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The kinetic theory of electromagnetic radiation

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Abstract It is shown that Planck's energy distribution for a black-body radiation field can be simply derived for a gas-like ether with Maxwellian statistics. The gas consists of an infinite variety of particles, whose masses are integral multiples n of the mass of the unit particle, the abundance of n-particles being proportional to n^{-4}. The frequency of electromagnetic waves correlates with the energy per unit mass of the particles, not with their energy, thus differing from Planck's quantum hypothesis. Identifying the special wave-speed, usually called the speed of light, with the wave-speed in the 2.7^oK background radiation field, leads to a mass $\frac{1}{2} \times 10^{-39}(kg)$ for the unit ether-particle, and an average number of about 360 ether particles per cubic centimetre in the background radiation field, whose density is about $0.2 \times 10^{-30}(kg)/m^3$.

'There fields of light and liquid ether flow' (Dryden).

1 Introduction

The question, whether or not there is a physical ethereal medium in which electromagnetic waves propagate, has been asked for many centuries. On the one hand, there have always been those who have maintained that it is not a sensible question to ask, since radiation is observed to have many physical properties and cannot, therefore, exist in a true vacuum or void which is, by definition, the total absence of anything physical. On the other hand, for about the last hundred years, it has come to be largely accepted that there is no physical ethereal medium, and the physical properties of radiation have been transmogrified into waves, and energy parcels or photons, in a space-time metric.

The arguments for the denial of a physical ethereal medium are manifold (see, for example, Whittaker[1]). One of these asserts that Maxwell's equations show that electromagnetic waves are transverse and that, therefore, any ethereal medium must behave like an elastic solid. This argument is invalid, since Maxwell's equations only show that the oscillating electric and magnetic fields are transverse to the direction of wave propagation, and can say nothing whatsoever about any condensational oscillations of any possible physical medium in which the waves are propagating. In fact, the deduction, from Maxwell's equations, that electromagnetic waves are entirely transverse, is no more than a restatement of an assumption that there is no physical ethereal medium. On the contrary, if there is such a medium, one would deduce from Maxwell's equations, since electric field, magnetic field and motion are mutually perpendicular for plane waves, that its condensational oscillations are longitudinal, in exact analogy with sound waves in a fluid.

Another argument against the existence of a physical ethereal medium is that Planck's empirical formula, for the energy distribution in a black-body radiation field, cannot be derived from the kinetic theory of a gas with Maxwellian statistics. Indeed, it is well-known that kinetic theory and Maxwellian statistics lead to an energy distribution which is a sum of Wien-type distributions, for a gas mixture with any number of different kinds of atoms or molecules. But this only establishes the impossibility of so deriving Planck's distribution for a gas with a *finite* variety of atoms or molecules. To assert the complete impossibility of so deriving Planck's distribution it is essential to eliminate the case of a gas with an infinite variety of atoms or molecules, i.e. infinite in a mathematical sense, but physically, in practice, a very large variety. The burden of the present paper is to show that this possibility cannot be eliminated, but rather that it permits a far simpler derivation of Planck's energy distribution than has been given anywhere heretofore.

2 Ethereal thermodynamics

If the ethereal medium is particulate like a gas, the black-body state may be taken to correspond with thermodynamic equilibrium in a gas. Observation (or, perhaps, lack of observation) then indicates that the size and mass of ether particles must be at least orders of magnitude lower than the size and mass of even the fundamental particles, and suggests, therefore, that such a medium may be expected to behave like an ideal gas.

A well-known property of black-body radiation gives, for the energy density

$$\frac{E}{v} = A_0 T^4 \tag{2.1}$$

where A_0 is called the radiation density 'constant'. (Here, the notation used is appropriate to fluid thermodynamics, namely pressure p, specific volume v, specific entropy S, temperature T, intrinsic energy per unit mass E; so that energy density, i.e. intrinsic energy per unit volume, becomes E/v.)

The thermodynamic properties of an ideal gas are completely specified by the expression for E as a function of v and S, namely

$$E = \frac{K\bar{c}_v}{v^{\omega-1}} \exp\left(\frac{S}{\bar{c}_v}\right) \tag{2.2}$$

where $K, \bar{c}_v$, and ω are constants. (The E, v, S system of thermodynamics is used here, in which E is the primary dependent variable, v and S the independent variables. Partial differentiation is confined to v and S, so that suffices v and S may be used to denote partial derivatives.) (See, for example, Thornhill[2] or Swan and Thornhill[3].)

The thermodynamic identities $\gamma \equiv vE_{vv}/E_v$, and $c_v \equiv E_S/E_{SS}$ show that ω is the constant first adiabatic index, and $\bar{c}_v$ is the constant specific heat at constant volume. The temperature T is given by

$$T \equiv E_S = \frac{K}{v^{\omega-1}} \exp\left(\frac{S}{\overline{c}_v}\right) \tag{2.3}$$

The relations (2.2) and (2.3) yield very simply,

$$\frac{E}{v} = \overline{c}_v \left[K \exp\left(\frac{S}{\overline{c}_v}\right)\right]^{\frac{-1}{(\omega-1)}} T^{\frac{\omega}{(\omega-1)}} \tag{2.4}$$

It is now clear that Eqs (2.1) and (2.4) are equivalent if the 'constancy' of A_0 is attributed to a constancy of entropy in the black-body radiation fields to which Eq.(2.1) is being applied. And, if this is done, it then follows that the first adiabatic index ω of the ether, and the number of degrees of freedom α of ether particles, are given by

$$\frac{\omega}{\omega-1} = 4, \quad \text{whence} \quad \omega = \frac{4}{3}$$

and

$$\omega = \frac{(\alpha+2)}{\alpha}, \quad \text{whence} \quad \alpha = 6$$

Thus, the quest for a gas-like ethereal medium, satisfying Planck's form for the energy distribution, is directed to an ideal gas formed by an infinite variety of particles, all having six degrees of freedom.

3 Kinetic theory

The simplest and most obvious approach to the problem under consideration is first to choose an infinite variety of ether particles, and then to try to determine a mixture of them which will yield Planck's energy distribution. A clue to the choice of infinite variety comes directly from observation of the photoelectric effect, for this indicates that, in interactions between matter and radiation, energy exchanges occur, at any frequency, ν, in integral multiples of some minimum quantity, $h_0\nu$. This suggests the choice of a single infinity of ether particles, whose masses are integral multiples of some minimum mass m, but which, for the purposes of kinetic theory, at least, may be considered as otherwise identical.

Consider then, a gas, occupying a volume V, whose particles all have three degrees of freedom of translational motion, and three other degrees of freedom; and whose particles are all identical except for (size? and) mass, their masses being integral multiples $n, (n = 1, ..., \infty)$ of an *absolute* mass quantum m, the mass of a unit particle.

Let ϵ denote energy per unit mass, $N_n(\epsilon)$ the number of n-particles in the range $(0, \epsilon), \underline{N}_n$ the total number of n-particles, and c_n^2 the mean value of the square of the translational speed of n-particles. Then, for Maxwellian statistics (see Appendix),

$$\frac{\partial N_n(\epsilon)}{\partial \epsilon} = \frac{27\underline{N}_n}{2c_n^6} \epsilon^2 \exp\left(\frac{-3\epsilon}{c_n^2}\right) \tag{3.1}$$

whilst, for equi-partition of energy, i.e. all six degrees of freedom of all particles, whatever their masses, have the same mean energy, mnc_n^2 must be the same for all n, so that temperature T may be defined by

$$\frac{mnc_n^2}{3} = kT \tag{3.2}$$

where k is a universal constant.

If, now $\mathbb{E}(\epsilon, T)$ denotes the total energy of all particles in the range $(0, \epsilon)$, then

$$\frac{\partial \mathbb{E}(\epsilon, T)}{\partial \epsilon} = \sum_{n=1}^{\infty} mn\epsilon \frac{\partial N_n(\epsilon)}{\partial \epsilon} = \sum_{n=1}^{\infty} \left(\frac{\underline{N}_n m^4 n^4}{2k^3 T^3} \right) \epsilon^3 \exp\left(\frac{-mn\epsilon}{kT} \right) \tag{3.3}$$

A mixture of the particles must now be specified, i.e. an abundance function $\underline{N}_n$ must be specified, and then the summation in Eq. (3.3) can be performed and the resulting expression for $\partial \mathbb{E}/\partial \epsilon$ compared with Planck's energy distribution. But it is, by now, already obvious that the simplest choice for $\underline{N}_n$, namely $\underline{N}_n \propto 1/n^4$, will succeed.

For, if $\underline{N}_n = \delta/n^4$, then the total number of all the particles, $\underline{N}$, is given by

$$\underline{N} = \sum_{n=1}^{\infty} \frac{\delta}{n^4} = \frac{\pi^4 \delta}{90} \tag{3.4}$$

and the total mass of the gas, denoted by $\underline{\underline{N}}m$, is given by

$$\underline{\underline{N}}m = \sum_{n=1}^{\infty} \frac{m\delta}{n^3} = m\delta\zeta(3) \tag{3.5}$$

where ζ denotes Riemann's zeta-function. Thus,

$$\delta = \frac{\underline{\underline{N}}}{\zeta(3)} \tag{3.6}$$

and the mean mass $\overline{m}$ of all the particles satisfies

$$\overline{m} = \frac{\underline{\underline{N}}m}{\underline{N}} = \frac{90\zeta(3)m}{\pi^4} \tag{3.7}$$

Substituting these values in Eq. (3.3) then leads to

$$\frac{\partial \mathbb{E}(\epsilon, T)}{\partial \epsilon} = \sum_{n=1}^{\infty} \left[\frac{m^4 \underline{\underline{N}}}{2k^3 T^3 \zeta(3)} \right] \epsilon^3 \exp\left(\frac{-mn\epsilon}{kT} \right)$$

whence

$$\frac{\partial \mathbb{E}(\epsilon, T)}{\partial \epsilon} = \left[\frac{m^4 \underline{\underline{N}}}{2k^3 T^3 \zeta(3)} \right] \frac{\epsilon^3}{\left[\exp\left(\frac{m\epsilon}{kT} \right) - 1 \right]} \tag{3.8}$$

The total energy of the gas is

$$\mathbb{E}(\infty, T) = \int_0^\infty \frac{\partial \mathbb{E}(\epsilon, T)}{\partial \epsilon} d\epsilon = \frac{\pi^4 \underline{\underline{N}} kT}{30\zeta(3)} \tag{3.9}$$

and the mean energy per unit mass is thus

$$E = \frac{\mathbb{E}(\infty, T)}{\underline{\underline{N}} m} = \left[\frac{\pi^4}{30\zeta(3)}\right] \frac{kT}{m} = \frac{3kT}{\overline{m}} = \frac{9c^2}{4} = \overline{c}^2 \tag{3.10}$$

where c is the wave speed, and $\overline{c}$ is the r.m.s. translational speed of all the particles. So

$$\overline{c} = \frac{3c}{2} \tag{3.11}$$

4 Black-body radiation

For black-body radiation, $\mathbb{E}$ is found, by observation, to depend on the temperature T, and on the frequency ν of electromagnetic waves, in accordance with the empirical relation first suggested by Planck, namely

$$\left(\frac{1}{V}\right) \frac{\partial \mathbb{E}(\nu, T)}{\partial \nu} = \left(\frac{8\pi h_0}{c_0^3}\right) \frac{\nu^3}{\left[\exp\left(\frac{h_0 \nu}{kT}\right) - 1\right]} \tag{4.1}$$

where h_0 is called Planck's 'constant' and c_0 is called the 'speed of light'.

In order to reconcile Eq. (4.1) with the above result, Eq. (3.8), two conditions must be satisfied. they are

$$\epsilon = \frac{h_0 \nu}{m} \tag{4.2}$$

and

$$\frac{\underline{\underline{\underline{N}}}}{V} = \frac{16\pi k^3 T^3 \zeta(3)}{c_0^3 h_0^3}$$

or

$$\underline{N}_n = \left(\frac{16\pi k^3 T^3 V}{c_0^3 h_0^3}\right) \frac{1}{n^4} \tag{4.3}$$

If also, now, ν is written for the specific volume $(V/\underline{\underline{N}}m)$ of the medium, then the last relation, Eq. (4.3), becomes

$$v = \frac{c_0^3 h_0^3}{16\pi m k^3 T^3 \zeta(3)} = \frac{45 c_0^3 h_0^3}{8\pi^5 k^3 T^3 \overline{m}} \tag{4.4}$$

so that, from Eq. (3.10),

$$E = \frac{3kT}{\overline{m}} = \left(\frac{8\pi^5 k^4}{15 c_0^3 h_0^3}\right) v T^4 \tag{4.5}$$

i.e.

$$\frac{E}{v} = A_0 T^4$$

where

$$A_0 = \frac{8\pi^5 k^4}{15 c_0^3 h_0^3} \tag{4.6}$$

gives the radiation-density 'constant'.

It is now clear that, if there is a gas-like ethereal medium, such as has now been derived, then c_0 must be a special wave-speed, and h_0 a special value of some quantity, both of which may vary with position and time in the Universe. The first and most obvious natural conclusion is that c_0 is the wave-speed which obtains in our galactic neighbourhood, at the present epoch, in the background radiation field, i.e. the 2.7^oK microwave background black-body radiation field; and that h_0, or Planck's 'constant' is, similarly, the contemporary value in our galactic neighbourhood of a quantity which may vary both with position and time in the Universe.

On this basis, taking

$$h_0 = 662.56 \times 10^{-35} \text{ (kg)m/sec}^2$$

$$c_0 = 0.3 \times 10^9 \text{ m/sec}, \; T_0 = 2.7^o\text{K}, \; \zeta(3) = 1.202$$

and, of course, identifying the Universal constant k as Boltzmann's constant,

$$k = 13.8054 \times 10^{-24} \text{ (kg)m}^2/\text{sec}^2 \cdot \text{deg C}$$

then it is found that the absolute mass quantum m, the mass of a unit ether-particle, is

$$m = 0.497 \times 10^{-39} \quad \text{(kg)} \tag{4.7}$$

and

$$\overline{m} = 0.552 \times 10^{-39} \quad \text{(kg)} \tag{4.8}$$

whilst v_0 and p_0, the specific volume and pressure, respectively, of the local contemporary background radiation field, are

$$v_0 = \frac{40 h_0^3}{3\pi^5 c_0^3 \overline{m}^4} = 5.05 \times 10^{30} \text{ m}^3/\text{(kg)} \tag{4.9}$$

or

$$\rho_0 = \frac{1}{v_0} = 0.198 \times 10^{-30} \text{ (kg)/m}^3 \tag{4.10}$$

and

$$p_0 = 13.37 \times 10^{-15} \text{ (kg)/m} \cdot \text{sec}^2 \quad \text{or} \quad 133.7 \times 10^{-21} \text{ bars} \tag{4.11}$$

It remains to summarise the implications of the results now obtained for a particulate gas-like ethereal medium. Equation (4.2) differs vitally from Planck's quantum hypothesis and shows that it is unnecessary to make any hypothesis of this kind. For the result, Eq. (4.2), is *derived* from observation and implies that a particular frequency ν is not associated, as Planck postulated, with a particular value of a continuously variable energy quantum $h_0\nu$, but rather that a particular frequency ν is associated with a particular value of energy per unit mass ϵ, i.e. any particular frequency is associated with all those ether particles, whatever their masses, which have a particular energy per unit mass, $\epsilon = h_0\nu/m$, and which thus have different energies $nh_0\nu$ corresponding to their different masses nm.

Equation (4.4) implies that $\nu T^3/c_0^3 h_0^3$ or, by Eq. (4.6), $\nu T^3 A_0$ is a universal constant. Now, in an ideal gas with constant first adiabatic index 4/3, constant νT^3 implies constant entropy so that Eq. (4.4) accords, as it must, with the assumption made in Section 3 above that A_0 (or $c_0 h_0$) is a function of entropy.

But further, if there is such an ethereal medium, the Universe must consist of an expanding flow of ether in which matter is suspended. In this case, if there are no ethereal shock waves, what are usually called 'world-lines' in unsteady fluid dynamics, but which are now become 'Universal-lines', will be isentropic, and Eq. (4.4) will then imply that, to an observer travelling with the ethereal flow, A_0 or $c_0 h_0$ will have a constant value for all time. And still further, if not only are the Universal-lines isentropic, but the whole ethereal flow of the Universe is homentropic, then Eq. (4.4) will imply that A_0 or $c_0 h_0$ is a Universal constant.

The calculated value, Eq. (4.7), for the absolute mass quantum m implies that the mass of an electron is about 2×10^9 times that of a unit ether-particle; whilst Eq. (3.10) implies that the constant specific heat at constant volume of the ether is

$$\bar{c}_v = \frac{3k}{\overline{m}} = 75 \times 10^{15}\ \mathrm{m^2/sec^2 \cdot deg\ C}$$

or about 18×10^{12} cal/g·deg C.

Equation (4.2) implies that, for instance, the frequency of red light, $\nu \approx \frac{1}{2} \times 10^{18}\ \mathrm{sec}^{-1}$ is associated with an energy per unit mass

$$\epsilon = \frac{2}{3} \times 10^{24}\ \mathrm{m^2/sec^2} \quad \text{or} \quad 0.16 \times 10^{21}\ \mathrm{cal/g}$$

Equations (4.8) and (4.9) together imply that, in the local contemporary background radiation field, there are, on average, at any given time, 0.359×10^9 ether-particles per cubic metre, or about 360 per cubic centimetre.

5 Historical note

It is of considerable historical interest to observe that de Broglie touched upon the possibility of an infinite variety of light-quanta or photo-molecules, each of energy $nh_0\nu$, an integral multiple of Planck's energy quantum $h_0\nu$. He noted,[4,5] that Planck's distribution could be expanded as an infinite series of terms in $\nu^3 \exp\left(-nh_0\nu/kT\right)$ corresponding to Eq. (3.3) above, each term having the form of Wien's distribution. Einstein[6] had derived the mean square of the fluctuation of energy per unit volume, from Planck's distribution, as the sum of two terms which were, respectively, the values which would have been obtained by starting with Wien's distribution or Rayleigh's distribution, rather than with Planck's distribution. de Broglie also expanded Einstein's result for the fluctuations as an infinite series, and found that it corresponded, term by term, with the fluctuations calculated individually for the Wien-type terms in the expansion of Planck's distribution. Thus the terms of each series could be regarded as corresponding to energy quanta $nh_0\nu$, and this suggested the possibility of obtaining both Planck's distribution and Einstein's fluctuations on the basis of a corpuscular or particulate theory of electromagnetic radiation, provided a suitably weighted mixture could be determined for these different corpuscles with energies $nh_0\nu$. Such a corpuscular theory had already been proposed earlier by Wolfke.[7]

de Broglie does not appear to have pursued these suggestions further, or attempted to determine whether a possible mixture of 'n-quanta' existed. However, Bothe,[8] apparently independently, since he does not refer to de Broglie, went considerably further. He made use of Einstein's hypothesis[9] regarding the emission and absorption of light by material molecules, in order to derive the number of 'n-quanta' in black-body radiation, and obtained the result, *cf.* Eq. (3.1) above,

$$\frac{1}{V}\frac{\partial N_n(\nu)}{\partial \nu} = \left(\frac{8\pi}{nc_0^3}\right)\nu^2 \exp\left(\frac{-nh_0\nu}{kT}\right) \tag{5.1}$$

On the hand, he proceeded to show that this led to the result, *cf.* Eq. (3.3) above,

$$\frac{\partial \mathbb{E}(\nu,T)}{\partial \nu} = \sum_{n=1}^{\infty} nh_0\nu \frac{\partial N_n(\nu)}{\partial \nu} = \left(\frac{8\pi h_0 V}{c_0^3}\right)\sum_{n=1}^{\infty}\nu^3 \exp\left(\frac{-nh_0\nu}{kT}\right) \tag{5.2}$$

in agreement with Planck's distribution as expanded in series form by de Broglie. On the other hand he did not, surprisingly, integrate his result, Eq. (5.1), to obtain quite simply

$$\underline{N}_n = \int_0^{\infty}\left[\frac{\partial N_n(\nu)}{\partial \nu}\right]d\nu = \left(\frac{16\pi k^3T^3V}{c_0^3h_0^3}\right)\frac{1}{n^4} \tag{5.3}$$

in precise agreement with Eq. (4.3) above.

Neither de Broglie nor Bothe remarked upon the fact that their concept of 'n-quanta' or photo-molecules implied a vital emendation of Planck's quantum hypothesis, in that it required an association of wave-frequency ν, not with a quantity of energy, but with energy per n of such photo-molecules. And, since their approach was based on such a quantum hypothesis and the associated ideas of photons or energy-packets without mass, they both failed to recognise, even though Bothe succeeded in working backwards to the solution given here, that their concepts of a corpuscular theory, and their results, could be derived by working forwards, without hypothesis, from the kinetic theory of a gas with Maxwellian statistics.

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APPENDIX The specific energy distribution in a gas with an arbitrary number of degrees of molecular freedom

For a gas whose molecules have α degrees of freedom, let the component speeds of the molecules be denoted by l_1, l_2, l_3, and let the contributions to the energy per unit mass, ϵ of the remaining $(\alpha - 3)$ degrees of freedom be written as $\frac{1}{2}l_4^2, \frac{1}{2}l_5^2, ..., \frac{1}{2}l_\alpha^2$, so that

$$\epsilon = \sum_{i=1}^{\alpha} \frac{1}{2} l_i^2 \tag{A.1}$$

If $N(l_1, l_2, l_3, ..., l_\alpha)$ is the number of molecules whose specific energy components lie in the ranges $(0, l_1), (0, l_2), ..., (0, l_\alpha)$, then, for Maxwellian statistics

$$\frac{\partial^\alpha N(l_1, l_2, l_3, ..., l_\alpha)}{\partial l_1 \partial l_2 \partial l_3 ... \partial l_\alpha} = K' \exp\left(-\sum_{i=1}^{\alpha} \delta_i l_i^2\right) \tag{A.2}$$

for some K' and δ_i. The mean square values of l_i are given by

$$L_i^2 = \frac{\int_0^\infty \exp\left(-\delta_i l_i^2\right) l_i^2 dl_i}{\int_0^\infty \exp\left(-\delta_i l_i^2\right) dl_i} = \frac{1}{2}\delta_i \tag{A.3}$$

and, for equipartition of energy between all the α degrees of freedom, these must all be equal, so that $L_i^2 = \overline{c}^2/3, \delta_i = 3/2\overline{c}^2$ for all values of i, where $\overline{c}$ is the r.m.s. speed of the molecules. Then,

$$\frac{\partial^\alpha N(l_1, l_2, l_3, ..., l_\alpha)}{\partial l_1 \partial l_2 \partial l_3 ... \partial l_\alpha} = K' \exp\left(-\sum_{i=1}^{\alpha} \frac{3 l_i^2}{2\overline{c}^2}\right) \tag{A.4}$$

or

$$N(l_1, l_2, l_3, ..., l_\alpha) = K' \int_0^{l_1} \int_0^{l_2} \int_0^{l_3} ... \int_0^{l_\alpha} \exp\left(-\sum_{i=1}^{\alpha} \frac{3 l_i^2}{2\overline{c}^2}\right) dl_1 dl_2 dl_3 ... dl_\alpha \tag{A.5}$$

The multiple integral may be transformed by means of the generalised spherical 'polar' co-ordinates,

$$
\begin{aligned}
l_1 &= u\sin\theta_1\cos\theta_2\cos\theta_3...\cos\theta_{\alpha-1}\\
l_2 &= u\sin\theta_1\sin\theta_2\cos\theta_3...\cos\theta_{\alpha-1}\\
l_3 &= u\sin\theta_1\sin\theta_3\cos\theta_4...\cos\theta_{\alpha-1}\\
&..........\\
&..........\\
l_{\alpha-2} &= u\sin\theta_1\sin\theta_{\alpha-2}\cos\theta_{\alpha-1}\\
l_{\alpha-1} &= u\sin\theta_1\sin\theta_{\alpha-1}\\
l_\alpha &= u\cos\theta_1
\end{aligned}
\tag{A.6}
$$

which satisfy

$$\sum_{i=1}^{\alpha} l_i^2 = u^2 = 2\epsilon \tag{A.7}$$

This enables the integrations with respect to the θ_i to be performed, giving $N(u)$ or $N(\epsilon)$, the number of molecules lying in the ranges $(0,u), (0,\epsilon)$. Thus

$$N(u) = K\int_0^u u^{\alpha-1}\exp\left(\frac{-3u^2}{2\overline{c}^2}\right)du \tag{A.8}$$

or

$$N(\epsilon) = 2^{\frac{(\alpha-2)}{2}}K\int_0^\epsilon \epsilon^{\frac{(\alpha-2)}{2}}\exp\left(\frac{-3\epsilon}{\overline{c}^2}\right)d\epsilon \tag{A.9}$$

The constant K may be evaluated by integrating Eq. (A.9) from 0 to ∞, to give the total number $\underline{N}$ of the molecules. Then, finally,

$$\frac{\partial N(\epsilon)}{\partial\epsilon} = \left[\frac{3^{\frac{1}{2}\alpha}\underline{N}}{\overline{c}^\alpha\Gamma\left(\frac{1}{2}\alpha\right)}\right]\epsilon^{\frac{(\alpha-2)}{2}}\exp\left(\frac{-3\epsilon}{\overline{c}^2}\right) \tag{A.10}$$

In the simplest case of a monatomic gas, for which $\alpha = 3$ and the energy of the atoms is entirely kinetic energy of motion, the general relation of Eq. (A.10) reduces to

$$\frac{\partial N(\epsilon)}{\partial\epsilon} = \left(\frac{6\sqrt{3}\underline{N}}{\pi^{\frac{1}{2}}\overline{c}^3}\right)\epsilon^{\frac{1}{2}}\exp\left(\frac{-3\epsilon}{\overline{c}^2}\right) \tag{A.11}$$

or, with $u^2 = 2\epsilon$

$$\frac{\partial N(u)}{\partial u} = \left(\frac{3\sqrt{6}\underline{N}}{\pi^{\frac{1}{2}}\overline{c}^3}\right) u^2 \exp\left(\frac{-3u^2}{2\overline{c}^2}\right) \tag{A.12}$$

In this particular simple case, and in this case only, u is the speed of the atoms, so that Eq. (A.12) must necessarily be the Maxwellian speed distribution. For a gas whose molecules have six degrees of freedom, $\alpha = 6$ and Eq. (A.10) reduces to

$$\frac{\partial N(\epsilon)}{\partial \epsilon} = \left(\frac{27\underline{N}}{2\overline{c}^6}\right) \epsilon^2 \exp\left(\frac{-3\epsilon}{\overline{c}^2}\right) \tag{A.13}$$

and this is the distribution, Eq. (3.1), required in Section 3 above, of the main part of the paper.