is interesting to note that the above describes the neutrino in terms of a four component spinor. Fermi describes the neutrino as a four component spinor. Is there a connection between the two descriptions?

We could do a similar calculation for each elementary particle, but we will only do one more in detail.

Computing the interaction of the γ_1 with a positron:

$$[\gamma_1, e^+] = [\gamma_1, X_{14} + iY_{14}] =$$

$$[\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), \left(x_1\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_1}\right)$$

$$+\left(y_1\frac{\partial}{\partial y_4} + y_4\frac{\partial}{\partial y_1}\right) + i\left(x_1\frac{\partial}{\partial y_4} + y_4\frac{\partial}{\partial x_1}\right) - i\left(x_4\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial x_4}\right)]$$

the interaction of the field with the particle distinguishes four separate parts of the anti-electron:

$$[\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), \left(x_1\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_1}\right)] =$$

$$-\left(x_4\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial x_4}\right)$$

$$[\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), \left(y_1\frac{\partial}{\partial y_4} + y_4\frac{\partial}{\partial y_1}\right)] =$$

$$\left(x_1\frac{\partial}{\partial y_4} + y_4\frac{\partial}{\partial x_1}\right)$$

$$[\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), \left(x_4\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial x_4}\right)] =$$

$$\left(x_1\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_1}\right)$$

$$[\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right), \left(x_1\frac{\partial}{\partial y_4} + y_4\frac{\partial}{\partial x_1}\right)] =$$

$$-\left(y_1\frac{\partial}{\partial y_4} + y_4\frac{\partial}{\partial y_1}\right)$$

Again as with the neutrino, the field permutes the four components. Now the question arises, do these four operators have any significance of their own, do their interactions play any role in the description of the electron? The only way to find out is to do the calculations and see what develops. Let's label the four parts:

$$\chi_{14} = \left(x_1 \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_1} \right)$$

$$\zeta_{14} = \left(y_1 \frac{\partial}{\partial y_4} + y_4 \frac{\partial}{\partial y_1} \right)$$

$$A_{14} = \left(x_4 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial x_4} \right)$$

$$B_{14} = \left(x_1 \frac{\partial}{\partial y_4} + y_4 \frac{\partial}{\partial x_1} \right)$$

Now we compute the rest of the operators in this "internal algebra":

$$[\chi_{14}, \zeta_{14}] = [\left(x_1 \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_1} \right), \left(y_1 \frac{\partial}{\partial y_4} + y_4 \frac{\partial}{\partial y_1} \right)] = 0$$

$$[\chi_{14}, A_{14}] = [\left(x_1 \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_1} \right), \left(x_4 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial x_4} \right)]$$

$$= -y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1} = \gamma_1$$

$$[\zeta_{14}, A_{14}] = [\left(y_1 \frac{\partial}{\partial y_4} + y_4 \frac{\partial}{\partial y_1} \right), \left(x_4 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial x_4} \right)]$$

$$= \left(y_4 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial y_4} \right) = -\gamma_4$$

$$[A_{14}, B_{14}] = 0$$

$$[\chi_{14}, B_{14}] = [\left(x_1 \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_1} \right), \left(x_1 \frac{\partial}{\partial y_4} + y_4 \frac{\partial}{\partial x_1} \right)]$$

$$= \left(x_4 \frac{\partial}{\partial y_4} - y_4 \frac{\partial}{\partial x_4} \right) = \gamma_4$$

$$[\zeta_{14}, B_{14}] = [\left(y_1 \frac{\partial}{\partial y_4} + y_4 \frac{\partial}{\partial y_1}\right), \left(x_1 \frac{\partial}{\partial y_4} + y_4 \frac{\partial}{\partial x_1}\right)]$$

$$= \left(y_1 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial y_1}\right) = -\gamma_1$$

Thus, the internal algebra includes the fields of the particle.

$$[\gamma_1, \chi_{14}] = [\left(x_1 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_1}\right), \left(x_1 \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_1}\right)]$$

$$= \left(-y_1 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial y_1}\right) = -A_{14}$$

$$[\gamma_4, \chi_{14}] = [\left(x_4 \frac{\partial}{\partial y_4} - y_4 \frac{\partial}{\partial x_4}\right), \left(x_1 \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_1}\right)]$$

$$= \left(-y_4 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial y_4}\right)] = -B_{14}$$

$$[\gamma_4, A_{14}] = [\left(x_4 \frac{\partial}{\partial y_4} - y_4 \frac{\partial}{\partial x_4}\right), \left(x_4 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial x_4}\right)]$$

$$= \left(y_1 \frac{\partial}{\partial y_4} + y_4 \frac{\partial}{\partial y_1}\right) = \zeta_{14}$$

$$[\gamma_1, A_{14}] = [\left(x_1 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_1}\right), \left(x_4 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial x_4}\right)]$$

$$= -\left(x_1 \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_1}\right) = -\chi_{14}$$

The internal algebra generators are then: χ_{14}, $A_{14} = -[\gamma_1, \chi_{14}]$, $B_{14} = -[\gamma_4, \chi_{14}]$, $\zeta_{21} = [\gamma_1, [\gamma_4, \chi_{14}]]$, γ_1, and γ_4.

The fields then act as the input of an anharmonic oscillator, setting up vibrational patterns:

$$\begin{array}{ccc} \chi_{14} & \leftarrow \gamma_1 \rightarrow & [\gamma_1, \chi_{14}] \\ \uparrow \gamma_4 \downarrow & & \uparrow \gamma_4 \downarrow \\ [\gamma_4, \chi_{14}] & \leftarrow \gamma_1 \rightarrow & [\gamma_1, [\gamma_4, \chi_{14}]] \end{array}$$

If our work on the Vortex-Spin Networks taught us anything it is that the arrows must go one direction for the particle, the other direction for the antiparticle. Then for the electron, we would have:

$$\begin{array}{ccc} \chi_{14} & \gamma_1 \rightarrow & [\gamma_1, \chi_{14}] \\ \uparrow \gamma_4 & & \gamma_4 \downarrow \\ [\gamma_4, \chi_{14}] & \leftarrow \gamma_1 & [\gamma_1, [\gamma_4, \chi_{14}]] \end{array}$$

while the antielectron would have a flow in the opposite direction:

$$\begin{array}{ccc} \chi_{14} & \leftarrow \gamma_1 & [\gamma_1, \chi_{14}] \\ \gamma_4 \downarrow & & \uparrow \gamma_4 \\ [\gamma_4, \chi_{14}] & \gamma_1 \rightarrow & [\gamma_1, [\gamma_4, \chi_{14}]] \end{array}$$

Just as the motion of a charged particle generates a magnetic field, so the internal motion within an elementary particle generates its fields. If we were to analyze a proton, we would also have four components which would evidently correspond to the idea of quarks. But here every particle has the four component internal structure.

In [35] I showed that the interaction of elementary particles can be interpreted in terms of the Lie bracket of vector fields on the complex space-time $QAdS = U(3,2)/U(3,1) \times U(1)$. The only problem with the analysis there is that the differential operator representation used there proved to be inadequate: the above diagram was not possible using those operators. In the next section we continue this analysis and show that we are forced to intepret the complex space-time $QAdS$ in terms of the modes of oscillation of fields on AdS.

15 Particles as Geometry

> At the turn of the century physicists began to be dissatisfied with the dualism of a theory admitting two kinds of fundamental physical reality: on the one hand the field and on the other hand the material particles. It is only natural that attempts were made to represent the material particles as structures in the field, that is, as places where the fields were exceptionally concentrated. Any such representation of particles as the basis of the field theory would have been a great achievement, but in spite of all efforts of

> science it has not been accomplished. It must even he admitted that this dualism is today sharper and more troublesome than it was ten years ago. This fact is connected with the latest impetus to developments in quantum theory, where the theory of the continuous (field theory) and the essentially discontinuous interpretation of the elementary structures and processes are fighting for supremacy.
>
> —Albert Einstein[11]

Wheeler [54] detailed his view of the role of geometry in physics:

> Is space-time only an arena within which fields and particles move about as "physical" and "foreign" entities? Or is the four-dimensional continuum all there is? Is curved empty geometry a kind of magic building material out of which everything in the physical world is made: (1) slow curvature in one region of space describes a gravitational field; (2) a rippled geometry with a different type of curvature somewhere else describes an electromagnetic field; (3) a knotted up region of high curvature describes a concentration of charge and mass-energy that moves like a particle? Are fields and particles entities immersed in geometry, or are they nothing but geometry?
>
> It would be difficult to name any issue more central to the plan of physics: whether space-time is only an arena or whether it is everything.

A complex space-time was required to realize what Wheeler [56] called

> ...the picture of Clifford and Einstein that particles originate from geometry; that there is no such thing as a particle immersed in geometry, but only a particle built out of geometry.

Wheeler was invoking Clifford's statements in his talk before the Cambridge Philosophical Society on February 21,1870 entitled "On the Space Theory of Matter":

> I hold in fact (1) That small portions of space are in fact of a nature analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not

valid in them. (2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave. (3) That this variation of the curvature of space is what really happens in that phenomenon which we call the motion of matter, whether ponderable or etherial. (4) That in the physical world nothing else takes place but this variation, subject (possibly) to the law of continuity.

In his vision of matter as curvature, Clifford presaged not only General Relativity but also quantum mechanics, since he saw the curvature propagating as a wave much like the currently popular Ricci flows.

This is a theme which Wheeler repeated many times, after the above quotation from Clifford, Rees, Ruffini and Wheeler [44] go on to:

> Ask if there is a sense in which to speak of a particle as constructed out of geometry. Or rephrase the question in updated language: "Is a particle a geometrodynamic exciton?" What else is there out of which to build a particle except geometry itself? And what else is there to give discreteness to such an object except the quantum principle? (p. 292)

According to Wheeler, "The most evident shortcoming of the geometrodynamical model as it stands is this, that it fails to supply any completely natural place for spin - in general and for the neutrino."

The geometric model of matter being developed here has a natural place for spin, charge, baryon number and lepton number as well as particle type.

Wheeler once more: "The Riemannian curvature tensor ... tells the response *of* matter *to* geometry." [55]

If it describes anything, the Riemannian curvature tensor describes the total geometric input from all particles. The problem with this is that the particle contributes to the geometry, but it does not respond to the geometry it creates. A particle creates the geometry which other particles respond to. It does not create the geometry it responds to. Which is why interaction via Lie bracket is essential to describing the physics: any particle bracketed with itself yields zero. This also touches on Wheeler's next question: "What about the converse issue, the response *of* geometry *to* matter?"

Wheeler [44] claims that "No inconsistency of principle has ever been found in Einstein's geometric theory of gravity." The self interaction problem is one, but there is a larger problem: conserved quantities in general relativity. In the standard treatment, a conserved quantity corresponds with a Killing field, a symmetry of the metric. A conservation law limits the behavior of matter, however the introduction of matter destroys the symmetry of the metric and hence also destroys the conservation law. I have to conclude that the metric cannot be the fundamental object and that the source of conservation laws is to be found elsewhere.

Specifically the conservation laws are found in the geometric setting of $U(3,2)/U(3,1) \times U(1)$. This an example of Theorem 8.1 of Hermann [26]:

> Let G be a semisimple Lie group, $M = G/L$ a coset space of G which admits a G-invariant canonical structure. Then, there is an element $Z \in G$ such that:
>
> $L =$ centralizer of Z in G,
>
> i.e. the set of $X \in G$ such that $[X, Z] = 0$. Further, the form ω determining the canonical structure is determined in terms of Z by
>
> $$\omega\left(X, Y\right) = B\left([Z, X], Y\right)$$

Here B is the Killing form.

Putting in the details of our specific model, with $Z = \gamma_5$:

$$\omega\left(X, Y\right) = B\left([\gamma_5, X], Y\right) = B\left([\gamma_5, [X, Y]\right)$$

Thus $\omega\left(X, Y\right) = 0$ except for

$$\omega\left(X_{i5}, Y_{i5}\right) = B\left([\gamma_5, X_{i5}], Y_{i5}]\right)$$

$$= B\left(-[Y_{i5}, [Y_{i5}\right) = \begin{cases} -1 \ldots i \neq 4 \\ 1 \ldots i = 4. \end{cases}$$

Let us recall the properties of the Killing form on a Lie algebra. The Killing form is defined by:

$$B(X, Y) = Tr(adXadY)$$

Thus, we would compute

$$[X, [Y, Z]]$$

for all Z in the Lie algebra. To evaluate the trace we take the sum of the eigenvalues.

The Killing form satisfies:

$$B([X,Y],Z) = B(X,[Y,Z])$$

The general curvature 4-tensor on a homogeneous space is given by

$$R(X,Y,U,V) = B(R(X,Y)U,V)$$

where B is the Killing form on $u(3,2)$. By elementary properties of the Killing form and using

$$R(X,Y)U = -[[X,Y],U]$$

which is valid for a reductive homogeneous space, we obtain

$$R(X,Y,U,V) = B(R(X,Y)U,V)$$

$$= -B([[X,Y],U],V)$$

$$= -B([X,Y],[U,V])$$

Thus the curvature is defined in terms of the Lie bracket.

In their foundational paper on axiomatic quantum field theory, Wightman and Garding [58] showed that relativistic quantum fields should be viewed as "operator valued distributions" In the present work, no distinction can be made between a particle and its fields. An element of the Lie algebra can be viewed as an operator, the corresponding Lie derivative. Thus tangent vectors are simply a geometric interpretation of "operator valued distributions".

The tangent vectors can also be taken as the foundation of a theory of gravity. Dirac [8] discussed the tetrad (a.k.a veirbein) formulation of General Relativity:

> For dealing with spinors in a Riemann space one must introduce a fourleg at each point described by field functions $h_{\mu a}$ satisfying
>
> $$h_a^\mu h_{\mu b} = \eta_{ab}, \qquad \eta^{ab} h_{\mu a} h_{\nu b} = g_{\mu\nu}$$
>
> where η_{ab} is the fundamental tensor of special relativity. The $h_{\mu a}$ become the fundamental field quantities of the gravitational field, instead of the $g_{\mu\nu}$.

The 'fourleg', a.k.a. tetrad or vierbein formalism finds its perfect fulfillment in the homogeneous space setting, where the vectors defining the tetrad are already present (as left invariant vectors) and do not need to be tacked on. The Killing form provides a natural background metric.

We have then the space of vector fields on $U(3,2)$ as the fundamental objects. These decompose into horizontal vectors in the tangent space of $U(3,2)/U(3,1) \times U(1)$ and the vertical vectors in $U(3,1) \times U(1)$.

The vertical vectors can be thought of as a supplement to the tangent space of space-time in the sense that Einstein and Mayer [12] and Rosen and Tauber [46] considered bundles of 4 + n dimensional vector spaces over space-time. The number of extra dimensions is arbitrary in their approach but is fixed geometrically here.

The vectors interact via Lie bracket.

The space of all vector fields generates the group of diffeomorphisms. The group of diffeomorphisms is related to the flow of a perfect fluid [9]. Gravitation has been described as a gauge theory of the group of diffeomorphisms.

With the 1-forms ω_μ dual to the basis X_μ, we have

$$\omega_\mu Z = B\left(X_\mu, Z\right)$$

where B is the Killing form.

Following Hermann [27], the Ricci-Tensor is defined by:

$$RI\left(X,Y\right)Z = \sum\nolimits_\mu \omega_\mu R\left(X_\mu, X\right)Z$$

$$= \sum\nolimits_\mu B\left(X_\mu, R\left(X_\mu, X\right)Z\right)$$

$$= \sum\nolimits_\mu B\left(X_\mu, [[X_\mu, X], Z]\right)$$

$$= -\sum\nolimits_\mu B\left([X_\mu, [X_\mu, X]], Z\right)$$

$$= -B\left(\sum\nolimits_\mu [X_\mu, [X_\mu, X]], Z\right)$$

For the Einstein equations, the range of the sum is one to four. It is not clear if we should extend the range of the indices to include the space-time Y_μ, or if we include all the indices of the this last sum: $\sum_\mu [X_\mu, [X_\mu,$ is essentially (a multiple of) the Casimir operator of $u(3,2)$. The Casimir operator is an invariant of $U(3,2)$. Thus the Einstein equations could be written as:

$$RI\left(X,Y\right) = \sum\nolimits_g \omega_\mu R\left(X_\mu, X\right)Z = 0$$

Where the sum is over all the indices of $g = u(3,2)$.

Which can be decomposed as

$$\sum\nolimits_h \omega_\mu R\left(X_\mu, X\right) Z + \sum\nolimits_T \omega_\mu R\left(X_\mu, X\right) Z = 0$$

where the sum over h is the sum over the indices of $h = u(3,1) \times u(1)$ and the sum over T is the sum over the tangent space of $U(3,2)/U(3,1) \times U(1)$.

Following Lurcat [37] this seems to require the Einstein Equations on the full group manifold. It seems reasonable to replace the Ricci tensor by the Casimir operator.

This presentation has shown us how to find the right hand side of the Einstein equations.

16 The Geometry of Elementary Particle Interactions

Now let us consider one of the secondary interactions:

$$e^- \bar{n} \leftrightarrow p^- \nu$$

In the principle fiber bundle $U(3,2) \to U(3,2)/U(3,1) \times U(1)$, the bracket of two vectors in the base space yields a vector in the bundle. The vectors in the bundle represent particles and in this instance:

$$e^- = X_{14} - iY_{14} =$$

$$\frac{-1}{2}\left[X_{15} - iY_{15}, X_{45} + iY_{45}\right]$$

$$\bar{n} = X_{23} + iY_{23}$$

$$\frac{1}{2}\left[X_{25} + iY_{25}, X_{35} - iY_{35}\right]$$

$$p^- = X_{34} - iY_{34}$$

$$\frac{-1}{2}\left[X_{35} - iY_{35}, X_{45} + iY_{45}\right]$$

$$\nu = X_{12} - iY_{12}$$

$$\frac{1}{2}\left[X_{15}-iY_{15},X_{25}+iY_{25}\right]$$

Thus, the interaction of e^- with $\bar{n}$ (dropping the numerical factors) would go:

$$\begin{aligned}&[X_{15}-iY_{15},X_{45}+iY_{45}]\otimes[X_{25}+iY_{25},X_{35}-iY_{35}]\\ &=(X_{15}-iY_{15})\otimes(X_{45}+iY_{45})\otimes(X_{25}+iY_{25})\otimes(X_{35}-iY_{35})\\ &=(X_{15}-iY_{15})\otimes(X_{25}+iY_{25})\otimes(X_{45}+iY_{45})\otimes(X_{35}-iY_{35})\\ &=(X_{15}-iY_{15})\otimes(X_{25}+iY_{25})\otimes(X_{45}+iY_{45})\otimes(X_{35}-iY_{35})\\ &=[X_{15}-iY_{15},X_{25}+iY_{25}]\otimes[X_{45}+iY_{45},X_{35}-iY_{35}]\end{aligned}$$

Beginning with e^- and $\bar{n}$ and following the rules for exchange of field quanta we arrived at p^- and ν. Thus, the change in particle type may be interpreted as an interaction involving the exchange of the field quanta. In the same way, all of the above interactions may be interpreted as interactions involving the exchange of the field quanta.

Thus, it seems that particles can be interpreted as being composed of the background field quanta and the dynamics required is the dynamics of the background field, exactly as Einstein envisioned. Except now, the background involved is that of $U(3,2)$, not just space-time, and there are five interactions involved, not just gravitation. Just as Lurcat [37] used the group manifold of the *Poincaré* Group to obtain a dynamical role for spin, so here we use the group manifold of $U(3,2)$ to obtain the a dynamical role for all the quantum numbers of a totally unified field theory.

17 Complex structures

When we went from the matrix representation to the differential operator representation, the complex number i in the matrix representation was replaced by a differential operator, in fact it was replaced by 5 different operators, depending on its location on the diagonal.

The complex number i can be considered to be the infinitesimal generator of the unit circle as a Lie group since

$$e^{i\theta}=\cos\theta+i\sin\theta$$

Now, if

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$\tan\theta = \tfrac{y}{x}$$

Then the operator

$$\frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y}$$
$$= -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}$$
$$= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

can also be viewed as the infinitesimal generator of the unit circle $U(1)$.

This seems to imply that in some of the differential equations of physics the factor of i should be replaced by an operator of the form

$$-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

as in these equations.

This leads us to a study of complex structures. An almost complex structure on a vector space V is a linear automorphism

$$J : V \to V$$

satisfying

$$J^2 = -1$$

From the characteristic equation of J,

$$1 + J^2 = (1 + iJ)(1 - iJ) = 0$$

we conclude that J has two eigenvalues, i and $-i$.

To obtain the eigenvectors of J we must extend the domain of J to the complexification of V, $V_C = C \otimes V$, for $v \in V$, we can form the vector $(1 + iJ)\,v = v + iJv$ then we calculate:

$$J(v + iJv) = Jv + iJ^2v = Jv - iv = -i(v + iJv),$$

Likewise using $(1 - iJ)\,v = v - iJv$

$$J(v - iJv) = Jv - iJ^2v = Jv + iv = i(v - iJv,)$$

Thus V is decomposed into two subspaces according the eigenvalue. Since

$$v + iJv = (1 + iJ)\, v,$$

we can define

$$V_- = (1 + iJ)\, V.$$

This is the subspace of V_C with eigenvalue $-i$.

Likewise,

$$V_+ = (1 - iJ)\, V$$

is the subspace of V_C with eigenvalue $+i$.

But another characterization is possible since

$$(1 - iJ)(1 + iJ)\, V = (1 + iJ)(1 - iJ)\, V = \left(1 - i^2 J^2\right) V = 0$$

Thus $V_- = ker\,(1 - iJ)$ and $V_+ = ker\,(1 + iJ)$.

$$(1 - iJ)^2\, V = \left(1 - 2iJ + i^2 J^2\right) V = (1 - 2iJ + 1)\, V = 2\,(1 - iJ)\, V$$

Thus

$$P_+ = \frac{(1 - iJ)}{2}$$

is a projection of V onto V_+. And:

$$P_- = \frac{(1 + iJ)}{2}.$$

is a projection of V onto V_-.

This is all standard fare.

18 The Complex Structure of $QAdS$

In the geometry of $QAdS$, an almost complex structure is given by

$$Jv = [\gamma_5, v]$$

with

$$\gamma_5 = \left(x_5 \frac{\partial}{\partial y_5} - y_5 \frac{\partial}{\partial x_5}\right)$$

The tangent space of QAdS is spanned by:

$$X_{15} = x_1\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial y_1}$$

$$X_{25} = x_2\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial y_2}$$

$$X_{35} = x_3\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial y_3}$$

$$X_{45} = x_4\frac{\partial}{\partial x_5} - x_5\frac{\partial}{\partial x_4} + y_4\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial y_4}$$

$$Y_{15} = x_1\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial x_1} - x_5\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_5}$$

$$Y_{25} = x_2\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial x_2} - x_5\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_5}$$

$$Y_{35} = x_3\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial x_3} - x_5\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_5}$$

$$Y_{45} = x_4\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_4} + x_5\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_5}$$

A direct calculation shows that:

$$[\gamma_5, X_{i5}] = -Y_{i5}$$

$$[\gamma_5, Y_{i5}] = X_{i5}$$

$$[\gamma_5, [\gamma_5, X_{i5}]] = -X_{i5}$$

$$[\gamma_5, [\gamma_5, Y_{i5}]] = -Y_{i5}$$

Thus, $([\gamma_5,)$ defines an almost complex structure.

Theorem: If an almost complex structure on a reductive homogeneous space is defined by $Jv = [\gamma, v]$, then J is integrable.

Proof: On a homogeneous space, G/H, the almost complex structure J is integrable iff the torsion tensor of J:

$$S(X,Y) = -2([X,Y] + J[JX,Y] + J[X,JY] - [JX,JY])$$

is an element of the Lie algebra of H (Kobayashi and Nomizu [31], p. 217). In the case at hand,

$$S(X,Y) = -2([X,Y] + [\gamma_5,[[\gamma_5,X],Y]] + [\gamma_5,[X,[\gamma_5,Y]]] - [[\gamma_5,X],[\gamma_5,Y])$$

We will examine each term in turn:

Since $X, Y \in T(G/H)$ and since the space is reductive, $[X,Y] \in H$.

Since $X, Y \in T(G/H)$ and J is an A.C.S.$[\gamma_5, X] \in T(G/H)$ and since the space is reductive, $[[\gamma_5, X], Y] \in H$, then $[\gamma_5,[[\gamma_5,X],Y]] = 0$.

For the third term: $[\gamma_5, Y] \in T(G/H)$ implies $[X,[\gamma_5,Y]] \in H$ and thus $[\gamma_5,[X,[\gamma_5,Y]]] = 0$.

The last term is treated in the same manner:

Since $X, Y \in T(G/H)$, $[\gamma_5, X] \in T(G/H)$ and $[\gamma_5, Y] \in T(G/H)$ then $[[\gamma_5,X],[\gamma_5,Y] \in H$ since the space is reductive.

Thus each term is in H so the sum is in H and the almost complex structure is integrable.

Theorem: If an almost complex structure on a homogeneous space is defined by $Jv = [\gamma, v]$, then the action of e^{tJ} is given by:

$$\exp\left(t[\gamma,\right)v = (\cos t)\,v + (\sin t)\,[\gamma,v]$$

Furthermore,

$$exp\left(\frac{\pi}{2}J\right) = \exp\left(\frac{\pi}{2}[\gamma,\right)$$

is also an almost complex structure.

The proof is exactly that of proving $e^{i\theta} = \cos\theta + i\sin\theta$.

Corollary: $exp\,(2n\pi J) = 1$ and $exp\,((2n+1)\,\pi J) = -1$

Corollary: $exp(2n\pi J + \frac{\pi}{2}J)$ and $exp((2n+1)\pi J + (\frac{\pi}{2}J))$ define complex structures.

It is an exercise to show that the operator:

$$\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5$$

satisfies

$$[\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5, X] = 0$$

for all $X \in T(QAdS)$ and thus the operator:

$$\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$$

defines another complex structure. At this point it should be no surprise that other complex structures are lurking in the background.

Define

$$\iota\,(X_{I5}) = Y_{I5} \qquad 1 \le i \le 3$$

$$\iota\,(Y_{I5}) = -X_{I5}$$

$$\iota\,(X_{45}) = -Y_{45}$$

$$\iota\,(Y_{45}) = X_{45}$$

Clearly, ι is defined by bracket with:

$$\gamma_1 + \gamma_2 + \gamma_3 - \gamma_4$$

And we can define other complex structures by changing the other signs:

$$\pm\gamma_1 \pm \gamma_2 \pm \gamma_3 \pm \gamma_4$$

Then we can exponentiate each γ_i individually to obtain different winding numbers for each γ_i. This gives us an infinite number of complex structures on $QAdS$.

In the standard theory of complex structures on a manifold [31],

> A real vector space with a complex structure J can be turned into a complex vector space by defining scalar multiplication by complex numbers as follows:
>
> $$(a + ib)X = aX + bJX$$

There are problems implementing this procedure in our setting. In the standard treatment of Lie Algebras, multiplying a generator by i changes it from compact to noncompact and vice-versa. That does not hold for the complex structure defined by bracket with γ_5, both X_{15} and $[\gamma_5, X_{15}] =$

$-Y_{15}$ are noncompact, while both X_{45} and $[\gamma_5, X_{45}] = -Y_{45}$ are compact. Furthermore, we have multiple complex structures, which one would we have replace multiplication by i? The standard mathematics fails us at this point. We need all these complex structures and we need to have eigenvectors with i as an eigenvalue.

In [35], the operator $ix_5\frac{\partial}{\partial x_5}$ was used to define a complex structure on $QAdS$. That is not a viable option in the present setting.

Define:

$$\kappa_5 = i\left(x_5\frac{\partial}{\partial x_5} + y_5\frac{\partial}{\partial y_5}\right)$$

Then

$$[\kappa_5, X_{15}] =$$

$$[i\left(x_5\frac{\partial}{\partial x_5} + y_5\frac{\partial}{\partial y_5}\right), x_1\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial y_1}]$$

$$= i\left(x_5\frac{\partial}{\partial x_1} + y_5\frac{\partial}{\partial y_1} - x_1\frac{\partial}{\partial x_5} - y_1\frac{\partial}{\partial y_5}\right)$$

$$[\kappa_5, [\kappa_5, X_{15}]] =$$

$$[i\left(x_5\frac{\partial}{\partial x_5} + y_5\frac{\partial}{\partial y_5}\right), i\left(x_5\frac{\partial}{\partial x_1} + y_5\frac{\partial}{\partial y_1} - x_1\frac{\partial}{\partial x_5} - y_1\frac{\partial}{\partial y_5}\right)$$

$$= -x_1\frac{\partial}{\partial x_5} - x_5\frac{\partial}{\partial x_1} - y_1\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial y_1}$$

$$[\kappa_5, Y_{15}] =$$

$$[i\left(x_5\frac{\partial}{\partial x_5} + y_5\frac{\partial}{\partial y_5}\right), x_1\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial x_1} - x_5\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_5}] =$$

$$i\left(-x_1\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial x_1} - x_5\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial x_5}\right)$$

$$[\kappa_5, [\kappa_5, Y_{15}]] =$$

$$[i\left(x_5\frac{\partial}{\partial x_5} + y_5\frac{\partial}{\partial y_5}\right), i\left(-x_1\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial x_1} - x_5\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial x_5}\right)] =$$

$$-x_1\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_1} + x_5\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial x_5} = -Y_{15}$$

Indeed, while the operator squared is -1, there is a problem: in the intermediate step, we left the tangent space of $QAdS$. This operator does not define a complex structure.

Define the operators:

$$\kappa_1 = i\left(x_1\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_1}\right)$$

$$\kappa_2 = i\left(x_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2}\right)$$

$$\kappa_3 = i\left(x_3\frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial y_3}\right)$$

$$\kappa_4 = i\left(x_4\frac{\partial}{\partial x_4} + y_4\frac{\partial}{\partial y_4}\right)$$

It is a exercise to show that the operator:

$$K = \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 + \kappa_5$$

satisfies:

$$[K, T] = 0$$

for all $T \in u(3,2)$. This is i times the $u(3,2)$ version of the dilation operator. The factor of i is necessary in order to make the operator compact. Since bracketing with the individual κ_i take us out of the tangent space, it is not obvious whether or not the sum, e.g. the dilation operator should be allowed.

If we allow K, we would then have two diagonal operators, i.e. two first order operators which commute with all of $u(3,2)$.

$$\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 = \tag{2}$$

$$x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_3}$$

$$+x_4\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_4} + x_5\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_5}$$

The existence of two operators which commute with every element of $u(3,2)$ make it clear that matix methods are inadequate for our study.

This result is surprising enough but now that a crack has appeared, the dam bursts:

For any value of n, the operator:

$$(\gamma_1)^n + (\gamma_2)^n + (\gamma_3)^n + (\gamma_4)^n + (\gamma_5)^n = \tag{3}$$

$$(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1})^n + (x_2\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_2})^n + (x_3\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_3})^n$$
$$+(x_4\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_4})^n + (x_5\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_5})^n$$

commutes with all of $u(3,2)$.

19 The Euler Degree Operator

The Euler degree operator has been misidentified in Twistor Theory. Let us begin with a quote from Roger Penrose's recent book, *The Road To Reality* [43], page 984:

> The operator
> $$Y = Z^\alpha \frac{\partial}{\partial Z^\alpha}$$
> is called the *Euler homogeneity operator.*

Unfortunately, this statement is not true, the problem is in the difference between use of complex variables and real variables. Let us switch to real variables and define

$$Y_R = X^\alpha \frac{\partial}{\partial X_\alpha}$$

Then Y_R is the Euler Degree operator, a.k.a. the Euler homogeneity operator, henceforth just Euler Operator.

$$Y_R(X^\beta)^n = X^\alpha \frac{\partial}{\partial X^\alpha}(X^\beta)^n = n(X^\beta)^n$$

The problem with applying Penrose's definition can be illustrated by applying it to $Z\bar{Z}$:

$$YZ\bar{Z} = Z^\alpha \frac{\partial}{\partial Z^\alpha} Z\bar{Z} = Z\bar{Z}$$

But we have

$$Z\bar{Z} = x^2 + y^2$$

Which is clearly of degree 2. However, with the real polynomial $x^2 + y^2$, the correct Euler operator is

$$Y_R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

Then we have

$$Y_R(x^2 + y^2) = (x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})(x^2 + y^2) = 2(x^2 + y^2)$$

In order to obtain the correct result in terms of complex variables, we must have:

$$Y_C = Z^\alpha\frac{\partial}{\partial Z^\alpha} + \bar{Z}^\alpha\frac{\partial}{\partial \bar{Z}^\alpha}$$

Converting to x and y variables:

$$Z\frac{\partial}{\partial Z} = \frac{1}{2}(x + iy)(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$$

$$= \frac{1}{2}(x\frac{\partial}{\partial x} - ix\frac{\partial}{\partial y} + iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$$

$$= \frac{1}{2}(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + i(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}))$$

Likewise:

$$\bar{Z}\frac{\partial}{\partial \bar{Z}} = \frac{1}{2}(x - iy)(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

$$= \frac{1}{2}(x\frac{\partial}{\partial x} + ix\frac{\partial}{\partial y} - iy\frac{\partial}{\partial x} - y\frac{\partial}{\partial y})$$

$$= \frac{1}{2}(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - i(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}))$$

And as advertised, we end up with

$$\bar{Z}\frac{\partial}{\partial \bar{\bar{Z}}} + \bar{Z}\frac{\partial}{\partial \bar{\bar{Z}}} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = Y_R \qquad (4)$$

We can construct five such operators:

$$Z_1\frac{\partial}{\partial Z_1} + \bar{Z}_1\frac{\partial}{\partial \bar{Z}_1} = x_1\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_1} \qquad (5)$$

$$Z_2\frac{\partial}{\partial Z_2} + \bar{Z}_2\frac{\partial}{\partial \bar{Z}_2} = x_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2} \tag{6}$$

$$Z_3\frac{\partial}{\partial Z_3} + \bar{Z}_3\frac{\partial}{\partial \bar{Z}_3} = x_3\frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial y_3} \tag{7}$$

$$Z_4\frac{\partial}{\partial Z_4} + \bar{Z}_4\frac{\partial}{\partial \bar{Z}_4} = x_4\frac{\partial}{\partial x_4} + y_4\frac{\partial}{\partial y_4} \tag{8}$$

$$Z_5\frac{\partial}{\partial Z_5} + \bar{Z}_5\frac{\partial}{\partial \bar{Z}_5} = x_5\frac{\partial}{\partial x_5} + y_5\frac{\partial}{\partial y_5} \tag{9}$$

We recognize these operators, up to the factor of i, as the κ_I previously constructed.

If instead of adding the two operators, we subtract, we obtain:

$$Z\frac{\partial}{\partial Z} - \bar{Z}\frac{\partial}{\partial \bar{Z}} = i(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})) \tag{10}$$

This is the operator which generates rotations in the $x - y$ plane.

We obtain all the $x_I - y_I$ rotations in five complex dimensions:

$$Z_1\frac{\partial}{\partial Z_1} - \bar{Z}_1\frac{\partial}{\partial \bar{Z}_1} = i(y_1\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial y_1})) \tag{11}$$

$$Z_2\frac{\partial}{\partial Z_2} - \bar{Z}_2\frac{\partial}{\partial \bar{Z}_2} = i(y_2\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial y_2})) \tag{12}$$

$$Z_3\frac{\partial}{\partial Z_3} - \bar{Z}_3\frac{\partial}{\partial \bar{Z}_3} = i(y_3\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial y_3})) \tag{13}$$

$$Z_4\frac{\partial}{\partial Z_4} - \bar{Z}_4\frac{\partial}{\partial \bar{Z}_4} = i(y_4\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial y_4})) \tag{14}$$

$$Z_5\frac{\partial}{\partial Z_5} - \bar{Z}_5\frac{\partial}{\partial \bar{Z}_5} = i(y_5\frac{\partial}{\partial x_5} - x_5\frac{\partial}{\partial y_5})) \tag{15}$$

We recognize these operators as the same operators as the γ_I, up to signs and the factor of i.

20 Dual Eigenvectors

We have two commuting complex structures on the tangent space of $QAdS$. Since we will encounter this situation in other cases, we will treat the general case of dual eigenvectors. Given two commuting complex structures on a vector space V, we construct a vector which is simultaneously an eigenvector of both operators.

Call the complex structures J_1 and J_2. Previously we saw that $v + iJ_1v$ is an eigenvector of J_1 and $v+iJ_2v$ is an eigenvector of J_2. We simply substitute the eigenvector of J_1 into the formula for the eigenvector of J_2 to obtain:

$$v + iJ_1v + iJ_2\,(v + iJ_1v) = v + iJ_1v + iJ_2v - J_1J_2v$$

Fortunately, if we substitute the eigenvector of J_2 into the formula for the eigenvector of J_1 we obtain the same vector. This is the result for both of the eigenvalues being minus i. For one eigenvalue of plus i, and one eigenvalue of minus i we obtain:

$$v + iJ_2v - iJ_1\,(v + iJ_2v) = v + iJ_2v - iJ_1v + J_1J_2v$$

Thus:

$$J_1(v + iJ_2v - iJ_1v + J_1J_2v) = J_1v + iJ_1J_2v - iJ_1J_1v + J_1J_1J_2v)$$

$$= J_1v + iJ_1J_2v + iv - J_2v) = i(v + iJ_2v - iJ_1v + J_1J_2v)$$

and

$$J_2(v + iJ_2v - iJ_1v + J_1J_2v) = J_2v + iJ_2J_2v - iJ_2J_1v + J_2J_1J_2v)$$

$$= J_2v - iv - iJ_2J_1v - J_1v) = -i(v + iJ_2v - iJ_1v + J_1J_2v)$$

For both eigenvalues plus i:

$$v - iJ_2v - iJ_1\,(v - iJ_2v) = v - iJ_2v - iJ_1v - J_1J_2v$$

21 Complex geometry without i

The calculations in the last few sections should have left the reader somewhat uncomfortable. Multiplication by i is not defined in the tangent space and replacing it by a complex structure is not tenable. The flow generated by a tangent vector X is e^{tX}, but what does e^{itX} mean on a curved manifold? In the operator representation of $u(3,2)$, we replaced the factors of i by differential operators. Can we do the same thing here? Since we have several complex structures available, we will attempt to re-do the sections above replacing multiplication by i with a second complex structure. So we begin with two complex structures J_1 and J_2 defined on TM. Separately, $J_1^2 = -1$ and $J_2^2 = -1$ Thus J_1 and J_2 have eigenvalues of $\pm i$, which is what we are trying to avoid. From the two complex structures, we can form $J_1 \pm J_2$ but these operators also have eigenvalues $\pm i$. That leaves us looking at the product $J_1 J_2$ which satisfies:

$$(J_1 J_2)^2 = 1$$

The characteristic equation is

$$(J_1 J_2)^2 - 1 = 0$$

which factors:

$$(J_1 J_2 - 1)(J_1 J_2 + 1) = 0$$

Imitating the construction of the eigenspaces in the complex case, let us define:

$$V_+ = \{J_1 J_2 v + v | v \in TM\}$$

We calculate:

$$J_1 J_2 (J_1 J_2 v + v) = (v + J_1 J_2 v)$$

So V_+ is the eigenspace with eigenvalue positive 1.

$$V_- = \{J_1 J_2 v - v | v \in TM\}$$

Another short calculation:

$$J_1 J_2 (J_1 J_2 v - v) = (v - J_1 J_2 v) = -(J_1 J_2 v - v)$$

shows that V_- is the eigenspace with eigenvalue negative 1.

As in the complex case, we have the projections:

$$P_- = \frac{1}{2}(J_1 J_2 - 1)$$

$$P_+ = \frac{1}{2}(J_1J_2 + 1)$$

Thus it seems that at least some of the results of complex geometry can be recovered without multiplying by i.

The interesting results appear when we apply the above calculations to $QAdS$.

On $TQAdS$, define

$$J_1 v = [\gamma_5, v]$$

and

$$J_2 v = -[\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4, v]$$

The minus sign is necessary to make the signs of the projections work. If J is a complex structure, so is $-J$, interchanging $-J$ and J interchanges P_+ and P_-. A few calculations seem in order to clarify matters. In order to calculate

$$P_- X_{15} = \frac{1}{2}(J_1J_2 - 1) X_{15}$$

We first calculate $J_1J_2X_{15}$

$$[\gamma_5, -[\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4,$$

$$x_1\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial y_1}]]$$

$$= -[\left(x_5\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial x_5}\right), [\left(x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}\right) + \left(x_2\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_2}\right) + \left(x_3\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial x_3}\right)$$

$$+ \left(x_4\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial x_4}\right), x_1\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial y_1}]]$$

$$= -\left(x_1\frac{\partial}{\partial x_5} + x_5\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial y_1}\right)$$

Thus $P_-X_{15} = -X_{15}$, likewise $P_-X_{I5} = -X_{I5}$, $P_-Y_{I5} = -Y_{I5}$ and $P_+X_{I5} = 0$, $P_+Y_{I5} = 0$. Surprisingly, this is where the V_{IJ} and W_{IJ} make their appearance: $P_-V_{I5} = 0$, $P_-W_{I5} = 0$ and $P_+V_{I5} = V_{I5}$, $P_+W_{I5} = W_{I5}$.

Thus, we can characterize $u(3,2)$ as those elements of Z_{ρ^2} which commute with the diagonal operator. Since the diagonal operator is the dilation operator and used in many models as a Hamiltonian. This discussion has shed some light on the mathematical meaning of the V_{IJ} and W_{IJ} but has not lead to any understanding of the physics they might generate.

22 The Schrödinger Equation

If we apply the formalism of complex structures to the Lie algebra $u(3,2)$, we run into a problem with the projection

$$P_+ = \frac{(1 - iJ)}{2}$$

since 1 is not an element of the Lie Algebra. However, suitably normalized, the quadratic Casimir operator of $u(3,2)$ is the identity operator, so we define the projection:

$$P_+ = \frac{(C_G - iJ)}{2}$$

where C_G is the normalized Casimir operator of $G = u(3,2)$. Allowing the operator to act on an object we will call ψ. we obtain:

$$P_+\psi = \frac{(C_G - iJ)}{2}\psi = \psi$$

$$(C_G - iJ)\,\psi = 2\psi$$

Now in order to specialize to the case of $QAdS = U(3,2)/U(3,1) \times U(1)$, we need to decompose the Casimir operator of $U(3,2)$ as a sum. Let H = $U(3,1) \times U(1)$ then $C_G = C_H + \Delta$. With C_H being the Casimir operator of H and Δ being the Laplace-Beltrami operator on $QAdS$. Then our equation becomes:

$$(C_H + \Delta - iJ)\,\psi = 2\psi$$

which has obvious similarities to the Schrödinger equation. The question then becomes which complex structure do we use? Suppose now we take advantage of the two complex structures on the tangent space of $QAdS$ and replace the iJ by $J_1 J_2$ to obtain:

$$(C_G - J_1 J_2)\,\psi = 2\psi$$

To be more specific, we would take $J_1 = [\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, and $J_2 = [\gamma_5$.

23 Dynamics

We have two different diagonal operators and we look at the sum:

$$H = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 + \kappa_5$$

$$= \left(x_1 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_1} \right) + \left(x_2 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial x_2} \right) +$$

$$\left(x_3 \frac{\partial}{\partial y_3} - y_3 \frac{\partial}{\partial x_3} \right) + \left(x_4 \frac{\partial}{\partial y_4} - y_4 \frac{\partial}{\partial x_4} \right) +$$

$$\left(x_5 \frac{\partial}{\partial y_5} - y_5 \frac{\partial}{\partial x_5} \right) + i \left(x_1 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_1} \right) +$$

$$i \left(x_2 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_2} \right) + i \left(x_3 \frac{\partial}{\partial x_3} + y_3 \frac{\partial}{\partial y_3} \right) +$$

$$i \left(x_4 \frac{\partial}{\partial x_4} + y_4 \frac{\partial}{\partial y_4} \right) + i \left(x_5 \frac{\partial}{\partial x_5} + y_5 \frac{\partial}{\partial y_5} \right)$$

Note that on the interchange of x_i and y_i, the κ_i are invariant while the γ_i change sign.

Up to sign and the factor of $\frac{1}{2}$, this is just

$$\sum \bar{Z}_I \frac{\partial}{\partial \bar{Z}_I} =$$

$$= \frac{1}{2} \sum (x_I \frac{\partial}{\partial x_I} + y_I \frac{\partial}{\partial y_I} - i(y_I \frac{\partial}{\partial x_I} - x_I \frac{\partial}{\partial y_I}))$$

This makes sense since a function in one variable is analytic if

$$\frac{\partial}{\partial \bar{Z}_I} f = 0$$

Just as the dilation operator provided the dynamics for Fubini, Hanson and Jackiw [17], this operator should provide the dynamics in the current setting.

24 Compact orbits

The hallmark of the compact generators in $u(3,2)$ is that the square of the operator has a negative eigenvalue. Bracketing with a compact operator oscillates between two values, one has a plus, the other a negative.

As an example, we begin with

$$[X_{12}, X_{15}] = -X_{25} \qquad [X_{12}, X_{25}] = X_{15}$$

Now we go on to compute the higher powers:

$$[X_{12}, [X_{12}, X_{15}]] = [X_{12}, -X_{25}] = -X_{15}$$

$$[X_{12}, [X_{12}, [X_{12}, X_{15}]]] = [X_{12}, [X_{12}, -X_{25}]] = [X_{12}, -X_{15}] = X_{25}$$

$$[X_{12}, [X_{12}, [X_{12}, [X_{12}, X_{15}]]]] = [X_{12}, [X_{12}, [X_{12}, -X_{25}]]]$$
$$= [X_{12}, [X_{12}, -X_{15}]] = [X_{12}, X_{25}] = X_{15}$$

By induction we have:

$$([X_{12},)^{4n} X_{15} = X_{15}$$

$$([X_{12},)^{4n+1} X_{15} = -X_{25}$$
$$([X_{12},)^{4n+2} X_{15} = -X_{15}$$
$$([X_{12},)^{4n+3} X_{15} = X_{25}$$

From which it follows that:

$$exp\theta \, ([X_{12},) \, X_{15} = (\cos\theta) \, X_{15} - (\sin\theta) \, X_{25}$$

Again by induction,

$$([X_{12},)^{4n} X_{25} = X_{25}$$
$$([X_{12},)^{4n+1} X_{25} = X_{15}$$
$$([X_{12},)^{4n+2} X_{25} = -X_{25}$$
$$([X_{12},)^{4n+3} X_{25} = -X_{15}$$

From which it follows that:

$$exp \, (\theta[X_{12},) \, X_{25} = (\cos\theta) \, X_{25} + (\sin\theta) \, X_{15}$$

By combining the above, we obtain:

$$exp\,(\theta[X_{12},)\,(X_{15}+iX_{25}) = exp\,(\theta[X_{12},)\,X_{15} + iexp\,(\theta[X_{12},)\,X_{25}$$

$$= (\cos\theta)\,X_{15} - (\sin\theta)\,X_{25} + i\,(\cos\theta)\,X_{25} + i\,(\sin\theta)\,X_{15}$$

$$= (\cos\theta)\,X_{15} + i\,(\sin\theta)\,X_{15} - (\sin\theta)\,X_{25} + i\,(\cos\theta)\,X_{25}$$

$$= (\cos\theta + i\sin\theta)\,X_{15} + i\,(\cos\theta + i\sin\theta)\,X_{25}$$

$$= e^{i\theta}\,(X_{15}+iX_{25})$$

$$exp\,(\theta[X_{12},)\,(X_{15}-iX_{25}) = exp\,(\theta[X_{12},)\,X_{15} - iexp\,(\theta[X_{12},)\,X_{25}$$

$$= (\cos\theta)\,X_{15} - (\sin\theta)\,X_{25} - i\,(\cos\theta)\,X_{25} - i\,(\sin\theta)\,X_{15}$$

$$= (cos\theta)\,X_{15} - i\,(sin\theta)\,X_{15} - (\sin\theta)\,X_{25} - i\,(\cos\theta)\,X_{25}$$

$$= (\cos\theta - i\sin\theta)\,X_{15} - i\,(\cos\theta - i\sin\theta)\,X_{25}$$

$$= e^{-i\theta}\,(X_{15}-iX_{25})$$

The following commutation relations exhibit the same pattern of one negative and one positive and hence have the same sort of orbits:

$$[X_{12},Y_{15}] = -Y_{25} \qquad [X_{12},Y_{25}] = Y_{15}$$

$$[X_{13},X_{15}] = -X_{35} \qquad [X_{13},X_{35}] = X_{15}$$
$$[X_{13},Y_{15}] = -Y_{35} \qquad [X_{13},Y_{35}] = Y_{15}$$
$$[X_{23},X_{25}] = -X_{35} \qquad [X_{23},X_{35}] = X_{25}$$
$$[X_{23},Y_{25}] = -Y_{35} \qquad [X_{23},Y_{35}] = Y_{25}$$
$$[Y_{12},Y_{15}] = -X_{25} \qquad [Y_{12},X_{25}] = Y_{15}$$
$$[Y_{12},Y_{25}] = -X_{15} \qquad [Y_{12},X_{15}] = Y_{25}$$
$$[Y_{13},Y_{15}] = -X_{35} \qquad [Y_{13},X_{35}] = Y_{15}$$
$$[Y_{13},Y_{35}] = -X_{15} \qquad [Y_{13},X_{15}] = Y_{35}$$
$$[Y_{23},Y_{25}] = -X_{35} \qquad [Y_{23},X_{35}] = Y_{25}$$
$$[Y_{23},Y_{35}] = -X_{25} \qquad [Y_{23},X_{25}] = Y_{35}$$
$$[\gamma_1,Y_{15}] = -X_{15} \qquad [\gamma_1,X_{15}] = Y_{15}$$

$$[\gamma_2, Y_{25}] = -X_{25} \qquad [\gamma_2, X_{25}] = Y_{25}$$
$$[\gamma_3, Y_{35}] = -X_{35} \qquad [\gamma_3, X_{35}] = Y_{35}$$
$$[\gamma_4, Y_{45}] = -X_{45} \qquad [\gamma_4, X_{45}] = Y_{45}$$
$$[\gamma_5, X_{15}] = -Y_{15} \qquad [\gamma_5, Y_{15}] = X_{15}$$
$$[\gamma_5, X_{25}] = -Y_{25} \qquad [\gamma_5, Y_{25}] = X_{25}$$
$$[\gamma_5, X_{35}] = -Y_{35} \qquad [\gamma_5, Y_{35}] = X_{35}$$
$$[\gamma_5, X_{45}] = -Y_{45} \qquad [\gamma_5, Y_{45}] = X_{45}$$

25 Noncompact generators

The hallmark of a noncompact operator in $u(3,2)$ is that the square of the operator has a positive eigenvalue. Thus, the signs in front of the two operators must agree. When the generator is noncompact, then we obtain a different set of commutators and a different sort of orbit:

$$[X_{14}, X_{15}] = X_{45} \qquad [X_{14}, X_{45}] = X_{15}$$

The pattern is both operators on the right have positive signs.

$$[X_{14}, [X_{14}, X_{15}]] = [X_{14}, X_{45}] = X_{15}$$

$$[X_{14}, [X_{14}, [X_{14}, X_{15}]]] = [X_{14}, [X_{14}, X_{45}]] = [X_{14}, X_{15}] = X_{45}$$

$$[X_{14}, [X_{14}, [X_{14}, [X_{14}, X_{15}]]]] = [X_{14}, [X_{14}, [X_{14}, X_{45}]]]$$
$$= [X_{14}, [X_{14}, X_{15}]] = [X_{14}, X_{45}] = X_{15}$$

By induction

$$([X_{14},)^{2n} X_{15} = X_{15}$$
$$([X_{14},)^{2n+1} X_{15} = X_{45}$$

From which it follows that:

$$exp\,(t[X_{14},)\, X_{15} = (\cosh t)\, X_{15} + (\sinh t)\, X_{45}$$

Likewise,

$$exp\,(t[X_{14},)\, X_{45} = (\cosh t)\, X_{45} + (\sinh t)\, X_{15}$$

Summing:

$$exp\,(t[X_{14},\,)\,(X_{15}+X_{45}) = (\cosh t)\,X_{15}+(\sinh t)\,X_{45}+(\cosh t)\,X_{45}+(\sinh t)\,X_{15}$$

$$= (\cosh t)\,X_{15} + (\sinh t)\,X_{15} + (\sinh t)\,X_{45} + (\cosh t)\,X_{45}$$
$$= (\cosh t + \sinh t)\,X_{15} + (\sinh t + \cosh t)\,X_{45}$$
$$= e^t\,(X_{15} + X_{45})$$

The same pattern is repeated with the two positives in the following commutation relations which then lead to the same type of orbit:

$$[X_{14}, Y_{15}] = Y_{45} \qquad [X_{14}, Y_{45}] = Y_{15}$$

$$[X_{24}, X_{25}] = X_{45} \qquad [X_{24}, X_{45}] = X_{25}$$
$$[X_{24}, Y_{25}] = Y_{45} \qquad [X_{24}, Y_{45}] = Y_{25}$$
$$[X_{34}, Y_{35}] = Y_{45} \qquad [X_{34}, Y_{45}] = Y_{35}$$
$$[Y_{24}, Y_{25}] = X_{45} \qquad [Y_{24}, X_{45}] = Y_{25}$$
$$[Y_{34}, Y_{35}] = X_{45} \qquad [Y_{34}, X_{45}] = Y_{35}$$
$$[Y_{14}, Y_{15}] = X_{45} \qquad [Y_{14}, X_{45}] = Y_{15}$$

The final pattern has both operators on the right with negative signs:

$$[Y_{14}, Y_{45}] = -X_{15} \qquad [Y_{14}, X_{15}] = -Y_{45}$$

Which yields a different orbit structure:

$$[Y_{14}, [Y_{14}, Y_{45}]] = [Y_{14}, -X_{15}] = Y_{45}$$

$$[Y_{14}, [Y_{14}, [Y_{14}, Y_{45}]]] = [Y_{14}, [Y_{14}, -X_{15}]] = [Y_{14}, Y_{45}] = -X_{15}$$
$$[Y_{14}, [Y_{14}, [Y_{14}, [Y_{14}, Y_{45}]]]] = [Y_{14}, [Y_{14}, [Y_{14}, -X_{15}]]]$$
$$= [Y_{14}, [Y_{14}, Y_{45}]] = [Y_{14}, -X_{15}] = Y_{45}$$

By induction

$$([Y_{14},\,)^{2n}\,Y_{45} = Y_{45}$$
$$([Y_{14},\,)^{2n+1}\,Y_{45} = -X_{15}$$

From which it follows that:

$$exp\,(t[Y_{14},)\,X_{15} = (\cosh t)\,X_{15} - (\sinh t)\,Y_{45}$$

Likewise,

$$exp\,(t[Y_{14},)\,Y_{45} = (\cosh t)\,Y_{45} - (\sinh t)\,X_{15}$$

Summing:

$$exp\,(t[Y_{14},)\,(X_{15} + Y_{45}) = (\cosh t)\,X_{15} - (sinht)\,Y_{45} + (\cosh t)\,Y_{45} - (\sinh t)\,X_{15}$$

$$= (\cosh t - \sinh t)\,X_{15} + (\cosh t - \sinh t)\,Y_{45}$$

$$= e^{-t}\,(X_{15} + Y_{45})$$

The following commutation relations exhibit the same behavior and hence have the same type of orbits:

$$[Y_{24}, Y_{45}] = -X_{25} \qquad [Y_{24}, X_{25}] = -Y_{45}$$

$$[Y_{34}, Y_{45}] = -X_{35} \qquad [Y_{34}, X_{35}] = -Y_{45}$$

26 The Wave Functions

Let us consider the problem of an electron interaction with a proton and include the wave functions. The electron is modeled as

$$\psi_{14}(X_{14} - iY_{14})$$

and the proton as

$$\psi_{34}(X_{34} + iY_{34})$$

The system of proton interacting with electron is given by the bracket:

$$\begin{aligned} &[\psi_{14}(X_{14} - iY_{14}), \psi_{34}(X_{34} + iY_{34})] = \\ &\psi_{14}\psi_{34}[(X_{14} - iY_{14}), X_{34} + iY_{34}] + \\ &[\psi_{14}((X_{14} - iY_{14})\psi_{34})X_{34} + iY_{34}] - \\ &\psi_{34}(X_{34} + iY_{34}\psi_{14})(X_{14} - iY_{14}) \end{aligned} \tag{16}$$

In the expansion of (16), the first term represents a hydrogen atom.

The second term represents a proton moving in the potential of an electron.

The third term represents an electron moving in the potential of a proton.

Comparing (16) with the solution to the Dirac equation found earlier (We showed that if A is an eigenfunction of H with eigenvalue e and E is an eigenfunction of H with eigenvalue m, then: $\psi = Eexp(A)$ is a solution to the equation $H\psi = (eA + m)\psi$, we see that if $\psi = Eexp(A)$ is the "electron moving in the potential of a proton" and so is

$$\psi_{34}(X_{34} + iY_{34})\psi_{14}(X_{14} - iY_{14},)$$

then we must identify $exp(A)$ as the wave function of the proton and $\psi_{34}(X_{34} + iY_{34})\psi_{14}$ must be E. As expected, the potential A due to the proton is related to the wavefunction ψ_{34} of the proton and A is identified as an eigenfunction of the dynamical operator H, thus ψ_{34} is a generalized eigenfunction of H. Not surprisingly, we are also forced to take ψ_{14}, the wavefunction of the electron as an eigenfunction of H. If we were to analyze the second term in the expansion instead, a parallel analysis shows that the wavefunction of the electron is the exponential of the classical potential of the electron. When all the details are put in we must exponentiate some constant times the classical potential to have a dimensionless quantity in the exponential. If the wave function, i.e., the field of an elementary particle is the exponential of the classical potential then we see why we cannot add wave functions, instead we want to add potentials and hence we need to multiply wave functions (or take the Lie bracket).

This has parallels in standard quantum theory, where as Wheeler [55] points out *"the exponent in the quantum mechanical propagator is* $(i/\hbar)$ *times the classical action."*

It seems from this analysis, that 'wave mechanics' was only an approximation—considering a classical point particle moving in the potential due to another. Now we must consider the problem of the 'potential of an electron moving in the potential of a proton'. The need for such a model was suggested by Sachs [48]:

> How can one accept the dualism of both the continuous field concept—to describe a part of the actual physical system called "influencer" –and the atomistic concept– to describe the rest of the system– the 'test body' called "influenced?". This division seems to me to be logically dichotomous.

Aharonov and Bohm [3] proposed

> . . . that, in quantum mechanics, the fundamental physical entities are the potentials, while the fields are derived from them by differentiation.

This has been experimentally confirmed, so the use of the potential as fundamental is natural.

27 On the use of Symmetries in Physics

The theory of matter introduced here uses symmetries differently than other theories and it seems to be a useful exercise to discuss those differences. In this discussion, I will follow the article "About Symmetries in Physics" by Gieres [18]. I chose this article simply because I had the idea of doing this comparison while reading it.

Let us begin with Gieres' definition:

> For the *classification of symmetries*, one distinguishes between those which operate on space-time coordinates, the so-called *geometric symmetries* and those which do not affect them, the *internal symmetries*.

These statements are not true in the present model. Since we view the space-time coordinates as a background field and the 'internal symmetries' as representing the particles, we have to allow the 'internal symmetries' to interact with the space-time coordinates so that the particles can interact with the background field. Here, all symmetries are space-time symmetries.

Gieres goes on to discuss the relation between symmetries and conservation laws as reflected in Noether's theorem:

> Covariance of the equations of motion with respect to a continuous transformation with n parameters implies the existence of n conserved quantities ('charges' or 'integrals of motion'), i.e. it implies conservation laws.

Gieres fails to mention that Noether's theorem holds only in Lagrangian theories. In the present theory, we do not use a Lagrangian and we obtain even more conserved quantities using the Casimir operators of $u(3,2)$.

Wigner classifed elementary particles in terms of their mass and spin, based on representations of the Poincaré group. We have replaced the Poincaré group with $u(3,2)$ and classified elementary particles in terms of their four quantum numbers.

Next Gieres discusses the super Poincaré algebra which extends the Lie algebra of the Poincaré group to include supersymmetries. We have no need for supersymmetries since the internal structure of $u(3,2)$ automatically carries a grading which distinguishes between bosons and fermions. Thus we have no need for anti-commutators.

> The supersymmetry generators are 'square roots' of translation generators.

If we take the Lie bracket of two space-time generators, we obtain an internal symmetry. For example:

$$[X_{15}, X_{45}] = -X_{14}$$

To return to the space-time, we must bracket again: Continuing with the same example:

$$[X_{15}, [X_{15}, X_{45}]] = -[X_{15}, X_{14}] = X_{45}$$

To remain in the base, perhaps we should take all the standard second order equations of physics and turn them into fourth order!

In gauge theories, such as Yang-Mills, the 'ordinary derivative' ∂_μ is replaced by the covariant derivative $\partial_\mu + A_\mu$ in our case the Lie derivative with respect to X_{ij} is replaced by the Lie derivative with respect to $X_{ij} + iY_{ij}$ which is very similar in form to the covariant derivative.

28 The Meaning of Quantum Gravity

In the previous section we discussed how the present theory differs from the standard approach in the use of symmetries. Here we will look at some other approaches to quantum gravity and discuss the differences and similarities. In the first part of this discussion, we will follow the recent book by Rovelli [47].

> We have learned from GR that spacetime is dynamical and we have learned from QM that any dynamical entity is made up of quanta and can be in probabilistic superposition states. [47](page 4)

For reasons previously discussed I strongly disagree with the concept of 'probabilistic superposition states.' But moreover, all the dynamics has been done in the tangent space (The Lie algebra) the space-time itself is not dynamical.

> The fact is that we do have plenty of information about quantum gravity, because we have QM and we have GR. Consistency with QM and GR is an extremly strict constraint. [47](page 5)

Despite Rovelli's affirmation of QM and GR, we have shown that both had to be modified to obtain our unification. It is impossible for any theory to be consistent with both QM and GR since QM and GR are inconsistent.

While there are many differences, we will discuss only a few. Rovelli correctly points out that:

> GR is the discovery that spacetime and the gravitational field are the same entity. What we call "spacetime" is itself a physical object, in many respects similar to the electromagnetic field. We can say that GR is the discovery that there is no spacetime at all. What Newton called "space," and Minkowski called "spacetime," is unmasked: it is nothing but a dynamical object- the gravitational field -in a regime in which we neglect its dynamics.... the Universe is not made up of fields on spacetime; it is made up of fields on fields. [47](page 9)

In the model presented here, the gravitational field is a vector field on the space-time, it is not the same entity as space-time. The problem is with Einstein's use of the metric as fundamental. Within GR, it seems that all vectors have unit length and that if we take

$$g_{ij}(X, Y)$$

and consider changing the length of the vectors to fX and hY:

$$g_{ij}(fX, hY)$$

then GR would interpret this as a new metric:

$$g'_{ij}(X, Y) = g_{ij} fh(X, Y)$$

Another point of strong disagreement lies with Rovelli's basic idea:

> *No unification.* Nowadays, a fashionable idea is that the problem of quantizing gravity has to be solved together with the problem of finding a unified description of all interactions. LQG is a solution of the first problem, not the second. [47](page 13)

The problem is this: There are many problems a complete theory must solve, how can we know that we have satisfactorily solved one problem until we have solved them all? Will the solution to a solitary question actually fit into a unified picture? Like the GUTS program which ignored gravitation could not possibly lead to a totally unified theory in which the particles arise from the gravitational field, Rovelli's LQG program cannot be made consistent with a unified theory of all interactions.

Rovelli claims:

> The fact that the notions of energy and vacuum are so ambiguous in GR should not be disconcerting. There is nothing essential in these notions: a quantum theory and its predictions are meaningful also in the absence of them. The notions of energy and vacuum play an important role in non-general-relativistic physics just because of the accidental fact that we live in a region of the Universe which happens to have a peculiar symmetry: translation invariance in newtonian or special-relativistic time. [47](page 204)

The fact that the conservation of energy, momentum and angular momentum are not well defined in GR is a *very* disconcerting notion. Rovelli's comments about the specialness of our region of the universe make no sense whatsoever since all regions of spacetime must be the same and all laws of physics must be valid everywhere in the universe.

Having exhausted interest in Rovelli's work, let us turn to Isham. Isham [28] asked several "Prima Facie Questions in Quantum Gravity" and it behooves us to see how the present theory answers his questions.

Isham notes that:

> The deep incompatibilities between the basic structures of general relativity and of quantum theory have lead many people to feel that the construction of a consistent theory of quantum gravity requires a profound revision of the most fundamental ideas of modern physics. The hope of securing such a paradigm shift has always been a major reason for studying the subject.

By questioning the foundations of both quantum theory and the general theory of relativity, we have created what Isham calls 'an iconoclastic theory.'

Isham asks "How much Spacetime Structure must be Fixed?" and we saw that the underlying homogeneous space $U(3,2)/U(3,1) \times U(1)$ must be fixed in order for the conservation laws to hold. What varies are the vector fields on $U(3,2)$. Isham questions "The Role of the Spacetime Diffeomorphism Group Diff(M)" and we saw that the vector fields on $U(3,2)$ generate the Diffeomorphism Group of $U(3,2)$, not just spacetime.

Isham notes that the 'problem of time' arises because "Time is not a physical observable in the normal sense since it not represented by an operator", but in the geometry of our theory, time is an operator.

29 Conclusions

> After Hertz, in the '80s of the last century, had confirmed the existence of the electro-magnetic waves and displayed their identity with light by means of his wonderful experiments, the great intellectual revolution in physics gradually became complete. People slowly accustomed themselves to the idea that the physical states of space itself were the final physical reality, especially after Lorentz had shown in his penetrating theoretical researches that even inside ponderable bodies the electro-magnetic fields are not to be regarded as states of the matter, but essentially as states of the empty space in which the material atoms are to be considered as loosely distributed.
> —Albert Einstein [11]

Each element in the spinor representation of $U(3,2)$ contains a generator from a copy of $so(3,2)$ involving the x_i and another generator from another copy of $so(3,2)$ involving the y_i (yielding two copies of anti-de Sitter space-time) plus two generators which involve both x_i and y_j and which generate a spiral flow between the two copies of space-time giving us a two space-time structure much like that introduced by Einstein and Rosen [13]:

> As u varies from $-\infty$ to $+\infty$, r varies from $+\infty$ to 2m and then again from 2m to $+\infty$. If one tries to interpret the regular solution (5a) in the space of r, θ, ϕ, t, one arrives at the following conclusion. The four-dimensional space is described mathematically by two congruent parts or "sheets," corresponding to $u > 0$ and $u < 0$, which are joined by a hyperplane $r = 2m$ or $u = 0$ in which g vanishes. We call such a connection between the two sheets a "bridge."

It seems that the present model is requiring us to embrace these "Einstein-Rosen bridges" in a very dramatic way. There are two copies of space-time, one in terms of the x_I and one in terms of the y_I. The fields are excited states of space-time and the different elementary particles form different types of bridges between these excited states.

These bridges have also been studied under the name of worm-holes. However, there is a difference in the present picture, the bridge connects two layers of space-time while "The wormhole or handle is envisaged as connecting two very different regions in the same space." [44] In the last few years the idea of "two sheets" has been revived with the new name of colliding "branes".

Einstein and Rosen glued together two copies of the Schwarzschild singularity to obtain a model of an elementary particle. If that is the true meaning of the Schwarzschild solution, then the meaning of other solutions is called into question. Is the model of an expanding universe really modeling the expansion of the gravitational field of an elementary particle? Is the initial singularity actually the creation of an elementary particle and not the creation of the universe? From the work done here, it seems that the answer to both questions is yes. An elementary particle is really light trapped in a vibratory state of spacetime, the light cannot get out, in other words, a particle is a black hole.

The picture presented here is incomplete. In a more complete work, each of the spinors representing an elementary particle will be multiplied by a suitable function, related somehow to the spinor derivative representing the particle family. We have yet to obtain any equations for these wave functions, other than to suggest that they be eigenfunctions of the Casimir operators of $u(3,2)$. Unfortunately, the standard theory of Casimir operators is wrong, as it predicts neither the correct operators nor the correct number of Casimir operators [36]. While some progress has been made, we are a long way from knowing all the generalized Casimir operators of $u(3,2)$.

The Copenhagen interpretation of the standard quantum mechanics requires point particles. So according to the Copenhagen intepretation, the elementary particles have no internal structure. The model of matter being developed here requires that particles be extended objects and that their wave functions be related to the dynamical flow of the complex space-time. But even this cursory treatment allows us to obtain a view of what the final picture will look like.

The internal workings of an elementary particle form two harmonic oscillators. The external fields act as forcing agents on the harmonic oscillators, thus we have two anharmonic oscillators. In turn, the internal oscillations of the elementary particle generate the field of the particle. Thus the electron has a complicated internal structure in spite of what many physicists once believed. Dirac's motivation was simplicity and beauty. But he still made mistakes in judgement. In a 1938 [7] paper he wrote:

> ... the electron is too simple a thing for the question of the laws governing its structure to arise, and thus, quantum mechanics should not be needed for the solution of the difficulty...

In way he was right, because something beyond quantum mechanics was required. In the present model, even the lowly neutrino has an internal structure. The internal strucure is necessary in order to explain how the elementary particle interacts with the fields of other elementary particles and how th elementary particle generates its own fields.

Many of the ideas incorporated into the present model have been examined before.

Vigier [53] suggested that there were internal motions of the elementary particles which could be described in terms of Lie Algebras.

In "The transactional interpretation of quantum mechanics", Cramer [5] suggested that the interaction of particles is accomplished by means of a 'space-time standing wave'. A particle sends out an "offer" wave, the other particle receives the offer wave and responds with an echo or "confirmation wave". This idea fits well with the present model.

There is a similiarity of philosophy (though not of detail) with the "Rotator Model of Elementary Particles Considered as Relativistic Extended Structures in Minkowski Space" of de Broglie, Bohm, Hillion, Halbwachs, Takabayasi and Vigier [4]. Indeed, the present model could be considered as the "Rotator Model of Elementary Particles Considered as Relativistic Extended Structures in Anti-de Sitter Space"

Albert Einstein and Nathan Rosen [13] discussed "The Particle Problem in the General Theory of Relativity":

> A complete field theory knows only fields and not the concepts of particle and motion. For these must not exist independently of the field but are to be treated as part of it. On the basis of the description of a particle without singularity one has the possibility of a logically more satisfactory treatment of the combined problem: The problem of the field and that of motion coincide.

We seem to be stuck in a semantic trap, what we call elementary particles are actually patterns of energy flows in what we normally call the field of the particle. This has been noted before.

Ohanian [41] shows that in 1937, Belinfante:

> ...established that the spin could be regarded as due to a circulating flow of energy, or a momentum density, in the electron wave field. He established that this picture of the spin is valid not only for electrons, but also for photons, vector mesons, and gravitons–in all cases the spin angular momentum is due to a circulating energy flow in the fields. Thus contrary to the common prejudice, the spin of the electron has a close classical analog: It is an angular momentum of exactly the same kind as carried by the fields of a circularly polarized electromagnetic wave.

In the present model, not only the spin, but also the electric charge, the baryon number, the meson number and the lepton number are "due to a circulating flow of energy". It seems reasonable to expect that the curvature of space-time is due to the rotation of these circulating flows of energy. These circulating flows of energy replace the idea of quarks. Quarks have not been isolated simply because they are standing wave patterns within the field of the particles, they cannot be isolated. In the present model, all elementary particles have the same type of substructure, not just the baryons and mesons of quark theory.

Given the relations between "Spinors, Minimal Surfaces, Torsion, Helicity, Chirality, Spin, Twistors, Orientation, Continuity, Fractals, Point Particles, Polarization, the Light Cone and the Hopf Map" as explored by R. M. Kiehn [29] it would seem that the spinors in our model generate a minimal hypersurface which connects the two regions of complex space-time. These are the Einstein-Rosen bridges.

James Clerk Maxwell wrote a series of papers "On Physical Lines of Force", several of which dealt with "The Theory of Molecular Vortices" [38]. The substructure of the internal symmetry groups has more than a passing semblence to these vortices and the vortex theory of atoms, popular in the 19^{th} century.

The point at which we have arrived seems to be less in agreement with standard quantum theory and more in accord with the views Schrödinger expressed in a letter to Einstein dated 19 July, 1939:

> Dear Einstein,
>
> A few months ago, a Dutch newspaper carried a report which sounded comparatively intelligent that you have discovered something important about the connection between gravitation and

> matter waves. I would be terribly interested in that because I have really believed for a long time that the Ψ waves are to be identified with waves representing disturbances of the gravitational potential; not of course with those you studied first, but rather with ones that transport real mass, i.e. a non-vanishing T_{ij}. That is, I believe that one has to introduce matter into the general theory of relativity, which contains the T_{ij} only as "asylum ignorantiae" (to use your expression), not as mass points or something like that, but rather, shall we say, as quantized gravitational waves.[14] , p. 33)

We also seem to be in accord with what Einstein [10] wrote about his vision:

> Since according to our present conceptions the elementary particles of matter are also, in their essence, nothing else than condensations of the electromagnetic field, our present view of the universe presents two realities which are completely separated from each other conceptually, although connected causally, namely, gravitational ether and electromagnetic field, or—as they might also be called—space and matter.
>
> Of course it would be a great advance if we could succeed in comprehending the gravitational field and the electromagnetic field together as one unified conformation. Then for the first time the epoch of theoretical physics founded by Faraday and Maxwell would reach a satisfactory conclusion. The contrast between ether and matter would fade away, and, through the general theory of relativity, the whole of physics would become a complete system of thought, like geometry, kinematics, and the theory of gravitation.

The model introduced here can be considered as a theory of elementary particles, a unified field theory or a quantum theory of gravity. While it explains several things, there are many questions left to answer about the model which will be addressed in subsequent papers. We have yet to examine the role of the V_{IJ}, and W_{IJ} in the physics. Are they allowed symmetries of the complex spacetime and thus carry new conserved quantities or do they represent a new family of elementary particles? This is really a question of whether the complex structure is fundamental. If the underlying manifold must be holomorphic, these vector fields will play no role. If the underlying manifold is real, they will be important. Which equations of standard physics do we expect to obtain? We are looking for the equations of individual particles and so we must question which of the equations of physics are valid for individuals and which are statistical? Do we look for the equations of the electric field γ_4 or of the electron (e^-, e^+) field and the proton ($p+$,p^-) field? Or both? We began by questioning Dirac's derivation of equations on Anti-de Sitter space, yet we haven't introduced the corrected versions of those equations. The present version of the model deals with two interacting particles with a common center. What happens when one is slightly offset? Further progress will require at least: (1) The tetrad formalism to obtain gravitation; (2) The coherent state formalism to obtain wave equations and (3) The eigenfunction representations of the Lie algebra $u(3,2)$ and (4) harmonic analysis on $QAdS$.

When I originally looked at the function

$$\rho^2 = z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 - z_4\bar{z}_4 - z_5\bar{z}_5,$$

I was imitating the construction of anti-de Sitter space, and thinking of $\rho = R$ as a model of the complex space-time $QAdS$. But then I encountered the following:

> The corresponding energy is easily seen to be $[(i) above]$
>
> $$H(x,y) = \frac{1}{2}\sum_{i=1}^{n} \alpha_i \left(x_i^2 + y_i^2\right)$$
>
> that is, X is the sum of n noninteracting harmonic oscillators.[1]

This leaves us in a quandry, is ρ^2 the energy of five harmonic oscillators or the metric of a complex space-time or does a level surface of ρ provide a model for a complex space-time? Are the above descriptions compatible? Could all three be true? Perhaps we should conclude that what we perceive to be space-time is the level hypersurface of the energy of a set of harmonic oscillators. Since the forces between the particles are not equal and the masses are far from being equal, it seems that we must give up the idea that a level surface of the function ρ is a representation of $QAdS$ as a complex space-time. How is it that the study of the Hamiltonian of 5 noninteracting harmonic oscillators led to the construction of 16 interacting harmonic oscillators? But confusion is good. Progress is made only by questioning the currently accepted theory to the point of becoming confused and then clarifying the issues which caused the confusion. To add to the confusion, if we paramentize ρ:

$$\rho^2 = \omega_1 z_1 \bar{z}_1 + \omega_2 z_2 \bar{z}_2 + \omega_3 z_3 \bar{z}_3 - \omega_4 z_4 \bar{z}_4 - \omega_5 z_5 \bar{z}_5,$$

we obtain something close to the moment of inertia. This interpretation seems to hold the most promise, for then inertial mass is just the moment of inertia in a higher dimensional space.

Does the process of constructing the Lie algebra Z_f work as a general procedure for quantizing Hamiltonian systems by finding the Casimir operators of Z_H?

Sophus Lie did all of his work on what we now call Lie algebras in terms of differential operators [22]. It is a sad state of affairs now that one can read many works on Lie algebras and Lie groups treated solely in terms of matrices and never encounter the idea of representing a Lie algebra in terms of differential operators. The results of this article, using differential operators, show that matrix representations are inadequate for modeling elementary particles since matrices do not have eigenfunctions. However, eigenfunction representations will inevitably play a major role in the identification of the wave functions.

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