Dmitri Rabounski

THE NEW ASPECTS OF GENERAL RELATIVITY

—2004—

This book has originally been published in CERN Document Server, Genève, EXT-2004-025

Copyright © D. D. Rabounski 2004

All rights reserved. Any electronic copying and printing of this book for individual or non-commercial use are free of any permission or charge. Any part of this book, being mentioned or used in any form in other publications, must be referred to this publication.

No part of this book may be reproduced in any form (including storing in any medium) for commercial use without the prior permission of the copyright holder. Applications for the copyright holder's permission to reproduce any part of this book for commercial use should be addressed to the copyright holder.

This book was typeset using $\mathcal{B}_{\mathcal{A}}\mathcal{K}_{\mathcal{O}}\mathcal{M}_{\mathcal{A}}$-TeX system

First Edition, April 28, 2004

Typeset and printed in Great Britain

Contents

Foreword

It is possible one thinks that the General Theory of Relativity is a fossilized science, all achievements of which were reached decades ago. In particular it is right — the mathematical apparatus of Riemannian geometry, being a base of the theory, remains unchanged. At the same time the mathematical technics have many varieties: general covariant methods, the tetrad method, etc. Developing the technics we can create new possibilities in theoretical physics, unknown before.

This book develops the mathematical methods of chronometric invariants — physical observable quantities in the General Theory of Relativity, which had been introduced by Abraham Zelmanov, a prominent cosmologist.

As you will see in this book, the mathematical methods create new directions in the theory of fields. In particular, those are the theory of the field of non-uniformities of time coordinates and the theory of vortical gravitational fields. The first case gives a possibility to express equations of the theory of gravitation in the form like as Maxwell's theory of electromagnetic fields. The theory of vortical gravitational fields gives an exact formula for the speed of gravitation and the new experimental statement to measure the speed. This new method is different from Weber's detector in principle and is free of its specific technical problems.

In conclusion of this brief foreword, I would like to remember Dr. Abraham Zelmanov, Prof. Kyril Stanyukovich, and Dr. Kyril Dombrowski. Many years of our conversations in friendly media, and their patient personal instructions reached me by all their experience in theoretical physics. Actually, I am beholden to them. They are gone, so any words of my acknowledgements can not be heard. I would like to do only one — to satisfy their hopes.

April 28, 2004 *Dmitri Rabounski*

Chapter 1

THE MATHEMATICAL APPARATUS

This Chapter gives the mathematical apparatus of chronometric invariants — physical observable quantities in the General Theory of Relativity, defining them as projections of four-dimensional quantities on time lines and the spatial section of a given observer. In addition, the main operators of tensor calculus are expressed through chronometrically invariant quantities. This update of tensor algebra and the analysis does calculations in the General Theory of Relativity simpler in order to see their physical sense. In particular, it gives templates to create field equations and other applications in the theory of fields.

§1.1 Introducing chronometric invariants

We are going to consider a four-dimensional pseudo-Riemannian space — the basic space-time of the General Theory of Relativity. Following Zelmanov [1, 2, 3, 4], we consider the space-time with signature $(+---)$, where time is real while spatial coordinates are imaginary. The main reason of this choice is that three-dimensional *observable impulse*, being the projection of four-dimensional impulse vector on the observer's spatial section, is positive in this case. Besides we sign the space-time indices Greek and the spatial indices Roman*.

What are physical observable quantities in the General Theory of Relativity? To answer the question is not a trivial problem, because we need to define those actual projections of four-dimensional quantities, which can be measured in practice. The problem had first been solved by Zelmanov in 1944 [1], who had built a complete

*It should be noted that Landau and Lifshitz in their famous *The Classical Theory of Fields* [5] stick to other signature $(-+++)$, where time is imaginary, spatial coordinates are real and three-dimensional *coordinate impulse* (the spatial part of four-dimensional impulse vector) is positive. Besides in their book the space-time indices are Roman, the spatial indices — Greek.

mathematical apparatus to calculate physical observable projections in a four-dimensional pseudo-Riemannian space. He called his mathematical apparatus the *theory of chronometric invariants*.

Other theorists of 1940's (see Landau and Lifshitz [5], for instance) also introduced observable time and observable spatial interval like as Zelmanov defined those. But they did not arrive to general mathematical methods to solve the problem, being limited themselves only to this particular case. Only Cattaneo, an Italian mathematician who developed his own approach to the problem, was not far from the Zelmanov solution. Cattaneo published his results on the theme in 1958 and later [6, 7, 8, 9]. Zelmanov appreciated his works highly. Cattaneo also referred to Zelmanov.

Here is an essence of the Zelmanov mathematical apparatus. From geometric viewpoint our three-dimensional space is a spatial section $x^0 = ct = const$, which may be placed in any point of space-time orthogonal to a time line $x^i = const$. If a spatial section is everywhere orthogonal to time lines, such space is known as *holonomic*. If a spatial section is only locally orthogonal to time lines, the space is *non-holonomic*. Frame of references of a real observer includes a coordinate net spanned over a real reference body and a real clock to which the observer refers his measurements. So, the observer's physical observable quantities should be projections of four-dimensional quantities on the time line and the coordinate net of his reference space. A four-dimensional monad (unit) vector

$$b^\alpha = \frac{dx^\alpha}{ds}\,, \qquad b_\alpha b^\alpha = +1\,, \tag{1.1}$$

of the observer's velocity in respect of his reference body, is the operator of projection on his time line. A four-dimensional symmetric tensor $h_{\alpha\beta}$, components of which are defined as

$$h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta\,, \quad h^{\alpha\beta} = -g^{\alpha\beta} + b^\alpha b^\beta, \quad h^\beta_\alpha = -g^\beta_\alpha + b_\alpha b^\beta, \tag{1.2}$$

is the operator of projection on his spatial section, because the vector b^α and the tensor $h_{\alpha\beta}$ have the necessary and sufficient properties of such projecting operators. Namely, the properties are

$$h^i_\alpha b^\alpha = 0\,, \qquad h^\alpha_i h^k_\alpha = \delta^k_i\,. \tag{1.3}$$

If an observer accompanies to his references ($b^i = 0$), then transformation of coordinates realizes only transition from one coordinate net to another within the same spatial section, transformation

of time displaces the observer into another spatial section. Therefore physical observable projections in the accompanying reference frame shall be invariant in respect of the transformation of time, so they shall be *chronometrically invariant quantities*. To calculate chr.inv.-projections Zelmanov set forth a theorem as follows:

THEOREM *We assume that* $A^{ij\dots k}_{00\dots0}$ *is the component of a world-tensor, all upper indices of which are significant, while all* m *the lower indices are zeroes. Next, we assume that* $B_{00\dots0}$ *is the time component of a covariant world-tensor of* n*-th rank. Then we have*

$$A^{ij\dots k\,\prime}_{00\dots0} = A^{ij\dots k}_{00\dots0}\left(\frac{\partial x^0}{\partial x^{0'}}\right)^m, \qquad B'_{00\dots0} = B_{00\dots0}\left(\frac{\partial x^0}{\partial x^{0'}}\right)^n, \tag{1.4}$$

so forth, using g_{00} *instead of* $B_{00\dots0}$*, we obtain that the quantity*

$$Q^{ij\dots k} = \frac{A^{ij\dots k}_{00\dots0}}{(g_{00})^{\frac{m}{2}}} \tag{1.5}$$

is the component of a contravariant chr.inv.-tensor.

In accordance with the Zelmanov theorem, chr.inv.-projections of an arbitrary four-dimensional vector Q^α on time lines and the spatial section are the quantities

$$b^\alpha Q_\alpha = \frac{Q_0}{\sqrt{g_{00}}}\,, \qquad h^i_\alpha Q^\alpha = Q^i, \tag{1.6}$$

chr.inv.-projections of any symmetric tensor of the 2nd rank $Q^{\alpha\beta}$ are the next three quantities

$$b^\alpha b^\beta Q_{\alpha\beta} = \frac{Q_{00}}{g_{00}}\,, \quad h^{i\alpha} b^\beta Q_{\alpha\beta} = \frac{Q^i_0}{\sqrt{g_{00}}}\,, \quad h^i_\alpha h^k_\beta Q^{\alpha\beta} = Q^{ik}, \tag{1.7}$$

in antisymmetric tensors of the 2nd rank just $Q_{00} = Q^{00} = 0$.

For instance, projecting an interval of four-dimensional coordinates dx^α on time lines in the accompanying reference frame, we obtain the chr.inv.-scalar

$$d\tau = \sqrt{g_{00}}\,dt + \frac{g_{0i}}{c\sqrt{g_{00}}}dx^i, \tag{1.8}$$

which is an interval of physical observable time, while its projection on the spatial section is the three-dimensional chr.inv.-vector dx^i, components of which coincide intervals of the spatial coordinates.

Projecting the fundamental metric tensor $g_{\alpha\beta}$ on the spatial section in the accompanying reference frame

$$h_i^\alpha h_k^\beta g_{\alpha\beta} = g_{ik} - b_i b_k = -h_{ik}\,, \tag{1.9}$$

we obtain that the chr.inv.-tensor

$$h_{ik} = -g_{ik} + b_i b_k \tag{1.10}$$

possesses all properties of the fundamental metric tensor in this spatial section. In particular, the h_{ik} can lift and lower indices of three-dimensional chr.inv.-quantities. So, it is the *metric chr.inv.-tensor* or, in other word, the *physical observable metric tensor*.

The square of space-time interval $ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$ in the terms of physical observable quantities is

$$ds^2 = b_\alpha b_\beta\, dx^\alpha dx^\beta - h_{\alpha\beta}\, dx^\alpha dx^\beta = c^2 d\tau^2 - d\sigma^2, \tag{1.11}$$

where the chr.inv.-scalar $d\sigma$ is a three-dimensional observable interval in this spatial section

$$d\sigma^2 = h_{\alpha\beta}\, dx^\alpha dx^\beta = h_{ik}\, dx^i dx^k. \tag{1.12}$$

As it had been found by Zelmanov, any real reference space, being the physical space of a real observer, has numerous observable properties, expressed by three-dimensional chr.inv.-quantities F_i, A_{ik}, D_{ik}, and Δ^i_{jk}. The quantities F_i and A_{ik} are introduced as follows. Chr.inv.-operators of derivation, marked with asterisk,

$$\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}\,, \qquad \frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{g_{0i}}{g_{00}}\frac{\partial}{\partial x^0} \tag{1.13}$$

are non-commutative

$$\begin{gathered} \frac{{}^*\partial^2}{\partial x^i \partial t} - \frac{{}^*\partial^2}{\partial t\, \partial x^i} = \frac{1}{c^2} F_i \frac{{}^*\partial}{\partial t}\,, \\ \frac{{}^*\partial^2}{\partial x^i \partial x^k} - \frac{{}^*\partial^2}{\partial x^k \partial x^i} = \frac{2}{c^2} A_{ik} \frac{{}^*\partial}{\partial t}\,, \end{gathered} \tag{1.14}$$

this non-commutativity defines the antisymmetric chr.inv.-tensor of angular velocities of the space rotation and the chr.inv.-vector of gravitational inertial force

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right)\,, \tag{1.15}$$

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \tag{1.16}$$

where w is the gravitational potential of the field of the reference body and v_i is the linear velocity of the space rotation*

$$\sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}\,, \qquad v_i = -c\,\frac{g_{0i}}{\sqrt{g_{00}}}\,. \tag{1.17}$$

So forth, components of the projecting operator b^α in the accompanying reference frame are

$$b^0 = \frac{1}{\sqrt{g_{00}}}\,, \qquad b_0 = \sqrt{g_{00}}\,, \qquad b^i = 0\,, \qquad b_i = -\frac{1}{c}\,v_i\,, \tag{1.18}$$

while components of the projecting operator $h_{\alpha\beta}$ are

$$h_{00} = 0\,, \qquad h_{0i} = 0\,, \qquad h_{ik} = -g_{ik} + \frac{1}{c^2}\,v_i v_k\,, \tag{1.19}$$

$$h^{00} = \frac{1}{g_{00}} - \frac{1 - \frac{1}{c^2}\,v_i v^i}{\sqrt{g_{00}}}\,, \quad h^{0i} = \frac{1}{c\sqrt{g_{00}}}\,v^i, \quad h^{ik} = -g^{ik}, \tag{1.20}$$

$$h^0_0 = 0\,, \quad h^i_0 = 0\,, \quad h^0_i = -\frac{1}{c\sqrt{g_{00}}}\,v_i\,, \quad h^i_k = -g^i_k = \delta^i_k\,. \tag{1.21}$$

A spatial section, being placed in a holonomic space, is everywhere orthogonal to time lines, i. e. there $g_{0i}=0$ is true. In the presence of $g_{0i}=0$ we have $v_i=0$, hence $A_{ik}=0$. This implies that non-holonomity of the space and its three-dimensional rotation are the same. In a non-holonomic space $g_{0i}\neq 0$ and $A_{ik}\neq 0$ are true. Hence $A_{ik}=0$ is the necessary and sufficient condition of holonomity of the space. So, the A_{ik} is the *tensor of the space non-holonomity*.

Zelmanov had also found that the chr.inv.-quantities F_i and A_{ik} are linked one to other by two identities

$$\frac{{}^*\partial A_{ik}}{\partial t} + \frac{1}{2}\left(\frac{{}^*\partial F_k}{\partial x^i} - \frac{{}^*\partial F_i}{\partial x^k}\right) = 0\,, \tag{1.22}$$

$$\frac{{}^*\partial A_{km}}{\partial x^i} + \frac{{}^*\partial A_{mi}}{\partial x^k} + \frac{{}^*\partial A_{ik}}{\partial x^m} + \frac{1}{2}\left(F_i A_{km} + F_k A_{mi} + F_m A_{ik}\right) = 0\,, \tag{1.23}$$

*As it could be proven from the main property of the fundamental metric tensor $g_{\alpha\sigma}g^{\sigma\beta}=\delta^\beta_\alpha$, being taken under the condition $\alpha=\beta=0$, the square of the velocity v_i equals $v^2=c^2(1-g_{00}g^{00})=h_{ik}v^iv^k$, where $v^i=-c\,g^{0i}\sqrt{g_{00}}$.

so we will refer to them as *Zelmanov's identities*.

Chr.inv.-derivatives of the metric chr.inv.-tensor with respect to time define the tree-dimensional symmetric chr.inv.-tensor D_{ik}, which is the rate of deformations of the space

$$D_{ik}=\frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}\,,\qquad D^{ik}=-\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t}\,,\qquad D=D_k^k=\frac{{}^*\partial\ln\sqrt{h}}{\partial t}\,,\tag{1.24}$$

where $h=\det\|h_{ik}\|$, $\sqrt{-g}=\sqrt{h}\,\sqrt{g_{00}}$, and $g=\det\|g_{\alpha\beta}\|$.

Observable non-uniformity of the space is defined by Christoffel's chr.inv.-symbols $\Delta^i_{jk}=h^{im}\Delta_{jk,m}$, which are set up similar to Christoffel's regular symbols $\Gamma^\alpha_{\mu\nu}=g^{\alpha\sigma}\Gamma_{\mu\nu,\sigma}$, namely

$$\Delta^i_{jk}=h^{im}\Delta_{jk,m}=\frac{1}{2}\,h^{im}\left(\frac{{}^*\partial h_{jm}}{\partial x^k}+\frac{{}^*\partial h_{km}}{\partial x^j}-\frac{{}^*\partial h_{jk}}{\partial x^m}\right),\tag{1.25}$$

$$\Gamma^\alpha_{\mu\nu}=g^{\alpha\sigma}\,\Gamma_{\mu\nu,\sigma}=\frac{1}{2}\,g^{\alpha\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu}+\frac{\partial g_{\nu\sigma}}{\partial x^\mu}-\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right).\tag{1.26}$$

From geometric viewpoint, the Christoffel symbols are coherence coefficients of the space. So, the Christoffel chr.inv.-symbols are *physical observable coherence coefficients* of the accompanying reference space of the given observer.

Components of the Christoffel regular symbols are linked to chr.inv.-chractersitics of the accompanying reference space of the given observer by the next correlations

$$D^i_k+A^{\cdot i}_{k\cdot}=\frac{c}{\sqrt{g_{00}}}\left(\Gamma^i_{0k}-\frac{g_{0k}\Gamma^i_{00}}{g_{00}}\right),\tag{1.27}$$

$$F^k=-\frac{c^2\,\Gamma^k_{00}}{g_{00}}\,,\qquad g^{i\alpha}g^{k\beta}\,\Gamma^m_{\alpha\beta}=h^{iq}h^{ks}\Delta^m_{qs}.\tag{1.28}$$

These are the main points of the Zelmanov mathematical apparatus of chronometric invariants. To do using the apparatus in the General Theory of Relativity easier, we need to express the basics of tensor algebra and the analysis in the terms of chronometric invariants. The next paragraphs will be focused on this problem.

§1.2 Tensor algebra

We assume a space with an arbitrary reference frame x^α. In some area of the space, there exists a geometric object G defined by

n functions of the coordinates x^α and the transformation rule to calculate these n functions in any other reference frame $\tilde{x}^\alpha$, located in this space.

A tensor object (*tensor*) of zero rank is any geometric object φ, transformable according to the rule

$$\tilde{\varphi}=\varphi\frac{\partial x^\alpha}{\partial\tilde{x}^\alpha}\,,\tag{1.29}$$

where the index one-by-one takes numbers of all coordinate axis. Any tensor of zero rank has a single component and is also known as *scalar*. From geometric viewpoint, scalar is a point to which a certain number is attributed.

Contravariant (upper-index) tensors of the 1st rank A^α and of higher ranks $A^{\alpha\ldots\sigma}$ are geometric objects with components, transformable according to the rules

$$\tilde{A}^\alpha=A^\mu\frac{\partial\tilde{x}^\alpha}{\partial x^\mu}\,,\qquad\tilde{A}^{\alpha\ldots\sigma}=A^{\mu\ldots\tau}\frac{\partial\tilde{x}^\alpha}{\partial x^\mu}\cdots\frac{\partial\tilde{x}^\sigma}{\partial x^\tau}\,.\tag{1.30}$$

For instance, contravariant tensor of the 1st rank is a vector, contravariant tensor of the 2nd rank (*bivector*) is a parallelogram constrained by two vectors, and so on.

Covariant (lower-index) tensors of the 1st rank A_α and of higher ranks $A_{\alpha\ldots\sigma}$ are geometric objects, transformable according to the rules

$$\tilde{A}_\alpha=A_\mu\frac{\partial x^\mu}{\partial\tilde{x}^\alpha}\,,\qquad\tilde{A}_{\alpha\ldots\sigma}=A_{\mu\ldots\tau}\frac{\partial x^\mu}{\partial\tilde{x}^\alpha}\cdots\frac{\partial x^\tau}{\partial\tilde{x}^\sigma}\,.\tag{1.31}$$

In particular, the gradient of a scalar field φ, i. e. the quantity $A_\alpha=\dfrac{\partial\varphi}{\partial x^\alpha}$, is covariant tensor of the 1st rank, because of taking into account that $\tilde{\varphi}=\varphi$ we have

$$\frac{\partial\tilde{\varphi}}{\partial\tilde{x}^\alpha}=\frac{\partial\tilde{\varphi}}{\partial x^\mu}\frac{\partial x^\mu}{\partial\tilde{x}^\alpha}=\frac{\partial\varphi}{\partial x^\mu}\frac{\partial x^\mu}{\partial\tilde{x}^\alpha}\,.\tag{1.32}$$

Mixed tensors are tensors of the 2nd rank or of higher ranks with both upper and lower indices. For instance, mixed symmetric tensor A^α_β is a geometric object, transformable according to the rule

$$\tilde{A}^\alpha_\beta=A^\mu_\nu\frac{\partial\tilde{x}^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial\tilde{x}^\beta}\,.\tag{1.33}$$

Tensor objects exist both in metric and non-metric spaces (in non-metric spaces the distance between any two points can not be measured). Every tensor has a^n components, where a is dimension of the tensor and n is the rank. So, a four-dimensional tensor of zero rank has 1 component, a tensor of the 1st rank has 4 components, a tensor of the 2nd rank has 16 components, and so on.

However the presence of indices (the number of axial components) are found not in tensors only, but in other geometric objects as well. Therefore, if we come across a quantity in by-component notation, this is not necessarily a tensor quantity. For instance, let us calculate Christoffel's symbols (the coherence coefficients of space) in a tilde-marked reference frame

$$\widetilde{\Gamma}^{\alpha}_{\mu\nu} = \tilde{g}^{\alpha\sigma}\,\widetilde{\Gamma}_{\mu\nu,\sigma}\,, \quad \widetilde{\Gamma}_{\mu\nu,\sigma} = \frac{1}{2}\left(\frac{\partial\tilde{g}_{\mu\sigma}}{\partial\tilde{x}^{\nu}} + \frac{\partial\tilde{g}_{\nu\sigma}}{\partial\tilde{x}^{\mu}} - \frac{\partial\tilde{g}_{\mu\nu}}{\partial\tilde{x}^{\sigma}}\right), \tag{1.34}$$

proceeding from the quantities in a non-marked reference frame. Because the fundamental metric tensor $g_{\alpha\beta}$, just like any other covariant tensor of the 2nd rank, is transformable to the rule

$$\tilde{g}_{\mu\sigma} = g_{\varepsilon\tau}\frac{\partial x^{\varepsilon}}{\partial\tilde{x}^{\mu}}\frac{\partial x^{\tau}}{\partial\tilde{x}^{\sigma}}\,, \tag{1.35}$$

we have

$$\frac{\partial g_{\varepsilon\tau}}{\partial\tilde{x}^{\nu}} = \frac{\partial g_{\varepsilon\tau}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial\tilde{x}^{\nu}}\,, \tag{1.36}$$

and the first term in the brackets of (1.34) is

$$\frac{\partial\tilde{g}_{\mu\sigma}}{\partial\tilde{x}^{\nu}} = \frac{\partial g_{\varepsilon\tau}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial\tilde{x}^{\nu}}\frac{\partial x^{\varepsilon}}{\partial\tilde{x}^{\mu}}\frac{\partial x^{\tau}}{\partial\tilde{x}^{\sigma}} + g_{\varepsilon\tau}\left(\frac{\partial x^{\tau}}{\partial\tilde{x}^{\sigma}}\frac{\partial^2 x^{\varepsilon}}{\partial\tilde{x}^{\nu}\partial\tilde{x}^{\mu}} + \frac{\partial x^{\varepsilon}}{\partial\tilde{x}^{\mu}}\frac{\partial^2 x^{\tau}}{\partial\tilde{x}^{\nu}\partial\tilde{x}^{\sigma}}\right). \tag{1.37}$$

Following the same way, deducing the rest of the terms of the tilde-marked Christoffel symbols (1.34), after transposition of free indices we arrive to

$$\widetilde{\Gamma}_{\mu\nu,\sigma} = \Gamma_{\varepsilon\rho,\tau}\frac{\partial x^{\varepsilon}}{\partial\tilde{x}^{\mu}}\frac{\partial x^{\rho}}{\partial\tilde{x}^{\nu}}\frac{\partial x^{\tau}}{\partial\tilde{x}^{\sigma}} + g_{\varepsilon\tau}\frac{\partial x^{\tau}}{\partial\tilde{x}^{\sigma}}\frac{\partial^2 x^{\varepsilon}}{\partial\tilde{x}^{\mu}\partial\tilde{x}^{\nu}}\,, \tag{1.38}$$

$$\widetilde{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\gamma}_{\varepsilon\rho}\frac{\partial\tilde{x}^{\alpha}}{\partial x^{\gamma}}\frac{\partial x^{\varepsilon}}{\partial\tilde{x}^{\mu}}\frac{\partial x^{\rho}}{\partial\tilde{x}^{\nu}} + \frac{\partial\tilde{x}^{\alpha}}{\partial x^{\gamma}}\frac{\partial^2 x^{\gamma}}{\partial\tilde{x}^{\mu}\partial\tilde{x}^{\nu}}\,, \tag{1.39}$$

so the Christoffel symbols are transformed not in the way tensors, hence they are not tensors.

Adding up two same-type tensors of the n-rank gives a new tensor of the same type and rank with components being sums of the respective components of the tensors added up. For instance

$$A^{\alpha} + B^{\alpha} = D^{\alpha}, \qquad A^{\alpha}_{\beta} + B^{\alpha}_{\beta} = D^{\alpha}_{\beta}\,. \tag{1.40}$$

Tensor multiplication is permitted not only for same-type, but for any tensors of any ranks. *External multiplication* of tensors of the n-rank and m-rank gives a tensor of the $(n+m)$-rank

$$A_{\alpha\beta}B_{\gamma} = D_{\alpha\beta\gamma}\,, \qquad A_{\alpha}B^{\beta\gamma} = D^{\beta\gamma}_{\alpha}. \tag{1.41}$$

Contraction is multiplication of the same-rank tensors, when indices are the same. Contraction of tensors by all indices gives a scalar quantity, for instance

$$A_{\alpha}B^{\alpha} = C\,, \qquad A^{\gamma}_{\alpha\beta}B^{\alpha\beta}_{\gamma} = D\,. \tag{1.42}$$

Often multiplication of tensors implies contraction by not all indices, so the result is not a scalar quantity. Such multiplication is referred to as *internal multiplication*, which implies contraction of only some indices inside the multiplication

$$A_{\alpha\sigma}B^{\sigma} = D_{\alpha}\,, \qquad A^{\gamma}_{\alpha\sigma}B^{\beta\sigma}_{\gamma} = D^{\beta}_{\alpha}\,. \tag{1.43}$$

In Riemannian spaces, the metric has Riemannian quadratic form $ds^2 = g_{\alpha\beta}\,dx^{\alpha}dx^{\beta}$. So the fundamental metric tensor, defining geometric properties of the spaces, is the tensor of the 2nd rank $g_{\alpha\beta}$. The metric tensor can lower or lift indices in geometric objects, for instance

$$g_{\alpha\beta}A^{\beta} = A_{\alpha}\,, \qquad g^{\mu\nu}g^{\sigma\rho}A_{\mu\nu\sigma} = A^{\rho}. \tag{1.44}$$

In Riemannian spaces the mixed fundamental metric tensor g^{β}_{α} equals to the four-dimensional unit tensor

$$g^{\beta}_{\alpha} = g_{\alpha\sigma}g^{\sigma\beta} = \delta^{\beta}_{\alpha}\,, \tag{1.45}$$

diagonal components of which are units, while the rest are zeroes. The spatial part of the unit world-tensor δ^{β}_{α} is the three-dimensional unit tensor δ^{k}_{i}, so that

$$\delta^{\beta}_{\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \delta^{k}_{i} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.46}$$

Using the unit tensors, we can replace indices. For instance, for four-dimensional quantities we can write down

$$\delta^{\beta}_{\alpha}A_{\beta} = A_{\alpha}\,, \qquad \delta^{\nu}_{\mu}\delta^{\sigma}_{\rho}A^{\mu\rho} = A^{\nu\sigma}, \tag{1.47}$$

while for three-dimensional quantities we have, respectively

$$\delta_i^k A_k = A_i \,, \qquad \delta_i^m \delta_k^n A^{ik} = A^{mn}. \tag{1.48}$$

Contraction of any tensor of the 2nd rank with the fundamental metric tensor gives a scalar quantity, known as the *tensor spur* or its *trace*

$$g^{\alpha\beta} A_{\alpha\beta} = A_\sigma^\sigma \,. \tag{1.49}$$

For instance, as it easy to see, the spur of the fundamental metric tensor in a four-dimensional Riemannian space equals the number of coordinate axes

$$g_{\alpha\beta}\, g^{\alpha\beta} = g_\sigma^\sigma = g_0^0 + g_1^1 + g_2^2 + g_3^3 = 4. \tag{1.50}$$

The metric chr.inv.-tensor h_{ik} (1.10) in the spatial section of an observer, who accompanies to his reference body*, possess all properties of the fundamental metric tensor $g_{\alpha\beta}$. Therefore the tensor h_{ik} can lower, lift or replace indices in chr.inv.-quantities. For instance, the spur of the tensor of the rate of the space deformations D_{ik} (1.24) equals

$$D = h^{ik} D_{ik} = D_n^n \,. \tag{1.51}$$

Scalar product of two vectors A^α and B^α in a four-dimensional pseudo-Riemannian space is the quantity

$$g_{\alpha\beta}\, A^\alpha B^\beta = A_\alpha B^\alpha = A_0 B^0 + A_i B^i. \tag{1.52}$$

If the both vectors are the same, their scalar product

$$g_{\alpha\beta}\, A^\alpha A^\beta = A_\alpha A^\alpha = A_0 A^0 + A_i A^i \tag{1.53}$$

is the square of the vector A^α. Consequently the length of any vector is scalar. For instance, the length of the A^α is

$$A = |A^\alpha| = \sqrt{g_{\alpha\beta}\, A^\alpha A^\beta}. \tag{1.54}$$

In three-dimensional Euclidean space scalar product of two vectors is a scalar quantity with module equal to the product of their lengths, multiplied by cosine of the angle between them

$$A_i B^i = \left|A^i\right| \left|B^i\right| \cos\left(\widehat{A^i; B^i}\right). \tag{1.55}$$

*From geometric viewpoint, "the spatial section of an observer, who accompanies to his reference body" implies "the three-dimensional observable space of the observer".

Theoretically at every point of any Riemannian space a tangential flat space can be set, which basic vectors will be tangential to the basic vectors of the Riemannian space in this tangential point. Then the metric of the tangential flat space will be the metric of the Riemannian space in this point. Therefore this statement is also true in the Riemannian space, if we consider the angle between spatial coordinate lines and replace Roman (three-dimensional) indices with Greek ones. From here we can see that scalar product of two orthogonal vectors is the projection of one vector on another and equals zero. If the vectors are the same, the vector is projected on itself.

Vector product of two vectors A^α and B^α is a tensor of the 2nd rank $V^{\alpha\beta}$, obtained from their external multiplication according to the specific rule

$$V^{\alpha\beta} = \left[A^\alpha; B^\beta\right] = \frac{1}{2}\left(A^\alpha B^\beta - A^\beta B^\alpha\right) = \frac{1}{2}\begin{vmatrix} A^\alpha & A^\beta \\ B^\alpha & B^\beta \end{vmatrix}. \tag{1.56}$$

As it easy to see, here the order in which vectors are multiplied does matter, i. e. the order in which we write down tensor indices. Therefore tensors obtained as vector products are *antisymmetric tensors*. In any antisymmetric tensor $V^{\alpha\beta} = -V^{\beta\alpha}$ indices being moved "reserve" their places as dots $g_{\alpha\sigma} V^{\sigma\beta} = V_{\alpha\cdot}^{\cdot\beta}$, thus showing from where the index was moved. In symmetric tensors there is no need of "reserving" places for moved indices, because the order in which they appear does not matter. In particular, the fundamental metric tensor $g_{\alpha\beta} = g_{\beta\alpha}$ is symmetric, while Riemann-Christoffel's tensor of the space curvature $R^{\alpha\cdots}_{\cdot\beta\gamma\delta}$ is symmetric in respect to transposition by pair of its indices and is antisymmetric inside each pair of the indices. It is evidently, only tensors of the 2nd rank or of higher ranks may be symmetric or antisymmetric.

All diagonal components of any antisymmetric tensor by its definition are zeroes, for example

$$V^{\alpha\alpha} = [A^\alpha; B^\alpha] = \frac{1}{2}\left(A^\alpha B^\alpha - A^\alpha B^\alpha\right) = 0\,. \tag{1.57}$$

In three-dimensional Euclidean space the numerical value of the vector product of two vectors is defined as the area of the parallelogram they make and equals the product of their modules, multiplied by sine of the angle between them

$$V^{ik} = \left|A^i\right| \left|B^k\right| \sin\left(\widehat{A^i; B^k}\right), \tag{1.58}$$

so any antisymmetric tensor of the 2nd rank is a pad, oriented in the space according to the directions of its forming vectors.

Because any antisymmetric tensor, $V_{\alpha\beta}$ for instance, possesses the properties $V_{\alpha\alpha}=0$ and $V_{\alpha\beta}=-V_{\beta\alpha}$, contraction of it with any symmetric tensor $A^{\alpha\beta}=A^{\alpha}A^{\beta}$ is zero

$$V_{\alpha\beta}A^{\alpha}A^{\beta} = V_{00}A^0A^0 + V_{0i}A^0A^i + V_{i0}A^iA^0 + V_{ik}A^iA^k = 0\,. \qquad (1.59)$$

Chr.inv.-projections of an arbitrary antisymmetric tensor of the 2nd rank $V^{\alpha\beta}$ are the quantities

$$\frac{V_{00}}{g_{00}} = 0\,, \qquad \frac{V_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} = -\frac{V_{\cdot 0}^{i\cdot}}{\sqrt{g_{00}}}\,, \qquad V^{ik} = -V^{ki}\,, \qquad (1.60)$$

where the 1st observable projection is zero, because in any antisymmetric tensor all diagonal components are zeroes.

§1.3 Pseudotensors

Asymmetry of tensor fields is defined by reference antisymmetric tensors. In a Galilean reference frame* such references are Levi-Civita's tensors, namely — the *four-dimensional completely antisymmetric unit tensor* $e^{\alpha\beta\mu\nu}$ for four-dimensional fields and the *three-dimensional completely antisymmetric unit tensor* e^{ikm} for three-dimensional fields. Components of the Levi-Civita tensors, which have all indices different, are either $+1$ or -1 depending upon the number of transpositions of indices. All the rest components, i. e. those having at least two coinciding indices, are zeroes. Moreover, for the signature $(+---)$ we are using all non-zero components have the sign opposite to their respective covariant components†. For instance, in the Minkowski space we have

$$\begin{gathered} g_{\alpha\sigma}g_{\beta\rho}g_{\mu\tau}g_{\nu\gamma}e^{\sigma\rho\tau\gamma} = g_{00}g_{11}g_{22}g_{33}e^{0123} = -e^{0123}, \\ g_{i\alpha}g_{k\beta}g_{m\gamma}e^{\alpha\beta\gamma} = g_{11}g_{22}g_{33}e^{123} = -e^{123} \end{gathered} \qquad (1.61)$$

*A Galilean frame of reference is the one that does not rotate, is not subject to deformations and falls freely in the flat space-time of the Special Theory of Relativity (Minkowski's space). There lines of time are linear and so are three-dimensional coordinate axes.

†Under the signature $(-+++)$ Landau and Lifshitz used in *The Classical Theory of Fields* [5] this is true for only the four-dimensional tensor $e^{\alpha\beta\mu\nu}$. Components of the three-dimensional tensor e^{ikm} will have same signs as well as the respective components of e_{ikm}.

due to the signature conditions $g_{00}=1$ and $g_{11}=g_{22}=g_{33}=-1$. So forth, components of the tensor $e^{\alpha\beta\mu\nu}$ equal

$$\begin{gathered} e^{0123}=+1, \quad e^{1023}=-1, \quad e^{1203}=+1, \quad e^{1230}=-1, \\ e_{0123}=-1, \quad e_{1023}=+1, \quad e_{1203}=-1, \quad e_{1230}=+1, \end{gathered} \qquad (1.62)$$

and components of the tensor e^{ikm} are

$$\begin{gathered} e^{123}=+1, \quad e^{213}=-1, \quad e^{231}=+1, \\ e_{123}=-1, \quad e_{213}=+1, \quad e_{231}=-1. \end{gathered} \qquad (1.63)$$

Because we have free choice for the sign of the first component, we assume $e^{0123}=-1$ and $e^{123}=-1$. The rest components will be changed subsequently. In general, the world-tensor $e^{\alpha\beta\mu\nu}$ is related to the spatial tensor e^{ikm} as $e^{0ikm}=e^{ikm}$.

Multiplying the four-dimensional antisymmetric unit tensor $e^{\alpha\beta\mu\nu}$ by itself, we obtain a regular tensor of the 8th rank with non-zero components, which are given in the matrix

$$e^{\alpha\beta\mu\nu}e_{\sigma\tau\rho\gamma} = -\begin{pmatrix} \delta^{\alpha}_{\sigma} & \delta^{\alpha}_{\tau} & \delta^{\alpha}_{\rho} & \delta^{\alpha}_{\gamma} \\ \delta^{\beta}_{\sigma} & \delta^{\beta}_{\tau} & \delta^{\beta}_{\rho} & \delta^{\beta}_{\gamma} \\ \delta^{\mu}_{\sigma} & \delta^{\mu}_{\tau} & \delta^{\mu}_{\rho} & \delta^{\mu}_{\gamma} \\ \delta^{\nu}_{\sigma} & \delta^{\nu}_{\tau} & \delta^{\nu}_{\rho} & \delta^{\nu}_{\gamma} \end{pmatrix}. \qquad (1.64)$$

The rest properties of the tensor $e^{\alpha\beta\mu\nu}$ are derived from the previous by means of contraction of indices

$$e^{\alpha\beta\mu\nu}e_{\sigma\tau\rho\nu} = -\begin{pmatrix} \delta^{\alpha}_{\sigma} & \delta^{\alpha}_{\tau} & \delta^{\alpha}_{\rho} \\ \delta^{\beta}_{\sigma} & \delta^{\beta}_{\tau} & \delta^{\beta}_{\rho} \\ \delta^{\mu}_{\sigma} & \delta^{\mu}_{\tau} & \delta^{\mu}_{\rho} \end{pmatrix}, \qquad (1.65)$$

$$e^{\alpha\beta\mu\nu}e_{\sigma\tau\mu\nu} = -2\begin{pmatrix} \delta^{\alpha}_{\sigma} & \delta^{\alpha}_{\tau} \\ \delta^{\beta}_{\sigma} & \delta^{\beta}_{\tau} \end{pmatrix} = -2\left(\delta^{\alpha}_{\sigma}\delta^{\beta}_{\tau} - \delta^{\beta}_{\sigma}\delta^{\alpha}_{\tau}\right), \qquad (1.66)$$

$$e^{\alpha\beta\mu\nu}e_{\sigma\beta\mu\nu} = -6\delta^{\alpha}_{\sigma}\,, \qquad e^{\alpha\beta\mu\nu}e_{\alpha\beta\mu\nu} = -6\delta^{\alpha}_{\alpha} = -24. \qquad (1.67)$$

Multiplying the three-dimensional antisymmetric unit tensor e^{ikm} by itself we obtain a regular tensor of the 6th rank

$$e^{ikm}e_{rst} = \begin{pmatrix} \delta^{i}_{r} & \delta^{i}_{s} & \delta^{i}_{t} \\ \delta^{k}_{r} & \delta^{k}_{s} & \delta^{k}_{t} \\ \delta^{m}_{r} & \delta^{m}_{s} & \delta^{m}_{t} \end{pmatrix}. \qquad (1.68)$$

The rest properties of the tensor e^{ikm} can be expressed as follows

$$e^{ikm}e_{rsm} = -\begin{pmatrix} \delta^i_r & \delta^i_s \\ \delta^k_r & \delta^k_s \end{pmatrix} = \delta^i_s\delta^k_r - \delta^i_r\delta^k_s\,, \tag{1.69}$$

$$e^{ikm}e_{rkm} = 2\delta^i_r\,, \quad e^{ikm}e_{ikm} = 2\delta^i_i = 6. \tag{1.70}$$

The completely antisymmetric unit tensor defines for a tensor object its respective *pseudotensor*, marked with asterisk. For instance, any four-dimensional scalar, vector and tensors of the 2nd, 3rd, and 4th ranks have respective four-dimensional pseudotensors of the following ranks

$$\begin{gathered} V^{*\alpha\beta\mu\nu} = e^{\alpha\beta\mu\nu}V, \quad V^{*\alpha\beta\mu} = e^{\alpha\beta\mu\nu}V_\nu\,, \quad V^{*\alpha\beta} = \frac{1}{2}\,e^{\alpha\beta\mu\nu}V_{\mu\nu}\,, \\ V^{*\alpha} = \frac{1}{6}\,e^{\alpha\beta\mu\nu}V_{\beta\mu\nu}\,, \quad V^* = \frac{1}{24}\,e^{\alpha\beta\mu\nu}V_{\alpha\beta\mu\nu}\,, \end{gathered} \tag{1.71}$$

where pseudotensors of the 1st rank $V^{*\alpha}$ are sometimes called *pseudovectors*, while pseudotensors of zero rank V^* are called *pseudoscalars*. Any tensor and its respective pseudotensor are referred to as *dual* to each other to emphasize their common genesis. In the same way, three-dimensional tensors have respective three-dimensional pseudotensors

$$\begin{gathered} V^{*ikm} = e^{ikm}V, \qquad V^{*ik} = e^{ikm}V_m\,, \\ V^{*i} = \frac{1}{2}\,e^{ikm}V_{km}\,, \quad V^* = \frac{1}{6}\,e^{ikm}V_{ikm}\,. \end{gathered} \tag{1.72}$$

Pseudotensors are called such because, in contrast to regular tensors, they do not change being reflected in respect of one of the axis. For instance, let us assume the reflection in respect of abscises axis $x^1 = -\tilde{x}^1$, $x^2 = \tilde{x}^2$, $x^3 = \tilde{x}^3$. Then the reflected component of an antisymmetric tensor V_{ik} orthogonal to x^1 axis is $\widetilde{V}_{23} = -V_{23}$, while its dual component is

$$\begin{gathered} V^{*1} = \frac{1}{2}\,e^{1km}V_{km} = \frac{1}{2}\left(e^{123}V_{23} + e^{132}V_{32}\right) = V_{23}\,, \\ \widetilde{V}^{*1} = \frac{1}{2}\,\tilde{e}^{1km}\widetilde{V}_{km} = \frac{1}{2}\,e^{k1m}\widetilde{V}_{km} = \frac{1}{2}\left(e^{213}\widetilde{V}_{23} + e^{312}\widetilde{V}_{32}\right) = V_{23}\,. \end{gathered} \tag{1.73}$$

Because four-dimensional antisymmetric tensors of the 2nd rank and their dual pseudotensors are of the same rank, their

contraction are pseudoscalars, so that

$$V_{\alpha\beta}V^{*\alpha\beta} = V_{\alpha\beta}\,e^{\alpha\beta\mu\nu}V_{\mu\nu} = e^{\alpha\beta\mu\nu}B_{\alpha\beta\mu\nu} = B^*. \tag{1.74}$$

The square of a pseudotensor $V^{*\alpha\beta}$ and the square of a pseudovector V^{*i}, expressed through their dual antisymmetric tensors of the 2nd rank, are

$$V_{*\alpha\beta}V^{*\alpha\beta} = e_{\alpha\beta\mu\nu}V^{\mu\nu}e^{\alpha\beta\rho\sigma}V_{\rho\sigma} = -24V_{\mu\nu}V^{\mu\nu}, \tag{1.75}$$

$$V_{*i}V^{*i} = e_{ikm}V^{km}e^{ipq}V_{pq} = 6V_{km}V^{km}. \tag{1.76}$$

In inhomogeneous anisotropic pseudo-Riemannian spaces we can not set a Galilean reference frame, so references of asymmetry of tensor fields will depend upon inhomogeneity and anisotropy of the space itself, which are defined by the fundamental metric tensor. In this case a reference antisymmetric tensor is the *four-dimensional completely antisymmetric discriminant tensor*

$$E^{\alpha\beta\mu\nu} = \frac{e^{\alpha\beta\mu\nu}}{\sqrt{-g}}\,, \qquad E_{\alpha\beta\mu\nu} = e_{\alpha\beta\mu\nu}\sqrt{-g}\,. \tag{1.77}$$

Here is the proof. Transformation of the unit completely antisymmetric tensor from a Galilean (non-tilde-marked) reference frame into an arbitrary (tilde-marked) reference frame is

$$\tilde{e}_{\alpha\beta\mu\nu} = e_{\sigma\gamma\varepsilon\tau}\frac{\partial x^\sigma}{\partial\tilde{x}^\alpha}\frac{\partial x^\gamma}{\partial\tilde{x}^\beta}\frac{\partial x^\varepsilon}{\partial\tilde{x}^\mu}\frac{\partial x^\tau}{\partial\tilde{x}^\nu} = J e_{\alpha\beta\mu\nu}\,, \tag{1.78}$$

where $J = \det\left\|\dfrac{\partial x^\alpha}{\partial\tilde{x}^\sigma}\right\|$ is called the *Jacobian of the transformation* (the determinant of Jacobi's matrix)

$$J = \det\begin{Vmatrix} \frac{\partial x^0}{\partial\tilde{x}^0} & \frac{\partial x^0}{\partial\tilde{x}^1} & \frac{\partial x^0}{\partial\tilde{x}^2} & \frac{\partial x^0}{\partial\tilde{x}^3} \\ \frac{\partial x^1}{\partial\tilde{x}^0} & \frac{\partial x^1}{\partial\tilde{x}^1} & \frac{\partial x^1}{\partial\tilde{x}^2} & \frac{\partial x^1}{\partial\tilde{x}^3} \\ \frac{\partial x^2}{\partial\tilde{x}^0} & \frac{\partial x^2}{\partial\tilde{x}^1} & \frac{\partial x^2}{\partial\tilde{x}^2} & \frac{\partial x^2}{\partial\tilde{x}^3} \\ \frac{\partial x^3}{\partial\tilde{x}^0} & \frac{\partial x^3}{\partial\tilde{x}^1} & \frac{\partial x^3}{\partial\tilde{x}^2} & \frac{\partial x^3}{\partial\tilde{x}^3} \end{Vmatrix}. \tag{1.79}$$

Because the metric tensor $g_{\alpha\beta}$ is transformable just like any covariant tensor of the 2nd rank, its determinant in the tilde-marked reference frame is

$$\tilde{g} = \det\|\tilde{g}_{\alpha\beta}\| = \det\left\|g_{\mu\nu}\frac{\partial x^\mu}{\partial\tilde{x}^\alpha}\frac{\partial x^\nu}{\partial\tilde{x}^\beta}\right\| = J^2 g\,. \tag{1.80}$$

Because in the Galilean (non-tilde-marked) frame of reference

$$g = \det \|g_{\alpha\beta}\| = \det \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix} = -1, \tag{1.81}$$

then $J^2 = -\tilde{g}^2$. Expressing $\tilde{e}_{\alpha\beta\mu\nu}$ in an arbitrary frame of reference as $E_{\alpha\beta\mu\nu}$ and writing down the metric tensor in a regular non-tilde-marked form, we obtain $E_{\alpha\beta\mu\nu} = e_{\alpha\beta\mu\nu}\sqrt{-g}$ (1.77). In the same way, we obtain the transformation rules for the components $E^{\alpha\beta\mu\nu}$, because for them $g = \tilde{g}\tilde{J}^2$, where $\tilde{J} = \det \left\| \frac{\partial \tilde{x}^\alpha}{\partial x^\sigma} \right\|$.

The discriminant tensor $E^{\alpha\beta\mu\nu}$ is not a physical observable quantity. A physical observable reference of asymmetry of tensor fields is the *discriminant chr.inv.-tensor*

$$\varepsilon^{\alpha\beta\gamma} = h^\alpha_\mu h^\beta_\nu h^\gamma_\rho b_\sigma E^{\sigma\mu\nu\rho} = b_\sigma E^{\sigma\alpha\beta\gamma}, \tag{1.82}$$

$$\varepsilon_{\alpha\beta\gamma} = h^\mu_\alpha h^\nu_\beta h^\rho_\gamma b^\sigma E_{\sigma\mu\nu\rho} = b^\sigma E_{\sigma\alpha\beta\gamma}\,, \tag{1.83}$$

which in the accompanying frame of reference, taking into account that $\sqrt{-g} = \sqrt{h}\,\sqrt{g_{00}}$, take the form

$$\varepsilon^{ikm} = b_0 E^{0ikm} = \sqrt{g_{00}}\, E^{0ikm} = \frac{e^{ikm}}{\sqrt{h}}\,, \tag{1.84}$$

$$\varepsilon_{ikm} = b^0 E_{0ikm} = \frac{E_{0ikm}}{\sqrt{g_{00}}} = e_{ikm}\sqrt{h}\,. \tag{1.85}$$

With its help we can build chr.inv.-pseudotensors. For instance, taking the antisymmetric chr.inv.-tensor of the space rotation A_{ik}, we obtain the chr.inv.-pseudovector of angular velocities of this rotation $\Omega^{*i} = \frac{1}{2}\,\varepsilon^{ikm} A_{km}$.

§1.4 Differential and derivative to the direction

In geometry a *differential* of a function is its variation between two infinitely close points with coordinates x^α and $x^\alpha + dx^\alpha$. Respectively, the *absolute differential* in a n-dimensional space is the variation of a n-dimensional quantity between two infinitely close points of n-dimensional coordinates in this space. In order to define the infinitesimal variation of a tensor quantity, we can not use simple "difference" between its numerical values in the points x^α and $x^\alpha + dx^\alpha$, because tensor algebra does not define the ratio between values of a tensor at different points in space. This ratio can be defined only using rules of transformation of tensors from one reference frame into another. So, differential operators and the results of their application to tensors must be tensors.

The absolute differential of a scalar φ is the scalar quantity

$$\mathrm{D}\varphi = \frac{\partial\varphi}{\partial x^\alpha}dx^\alpha, \tag{1.86}$$

which is the same that the regular differential $d\varphi$. Expressing the quantity with the terms of chronometric invariants, we obtain the formula

$$\mathrm{D}\varphi = \frac{{}^*\partial\varphi}{\partial t}d\tau + \frac{{}^*\partial\varphi}{\partial x^i}dx^i, \tag{1.87}$$

where is, aside for three-dimensional observable differential, also an additional term, which takes into account that the absolute displacement $\mathrm{D}\varphi$ is dependent from the flow of physical observable time $d\tau$.

The absolute differential of a contravariant vector A^α is

$$\mathrm{D}A^\alpha = \nabla_\sigma A^\alpha dx^\sigma = \frac{\partial A^\alpha}{\partial x^\sigma}dx^\sigma + \Gamma^\alpha_{\mu\sigma}A^\mu dx^\sigma = dA^\alpha + \Gamma^\alpha_{\mu\sigma}A^\mu dx^\sigma, \tag{1.88}$$

where $\nabla_\sigma A^\alpha$ is the absolute derivative of the vector A^α

$$\nabla_\sigma A^\alpha = \frac{\partial A^\alpha}{\partial x^\sigma} + \Gamma^\alpha_{\mu\sigma}A^\mu. \tag{1.89}$$

Let us deduce chr.inv.-projections of the absolute differential of the vector A^α

$$T = b_\alpha \mathrm{D}A^\alpha = \frac{g_{0\alpha}\mathrm{D}A^\alpha}{\sqrt{g_{00}}}\,, \qquad B^i = h^i_\alpha \mathrm{D}A^\alpha. \tag{1.90}$$

Denoting chr.inv.-projections of the vector A^α as follows

$$\varphi = \frac{A_0}{\sqrt{g_{00}}}\,, \qquad q^i = A^i, \tag{1.91}$$

we obtain the rest components of the vector A^α, which are

$$A^0 = \frac{\varphi + \frac{1}{c}\,v_i q^i}{\sqrt{g_{00}}}\,, \qquad A_i = -q_i - \frac{\varphi}{c}\,v_i\,. \tag{1.92}$$

Actually, to deduce the chr.inv.-projections (1.90) we need formulas for the Christoffel symbols in the accompanying reference frame. So forth, after some algebra we obtain the Christofel symbols of the 1st kind $\Gamma_{\alpha\mu,\nu}$ in the accompanying reference frame

$$\Gamma_{00,0} = -\frac{1}{c^3}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial \mathrm{w}}{\partial t}\,, \tag{1.93}$$

$$\Gamma_{00,i} = \frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2 F_i + \frac{1}{c^4}\,v_i\frac{\partial \mathrm{w}}{\partial t}\,, \tag{1.94}$$

$$\Gamma_{0i,0} = -\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial \mathrm{w}}{\partial x^i}\,, \tag{1.95}$$

$$\Gamma_{0i,j} = -\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(D_{ij}+A_{ij}+\frac{1}{c^2}F_j v_i\right)+\frac{1}{c^3}\,v_j\frac{\partial \mathrm{w}}{\partial x^i}\,, \tag{1.96}$$

$$\Gamma_{ij,0} = \frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left[D_{ij}-\frac{1}{2}\left(\frac{\partial v_j}{\partial x^i}+\frac{\partial v_i}{\partial x^j}\right)+\frac{1}{2c^2}\left(F_i v_j+F_j v_i\right)\right], \tag{1.97}$$

$$\begin{aligned}\Gamma_{ij,k} = -\Delta_{ij,k} + \frac{1}{c^2}\Bigg[v_i A_{jk} + v_j A_{ik} + \frac{1}{2}\,v_k\left(\frac{\partial v_j}{\partial x^i}+\frac{\partial v_i}{\partial x^j}\right) - \\ -\frac{1}{2c^2}\,v_k\left(F_i v_j + F_j v_i\right)\Bigg] + \frac{1}{c^4}F_k v_i v_j\,,\end{aligned} \tag{1.98}$$

and also the Christoffel symbols of the 2nd kind $\Gamma^{\alpha}_{\mu\nu}$, namely

$$\Gamma^0_{00} = -\frac{1}{c^3}\left[\frac{1}{\sqrt{g_{00}}}\frac{\partial \mathrm{w}}{\partial t}+\sqrt{g_{00}}\,v_k F^k\right], \tag{1.99}$$

$$\Gamma^k_{00} = -\frac{1}{c^2}\,g_{00}\,F^k, \tag{1.100}$$

$$\Gamma^0_{0i} = \frac{1}{c^2}\left[-\frac{1}{\sqrt{g_{00}}}\frac{\partial \mathrm{w}}{\partial x^i}+v_k\left(D^k_i + A_{i\cdot}^{\cdot k}+\frac{1}{c^2}\,v_i F^k\right)\right], \tag{1.101}$$

$$\Gamma^k_{0i} = \frac{1}{c}\,\sqrt{g_{00}}\left(D^k_i + A_{i\cdot}^{\cdot k}+\frac{1}{c^2}\,v_i F^k\right), \tag{1.102}$$

$$\begin{aligned}\Gamma^0_{ij} = -\frac{1}{c\sqrt{g_{00}}}\Bigg\{-D_{ij}+\frac{1}{c^2}\,v_n\,\times \\ \times\left[v_j\left(D^n_i + A_{i\cdot}^{\cdot n}\right)+v_i\left(D^n_j + A_{j\cdot}^{\cdot n}\right)+\frac{1}{c^2}\,v_i v_j F^n\right]+ \\ +\frac{1}{2}\left(\frac{\partial v_i}{\partial x^j}+\frac{\partial v_j}{\partial x^i}\right)-\frac{1}{2c^2}\left(F_i v_j + F_j v_i\right)-\Delta^n_{ij}\,v_n\Bigg\},\end{aligned} \tag{1.103}$$

$$\Gamma^k_{ij} = \Delta^k_{ij} - \frac{1}{c^2}\left[v_i\left(D^k_j + A_{j\cdot}^{\cdot k}\right)+v_j\left(D^k_i + A_{i\cdot}^{\cdot k}\right)+\frac{1}{c^2}\,v_i v_j F^k\right], \tag{1.104}$$

where Δ^i_{jk} are the Christoffel chr.inv.-symbols (1.25).

Substituting the formulas into (1.90), we arrive to the chr.inv.-projections of the absolute differential of A^α in the final form

$$T = b_\alpha\,\mathrm{D}A^\alpha = d\varphi + \frac{1}{c}\left(-F_i q^i d\tau + D_{ik} q^i dx^k\right), \tag{1.105}$$

$$\begin{aligned}B^i = h^i_\sigma\,\mathrm{D}A^\sigma = dq^i + \left(\frac{\varphi}{c}dx^k + q^k d\tau\right)\left(D^i_k + A_{k\cdot}^{\cdot i}\right) - \\ -\frac{\varphi}{c}F^i d\tau + \Delta^i_{mk}\,q^m dx^k.\end{aligned} \tag{1.106}$$

A *derivative to a direction* of a function is its change in respect of an elementary displacement along the given direction. The *absolute derivative to the direction* in a n-dimensional space is a change of a n-dimensional quantity in respect of an elementary n-dimensional interval along the given direction. For instance, the absolute derivative of a scalar function φ to a direction, defined by a curve $x^\alpha = x^\alpha(\rho)$, where ρ is a non-zero parameter along this curve, shows the "rate" of change of this function

$$\frac{\mathrm{D}\varphi}{d\rho} = \frac{\partial\varphi}{\partial x^\alpha}\frac{dx^\alpha}{d\rho} = \frac{d\varphi}{d\rho}\,, \tag{1.107}$$

that in the accompanying reference frame is

$$\frac{\mathrm{D}\varphi}{d\rho} = \frac{^*\partial\varphi}{\partial t}\frac{d\tau}{d\rho} + \frac{^*\partial\varphi}{\partial x^i}\frac{dx^i}{d\rho}\,. \tag{1.108}$$

The absolute derivative of a vector A^α to the direction of a curve $x^\alpha = x^\alpha(\rho)$ is the quantity

$$\frac{\mathrm{D}A^\alpha}{d\rho} = \nabla_\sigma A^\alpha\frac{dx^\sigma}{d\rho} = \frac{dA^\alpha}{d\rho} + \Gamma^\alpha_{\mu\sigma}A^\mu\frac{dx^\sigma}{d\rho}\,, \tag{1.109}$$

which chr.inv.-projections are

$$b_\alpha\frac{\mathrm{D}A^\alpha}{d\rho} = \frac{d\varphi}{d\rho} + \frac{1}{c}\left(-F_i q^i\frac{d\tau}{d\rho} + D_{ik} q^i\frac{dx^k}{d\rho}\right), \tag{1.110}$$

$$\begin{aligned}h^i_\sigma\frac{\mathrm{D}A^\sigma}{d\rho} = \frac{dq^i}{d\rho} + \left(\frac{\varphi}{c}\frac{dx^k}{d\rho} + q^k\frac{d\tau}{d\rho}\right)\left(D^i_k + A_{k\cdot}^{\cdot i}\right) - \\ -\frac{\varphi}{c}F^i\frac{d\tau}{d\rho} + \Delta^i_{mk}\,q^m\frac{dx^k}{d\rho}\,.\end{aligned} \tag{1.111}$$

§1.5 Absolute divergence and rotor

A *divergence* of a tensor field is its "change" along a coordinate axis. The *absolute divergence* of a n-dimensional tensor field is its divergence in a n-dimensional space.

Algebraically, a divergence of a tensor field is a result of contraction of the field tensor with the operator of absolute derivation ∇. So, the divergence of a vector field A^{σ} is the scalar

$$\nabla_{\sigma} A^{\sigma} = \frac{\partial A^{\sigma}}{\partial x^{\sigma}} + \Gamma^{\sigma}_{\sigma\mu} A^{\mu}, \tag{1.112}$$

while the divergence of a field of the 2nd rank is the vector

$$\nabla_{\sigma} F^{\sigma\alpha} = \frac{\partial F^{\sigma\alpha}}{\partial x^{\sigma}} + \Gamma^{\sigma}_{\sigma\mu} F^{\alpha\mu} + \Gamma^{\alpha}_{\sigma\mu} F^{\sigma\mu}, \tag{1.113}$$

where, as it can be proven, the $\Gamma^{\sigma}_{\sigma\mu}$ equals

$$\Gamma^{\sigma}_{\sigma\mu} = g^{\sigma\rho}\,\Gamma_{\mu\sigma,\rho} = \frac{1}{2}\, g^{\sigma\rho}\left(\frac{\partial g_{\mu\rho}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\rho}}{\partial x^{\mu}} - \frac{\partial g_{\mu\sigma}}{\partial x^{\rho}}\right) = \frac{\partial \ln\sqrt{-g}}{\partial x^{\mu}}\,. \tag{1.114}$$

To prove this, we take into account that σ and ρ here are free indices, so they can change their sites. As a result, after contraction with the tensor $g^{\rho\sigma}$, the first and the last terms in the brackets cancel each other and $\Gamma^{\sigma}_{\sigma\mu}$ takes the form

$$\Gamma^{\sigma}_{\sigma\mu} = \frac{1}{2}\, g^{\rho\sigma} \frac{\partial g_{\rho\sigma}}{\partial x^{\mu}}\,. \tag{1.115}$$

The quantities $g^{\rho\sigma}$ are components of a tensor reciprocal to the tensor $g_{\rho\sigma}$. For this reason, each component of the matrix $g^{\rho\sigma}$ is

$$g^{\rho\sigma} = \frac{a^{\rho\sigma}}{g}\,, \qquad g = \det\|g_{\rho\sigma}\|\,, \tag{1.116}$$

where $a^{\rho\sigma}$ is the algebraic cofactor of the matrix' element with indices $\rho\sigma$, equal to $(-1)^{\rho+\sigma}$, multiplied by the determinant of the matrix obtained by crossing the row and the column with the numbers σ and ρ out of the matrix $g_{\rho\sigma}$. As a result we obtain $a^{\rho\sigma} = g g^{\rho\sigma}$. Because the determinant of the fundamental metric tensor $g = \det\|g_{\rho\sigma}\|$ by definition is

$$g = \sum_{\alpha_0\ldots\alpha_3} (-1)^{N(\alpha_0\ldots\alpha_3)}\, g_{0(\alpha_0)}\, g_{1(\alpha_1)}\, g_{2(\alpha_2)}\, g_{3(\alpha_3)}\,, \tag{1.117}$$

the value dg will be $dg = a^{\rho\sigma} dg_{\rho\sigma} = g g^{\rho\sigma} dg_{\rho\sigma}$, that finally gives

$$\frac{dg}{g} = g^{\rho\sigma} dg_{\rho\sigma}\,. \tag{1.118}$$

Integration of the left part gives $\ln(-g)$, because the g is negative, while logarithm is defined for only positive functions. Then $d\ln(-g) = \frac{dg}{g}$. Taking $(-g)^{\frac{1}{2}} = \frac{1}{2}\ln(-g)$ into account, we arrive to

$$d\ln\sqrt{-g} = \frac{1}{2}\, g^{\rho\sigma} dg_{\rho\sigma}\,, \tag{1.119}$$

so $\Gamma^{\sigma}_{\sigma\mu}$ (1.115) takes form

$$\Gamma^{\sigma}_{\sigma\mu} = \frac{1}{2}\, g^{\rho\sigma} \frac{\partial g_{\rho\sigma}}{\partial x^{\mu}} = \frac{\partial \ln\sqrt{-g}}{\partial x^{\mu}}\,, \tag{1.120}$$

which has been proven (1.114).

Now we are going to deduce the divergence of a vector field A^{α} (1.112) and of a tensor field of the 2nd rank $F^{\alpha\beta}$ (1.113) in chr.inv.-form.

The divergence of a vector field A^{α} is scalar, hence $\nabla_{\sigma} A^{\sigma}$ can not be projected on time lines and the spatial section, while it is enough to express the formula through chr.inv.-quantities. Assuming φ and q^i notations for chr.inv.-projections of the vector A^{α} and taking into account that

$$\sqrt{-g} = \sqrt{h}\,\sqrt{g_{00}}\,, \tag{1.121}$$

after some algebra we have

$$\nabla_{\sigma} A^{\sigma} = \frac{1}{c}\left(\frac{{}^*\partial\varphi}{\partial t} + \varphi D\right) + \frac{{}^*\partial q^i}{\partial x^i} + q^i \frac{{}^*\partial \ln\sqrt{h}}{\partial x^i} - \frac{1}{c^2} F_i q^i, \tag{1.122}$$

where the third term equals the Christoffel chr.inv.-symbols Δ^k_{ji}, contracted by two symbols, so that

$$\frac{{}^*\partial \ln\sqrt{h}}{\partial x^i} = \Delta^j_{ji}\,. \tag{1.123}$$

Because of the evident analogy with the absolute divergence of a four-dimensional vector field, Zelmanov called

$${}^*\nabla_i\, q^i = \frac{{}^*\partial q^i}{\partial x^i} + q^i \frac{{}^*\partial \ln\sqrt{h}}{\partial x^i} = \frac{{}^*\partial q^i}{\partial x^i} + q^i \Delta^j_{ji} \tag{1.124}$$

the *chr.inv.-divergence* of a three-dimensional vector field q^i. In addition to this, he called

$$^*\widetilde{\nabla}_i\, q^i = {}^*\nabla_i\, q^i - \frac{1}{c^2} F_i q^i, \tag{1.125}$$

the *physical chr.inv.-divergence*, in which the 2nd term takes into account that the pace of time is different on the opposite walls of an elementary volume. Therefore, if we measure durations of time intervals at the opposite walls of the volume, the beginnings and the ends of the interval will not coincide making them invalid for comparison. Synchronization of clocks, realizing the property of chronometric invariance, at the opposite walls of the volume will give the true picture, i. e. the measured durations of the intervals will be different.

The final equation for $\nabla_\sigma A^\sigma$ will be

$$\nabla_\sigma A^\sigma = \frac{1}{c}\left(\frac{^*\partial\varphi}{\partial t} + \varphi D\right) + {}^*\widetilde{\nabla}_i\, q^i. \tag{1.126}$$

The divergence of an antisymmetric tensor field $F^{\alpha\beta} = -F^{\beta\alpha}$ can be represented in the form

$$\nabla_\sigma F^{\sigma\alpha} = \frac{\partial F^{\sigma\alpha}}{\partial x^\sigma} + \Gamma^\sigma_{\sigma\mu} F^{\alpha\mu} + \Gamma^\alpha_{\sigma\mu} F^{\sigma\mu} = \frac{\partial F^{\sigma\alpha}}{\partial x^\sigma} + \frac{\partial \ln\sqrt{-g}}{\partial x^\mu} F^{\alpha\mu}, \tag{1.127}$$

where the third term $\Gamma^\alpha_{\sigma\mu} F^{\sigma\mu}$ is zero because of contraction of the Christoffel symbols $\Gamma^\alpha_{\sigma\mu}$, symmetric by lower indices $\sigma\mu$, and the antisymmetric tensor $F^{\sigma\mu}$ is zero (just like as a symmetric and an antisymmetric tensors).

The divergence $\nabla_\sigma F^{\sigma\alpha}$ is four-dimensional vector, so its chr.inv.-projections are

$$T = b_\alpha \nabla_\sigma F^{\sigma\alpha}, \qquad B^i = h^i_\alpha \nabla_\sigma F^{\sigma\alpha} = \nabla_\sigma F^{i\alpha}. \tag{1.128}$$

Denoting chr.inv.-projections of the field tensor $F^{\alpha\beta}$ as

$$E^i = \frac{F_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}}, \qquad H^{ik} = F^{ik}, \tag{1.129}$$

we obtain the rest non-zero components of the tensor

$$F_{0\cdot}^{\cdot 0} = \frac{1}{c} v_k E^k, \tag{1.130}$$

$$F_{k\cdot}^{\cdot 0} = \frac{1}{\sqrt{g_{00}}}\left(E_i - \frac{1}{c} v_n H_{k\cdot}^{\cdot n} - \frac{1}{c^2} v_k v_n E^n\right), \tag{1.131}$$

$$F^{0i} = \frac{E^i - \frac{1}{c} v_k H^{ik}}{\sqrt{g_{00}}}, \qquad F_{0i} = -\sqrt{g_{00}}\, E_i\,, \tag{1.132}$$

$$F_{i\cdot}^{\cdot k} = -H_{i\cdot}^{\cdot k} - \frac{1}{c} v_i E^k, \quad F_{ik} = H_{ik} + \frac{1}{c}\left(v_i E_k - v_k E_i\right), \tag{1.133}$$

and the square of the tensor $F^{\alpha\beta}$ is

$$F_{\alpha\beta} F^{\alpha\beta} = H_{ik} H^{ik} - 2 E_i E^i. \tag{1.134}$$

Substituting these components into (1.128), after some algebra we arrive to

$$T = \frac{\nabla_\sigma F_{0\cdot}^{\cdot\sigma}}{\sqrt{g_{00}}} = \frac{^*\partial E^i}{\partial x^i} + E^i \frac{^*\partial \ln\sqrt{h}}{\partial x^i} - \frac{1}{c} H^{ik} A_{ik}\,, \tag{1.135}$$

$$\begin{aligned} B^i = \nabla_\sigma F^{\sigma i} = \frac{^*\partial H^{ik}}{\partial x^k} + H^{ik} \frac{^*\partial \ln\sqrt{h}}{\partial x^k} - \frac{1}{c^2} F_k H^{ik} - \\ - \frac{1}{c}\left(\frac{^*\partial E^i}{\partial t} + D E^i\right), \end{aligned} \tag{1.136}$$

where A_{ik} is the antisymmetric chr.inv.-tensor of the space rotation. In the other notation, we have

$$T = {}^*\nabla_i\, E^i - \frac{1}{c} H^{ik} A_{ik}\,, \tag{1.137}$$

$$B^i = {}^*\widetilde{\nabla}_k\, H^{ik} - \frac{1}{c}\left(\frac{^*\partial E^i}{\partial t} + D E^i\right). \tag{1.138}$$

So forth, we deduce chr.inv.-projections of the divergence of a pseudotensor $F^{*\alpha\beta}$, dual to the given antisymmetric tensor $F^{\alpha\beta}$

$$F^{*\alpha\beta} = \frac{1}{2} E^{\alpha\beta\mu\nu} F_{\mu\nu}\,, \quad F_{*\alpha\beta} = \frac{1}{2} E_{\alpha\beta\mu\nu} F^{\mu\nu}. \tag{1.139}$$

We denote chr.inv.-projections of the pseudotensor $F^{*\alpha\beta}$ as

$$H^{*i} = \frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}}, \qquad E^{*ik} = F^{*ik}, \tag{1.140}$$

so there are evident correlations $H^{*i} \sim H^{ik}$ and $E^{*ik} \sim E^i$ between the quantities (1.140) and the chr.inv.-projections of the antisymmetric tensor $F^{\alpha\beta}$ (1.129), because of duality of the given tensors $F^{\alpha\beta}$ and $F^{*\alpha\beta}$. Therefore, given that

$$\frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}} = \frac{1}{2}\,\varepsilon^{ipq} H_{pq}\,, \qquad F^{*ik} = -\,\varepsilon^{ikp} E_p\,, \tag{1.141}$$

the rest components of the pseudotensor $F^{*\alpha\beta}$ are

$$F_{0\cdot}^{*\cdot 0} = \frac{1}{2c}\,v_k\,\varepsilon^{kpq}\left[H_{pq} + \frac{1}{c}\left(v_p E_q - v_q E_p\right)\right], \tag{1.142}$$

$$\begin{aligned} F_{i\cdot}^{*\cdot 0} = \frac{1}{2\sqrt{g_{00}}}\Big[\varepsilon_{i\cdot}^{\cdot pq} H_{pq} + \frac{1}{c}\,\varepsilon_{i\cdot}^{\cdot pq}\left(v_p E_q - v_q E_p\right) - \\ -\frac{1}{c^2}\,\varepsilon^{kpq} v_i v_k H_{pq} - \frac{1}{c^3}\,\varepsilon^{kpq} v_i v_k \left(v_p E_q - v_q E_p\right)\Big], \end{aligned} \tag{1.143}$$

$$F^{*0i} = \frac{1}{2\sqrt{g_{00}}}\,\varepsilon^{ipq}\left[H_{pq} + \frac{1}{c}\left(v_p E_q - v_q E_p\right)\right], \tag{1.144}$$

$$F_{*0i} = \frac{1}{2}\sqrt{g_{00}}\,\varepsilon_{ipq} H^{pq}, \tag{1.145}$$

$$F_{i\cdot}^{*\cdot k} = \varepsilon_{i\cdot}^{\cdot kp} E_p - \frac{1}{2c}\,v_i\,\varepsilon^{kpq} H_{pq} - \frac{1}{c^2}\,v_i v_m\,\varepsilon^{mkp} E_p\,, \tag{1.146}$$

$$F_{*ik} = \varepsilon_{ikp}\left(E^p - \frac{1}{c}\,v_q H^{pq}\right), \tag{1.147}$$

while its square is

$$F_{*\alpha\beta} F^{*\alpha\beta} = \varepsilon^{ipq}\left(E_p H_{iq} - E_i H_{pq}\right). \tag{1.148}$$

Then the chr.inv.-projections of the divergence $\nabla_\sigma F^{*\sigma\alpha}$ are

$$\frac{\nabla_\sigma F_{0\cdot}^{*\cdot\sigma}}{\sqrt{g_{00}}} = \frac{{}^*\partial H^{*i}}{\partial x^i} + H^{*i}\frac{{}^*\partial \ln\sqrt{h}}{\partial x^i} - \frac{1}{c}\,E^{*ik} A_{ik}\,, \tag{1.149}$$

$$\begin{aligned} \nabla_\sigma F^{*\sigma i} = \frac{{}^*\partial E^{*ik}}{\partial x^i} + E^{*ik}\frac{{}^*\partial \ln\sqrt{h}}{\partial x^k} - \frac{1}{c^2}\,F_k E^{*ik} - \\ -\frac{1}{c}\left(\frac{{}^*\partial H^{*i}}{\partial t} + D H^{*i}\right), \end{aligned} \tag{1.150}$$

or, in the other notation

$$\frac{\nabla_\sigma F_{0\cdot}^{*\cdot\sigma}}{\sqrt{g_{00}}} = {}^*\nabla_i\, H^{*i} - \frac{1}{c}\,E^{*ik} A_{ik}\,, \tag{1.151}$$

$$\nabla_\sigma F^{*\sigma i} = {}^*\widetilde{\nabla}_k\, E^{*ik} - \frac{1}{c}\left(\frac{{}^*\partial H^{*i}}{\partial t} + D H^{*i}\right). \tag{1.152}$$

Aside for these we actually need to know chr.inv.-projections of the divergence of symmetric tensors of the 2nd rank. So forth, denoting chr.inv.-projections of a symmetric tensor $T^{\alpha\beta}$ like as Zelmanov did it, namely

$$\frac{T_{00}}{g_{00}} = \rho\,, \qquad \frac{T_0^i}{\sqrt{g_{00}}} = K^i, \qquad T^{ik} = N^{ik}, \tag{1.153}$$

after some algebra we obtain

$$\frac{\nabla_\sigma T_0^\sigma}{\sqrt{g_{00}}} = \frac{{}^*\partial\rho}{\partial t} + \rho D + D_{ik} N^{ik} + c\,{}^*\nabla_i\, K^i - \frac{2}{c}\,F_i K^i, \tag{1.154}$$

$$\begin{aligned} \nabla_\sigma T^{\sigma i} = c\,\frac{{}^*\partial K^i}{\partial t} + c\,D K^i + 2c\left(D_k^i + A_{k\cdot}^{\cdot i}\right) K^k + \\ + c^2\,{}^*\nabla_k\, N^{ik} - F_k N^{ik} - \rho F^i. \end{aligned} \tag{1.155}$$

A *rotor* of a tensor field is that difference of covariant derivatives of the field tensor, which from geometric viewpoint is the vortex (rotation) of the field. Accordingly, the *absolute rotor* is the rotor of a n-dimensional tensor field in a n-dimensional space. The rotor of a four-dimensional vector field A^α is a covariant antisymmetric tensor of the 2nd rank, defined as follows*

$$F_{\mu\nu} = \nabla_\mu\, A_\nu - \nabla_\nu\, A_\mu = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\,, \tag{1.156}$$

where $\nabla_\mu\, A_\nu$ is the absolute derivative of the A_α with respect to the coordinate x^μ

$$\nabla_\mu\, A_\nu = \frac{\mathrm{D}A_\nu}{dx^\mu} = \frac{dA_\nu}{dx^\mu} - \Gamma_{\nu\tau}^\sigma A_\sigma \frac{dx^\tau}{dx^\mu} = \frac{\partial A_\nu}{\partial x^\mu} - \Gamma_{\nu\mu}^\sigma A_\sigma\,. \tag{1.157}$$

*For example, see §98 in Raschewski's well-known book *Riemannian Geometry and Tensor Analysis* [10]. However naturally, rotor is not the tensor (1.156), but its dual pseudotensor (1.158), because of the invariance in respect of reflection is necessary for any rotations.

The rotor, contracted with the four-dimensional absolutely antisymmetric discriminant tensor $E^{\alpha\beta\mu\nu}$ (1.77), is the pseudotensor

$$F^{*\alpha\beta} = E^{\alpha\beta\mu\nu}\left(\nabla_\mu A_\nu - \nabla_\nu A_\mu\right) = E^{\alpha\beta\mu\nu}\left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right). \qquad (1.158)$$

So forth, we deduce components of an arbitrary rotor $F_{\mu\nu}$. Taking into account that $F_{00} = F^{00} = 0$ just like for any other antisymmetric tensor, after some algebra we obtain

$$F_{0i} = \sqrt{g_{00}}\left(\frac{\varphi}{c^2}F_i - \frac{{}^*\partial\varphi}{\partial x^i} - \frac{1}{c}\frac{{}^*\partial q_i}{\partial t}\right), \qquad (1.159)$$

$$\begin{aligned} F_{ik} = \frac{{}^*\partial q_i}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^i} + \frac{\varphi}{c}\left(\frac{\partial v_i}{\partial x^k} - \frac{\partial v_k}{\partial x^i}\right) + \\ + \frac{1}{c}\left(v_i\frac{{}^*\partial\varphi}{\partial x^k} - v_k\frac{{}^*\partial\varphi}{\partial x^i}\right) + \frac{1}{c^2}\left(v_i\frac{{}^*\partial q_k}{\partial t} - v_k\frac{{}^*\partial q_i}{\partial t}\right), \end{aligned} \qquad (1.160)$$

$$F_{0\cdot}^{\cdot 0} = -\frac{\varphi}{c^3}\,v_k F^k + \frac{1}{c}\,v^k\left(\frac{{}^*\partial\varphi}{\partial x^k} + \frac{1}{c}\frac{{}^*\partial q_k}{\partial t}\right), \qquad (1.161)$$

$$\begin{aligned} F_{k\cdot}^{\cdot 0} = -\frac{1}{\sqrt{g_{00}}}\left[\frac{\varphi}{c^2}F_k - \frac{{}^*\partial\varphi}{\partial x^k} - \frac{1}{c}\frac{{}^*\partial q_k}{\partial t} + \right. \\ + \frac{2\varphi}{c^2}\,v^m A_{mk} + \frac{1}{c^2}\,v_k v^m\left(\frac{{}^*\partial\varphi}{\partial x^m} + \frac{1}{c}\frac{{}^*\partial q_m}{\partial t}\right) - \\ \left. - \frac{1}{c}\,v^m\left(\frac{{}^*\partial q_m}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^m}\right) - \frac{\varphi}{c^4}\,v_k v_m F^m\right], \end{aligned} \qquad (1.162)$$

$$\begin{aligned} F_{k\cdot}^{\cdot i} = h^{im}\left(\frac{{}^*\partial q_m}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^m}\right) - \frac{1}{c}\,h^{im}v_k\frac{{}^*\partial\varphi}{\partial x^m} - \\ - \frac{1}{c^2}\,h^{im}v_k\frac{{}^*\partial q_m}{\partial t} + \frac{\varphi}{c^3}\,v_k F^i + \frac{2\varphi}{c}A_{k\cdot}^{\cdot i}, \end{aligned} \qquad (1.163)$$

$$\begin{aligned} F^{0k} = \frac{1}{\sqrt{g_{00}}}\left[h^{km}\left(\frac{{}^*\partial\varphi}{\partial x^m} + \frac{1}{c}\frac{{}^*\partial q_m}{\partial t}\right) - \frac{\varphi}{c^2}F^k + \right. \\ \left. + \frac{1}{c}\,v^n h^{mk}\left(\frac{{}^*\partial q_n}{\partial x^m} - \frac{{}^*\partial q_m}{\partial x^n}\right) - \frac{2\varphi}{c^2}\,v_m A^{mk}\right], \end{aligned} \qquad (1.164)$$

$$\frac{F_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} = \frac{g^{i\alpha}F_{0\alpha}}{\sqrt{g_{00}}} = h^{ik}\left(\frac{{}^*\partial\varphi}{\partial x^k} + \frac{1}{c}\frac{{}^*\partial q_k}{\partial t}\right) - \frac{\varphi}{c^2}F^i, \qquad (1.165)$$

$$F^{ik} = g^{i\alpha}g^{k\beta}F_{\alpha\beta} = h^{im}h^{kn}\left(\frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m}\right) - \frac{2\varphi}{c}A^{ik}, \qquad (1.166)$$

where (1.165) and (1.166) are chr.inv.-projections of the rotor $F_{\mu\nu}$. Respectively, chr.inv.-projections of its dual pseudotensor $F^{*\alpha\beta}$ are

$$\frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}} = \frac{g_{0\alpha}F^{*\alpha i}}{\sqrt{g_{00}}} = \varepsilon^{ikm}\left[\frac{1}{2}\left(\frac{{}^*\partial q_k}{\partial x^m} - \frac{{}^*\partial q_m}{\partial x^k}\right) - \frac{\varphi}{c}A_{km}\right], \qquad (1.167)$$

$$F^{*ik} = \varepsilon^{ikm}\left(\frac{\varphi}{c^2}F_m - \frac{{}^*\partial\varphi}{\partial x^m} - \frac{1}{c}\frac{{}^*\partial q_m}{\partial t}\right), \qquad (1.168)$$

where we deduced $F_{0\cdot}^{*\cdot i} = g_{0\alpha}F^{*\alpha i} = g_{0\alpha}E^{\alpha i\mu\nu}F_{\mu\nu}$, using already obtained components of the rotor $F_{\mu\nu}$.

§1.6 Laplace's and d'Alembert's operators

Laplace's operator is the three-dimensional operator of derivation

$$\Delta = \nabla\,\nabla = \nabla^2 = -g^{ik}\,\nabla_i\,\nabla_k\,, \qquad (1.169)$$

a four-dimensional generalization of which in pseudo-Riemannian spaces is *d'Alembert's operator*

$$\Box = g^{\alpha\beta}\,\nabla_\alpha\,\nabla_\beta\,. \qquad (1.170)$$

At first, let us apply the d'Alembert operator to a field of a four-dimensional scalar φ, because in this case the calculations will be much simpler (the absolute derivative of a scalar field $\nabla_\alpha\,\varphi$ does not contain the Christoffel symbols, becoming regular derivative)

$$\Box\varphi = g^{\alpha\beta}\,\nabla_\alpha\,\nabla_\beta\,\varphi = g^{\alpha\beta}\frac{\partial\varphi}{\partial x^\alpha}\left(\frac{\partial\varphi}{\partial x^\beta}\right) = g^{\alpha\beta}\frac{\partial^2\varphi}{\partial x^\alpha\partial x^\beta}\,. \qquad (1.171)$$

Components of the fundamental metric tensor $g^{\alpha\beta}$ will be formulated with chr.inv.-quantities — see formula (1.20) for the components g^{ik} and g^{0i}, and footnote in p. 10 for the component g^{00}. So forth, we substitute the necessary components of the metric tensor into the initial formula (1.171). As a result, we obtain a formula for d'Alembertian of the scalar field φ, where all terms are chr.inv.-quantities. This formula, taking asterisk as a brand of that all its terms are filled in chr.inv.-form, is

$${}^*\Box\,\varphi = \frac{1}{c^2}\frac{{}^*\partial^2\varphi}{\partial t^2} - h^{ik}\frac{{}^*\partial^2\varphi}{\partial x^i\partial x^k} = \frac{1}{c^2}\frac{{}^*\partial^2\varphi}{\partial t^2} - {}^*\Delta\,, \qquad (1.172)$$

where we denote ${}^*\!\Delta$ the Laplace chr.inv.-operator

$$ {}^*\!\Delta = -g^{ik}\,{}^*\nabla_i\,{}^*\nabla_k = h^{ik}\,\frac{{}^*\partial^2}{\partial x^i \partial x^k}\,. \tag{1.173}$$

Now we apply the d'Alembert operator again. In this time, the operator will be applied to an arbitrary four-dimensional vector field A^α, namely

$$ \Box\, A^\alpha = g^{\mu\nu}\,\nabla_\mu\,\nabla_\nu\,A^\alpha. \tag{1.174}$$

Because $\Box\, A^\alpha$ is four-dimensional vector, chr.inv.-projections of this quantity on time lines and the spatial section are

$$ T = b_\sigma\,\Box\, A^\sigma = b_\sigma\, g^{\mu\nu}\,\nabla_\mu\,\nabla_\nu\,A^\sigma, \tag{1.175}$$

$$ B^i = h^i_\sigma\,\Box\, A^\sigma = h^i_\sigma\, g^{\mu\nu}\,\nabla_\mu\,\nabla_\nu\,A^\sigma. \tag{1.176}$$

The chr.inv.-projections had been deduced by Borissova in 1998 as a result of very difficult algebra*. They have the form

$$ \begin{gathered} T = {}^*\Box\,\varphi - \frac{1}{c^3}\frac{{}^*\partial}{\partial t}\left(F_k q^k\right) - \frac{1}{c^3}F_i\frac{{}^*\partial q^i}{\partial t} + \frac{1}{c^2}F^i\frac{{}^*\partial\varphi}{\partial x^i} + \\ + h^{ik}\Delta^m_{ik}\frac{{}^*\partial\varphi}{\partial x^m} - h^{ik}\frac{1}{c}\frac{{}^*\partial}{\partial x^i}\left[\left(D_{kn}+A_{kn}\right)q^n\right] + \frac{D}{c^2}\frac{{}^*\partial\varphi}{\partial t} - \\ - \frac{1}{c}D^k_m\frac{{}^*\partial q^m}{\partial x^k} + \frac{2}{c^3}A_{ik}F^i q^k + \frac{\varphi}{c^4}F_i F^i - \frac{\varphi}{c^2}D_{mk}D^{mk} - \\ - \frac{D}{c^3}F_m q^m - \frac{1}{c}\Delta^m_{kn}D^k_m q^n + \frac{1}{c}h^{ik}\Delta^m_{ik}\left(D_{mn}+A_{mn}\right)q^n, \end{gathered} \tag{1.177}$$

$$ \begin{gathered} B^i = {}^*\Box\, A^i + \frac{1}{c^2}\frac{{}^*\partial}{\partial t}\left[\left(D^i_k + A^{\cdot i}_{k\cdot}\right)q^k\right] + \frac{D}{c^2}\frac{{}^*\partial q^i}{\partial t} + \\ + \frac{1}{c^2}\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\frac{{}^*\partial q^k}{\partial t} - \frac{1}{c^3}\frac{{}^*\partial}{\partial t}\left(\varphi F^i\right) - \frac{1}{c^3}F^i\frac{{}^*\partial\varphi}{\partial t} + \\ + \frac{1}{c^2}F^k\frac{{}^*\partial q^i}{\partial x^k} - \frac{1}{c}\left(D^{mi} + A^{mi}\right)\frac{{}^*\partial\varphi}{\partial x^m} + \frac{1}{c^4}q^k F_k F^i + \\ + \frac{1}{c^2}\Delta^i_{km}q^m F^k - \frac{\varphi}{c^3}DF^i + \frac{D}{c^2}\left(D^i_n + A^{\cdot i}_{n\cdot}\right)q^n - \\ - h^{km}\left\{\frac{{}^*\partial}{\partial x^k}\left(\Delta^i_{mn}q^n\right) + \frac{1}{c}\frac{{}^*\partial}{\partial x^k}\left[\varphi\left(D^i_m + A^{\cdot i}_{m\cdot}\right)\right] + \right. \\ + \left(\Delta^i_{kn}\Delta^n_{mp} - \Delta^n_{km}\Delta^i_{np}\right)q^p + \frac{\varphi}{c}\left[\Delta^i_{kn}\left(D^n_m + A^{\cdot n}_{m\cdot}\right) - \right. \\ \left. \left. - \Delta^n_{km}\left(D^i_n + A^{\cdot i}_{n\cdot}\right)\right] + \Delta^i_{kn}\frac{{}^*\partial q^n}{\partial x^m} - \Delta^n_{km}\frac{{}^*\partial q^i}{\partial x^n}\right\}, \end{gathered} \tag{1.178}$$

*To deduce chronometrically invariant d'Alembertian for a vector field in a pseudo-Riemannian space is not a trivial task, because the Christoffel symbols are not zeroes there. So, formulas for projections of the second derivatives take dozens of pages. This is one of the reasons why applications of the theory of electromagnetic field in *The Classical Theory of Fields* [5] and other books are mainly calculated in a Galilean reference frame in the Minkowski space (the space-time of the Special Theory of Relativity), where the Christoffel symbols are zeroes.

where ${}^*\Box\,\varphi$ and ${}^*\Box\,q^i$ are results of that we have applied the d'Alembert operator to the chr.inv.-projections φ and q^i of the vector A^α

$$ {}^*\Box\,\varphi = \frac{1}{c^2}\frac{{}^*\partial^2\varphi}{\partial t^2} - h^{ik}\frac{{}^*\partial^2\varphi}{\partial x^i\partial x^k}\,, \tag{1.179}$$

$$ {}^*\Box\,q^i = \frac{1}{c^2}\frac{{}^*\partial^2 q^i}{\partial t^2} - h^{km}\frac{{}^*\partial^2 q^i}{\partial x^k\partial x^m}\,. \tag{1.180}$$

D'Alembertian from a tensor field, being equalized to zero or not, gives the *d'Alembert equations* for the same field. From physical viewpoint these are the equations of propagation of waves of the field. If the d'Alembertian is not zero, the waves are enforced by the field sources like charges or currents (the d'Alembert equations with the sources). If the d'Alembert operator for the field is zero, then these are the equations of propagation of free waves of the given field, so the waves are not related to the field charges or currents. If the space-time area we are considering for the tensor field in this question is filled with also another medium, then the d'Alembert equations will gain an additional term in their right parts to characterize the media, which can be obtained from the equations that define it.

§1.7 Conclusions

Of course this brief account can not fully cover such a vast field like tensor calculus. Moreover, there is even no need in doing that here. Detailed accounts of tensor algebra and the analysis can be found in numerous mathematical books indirectly related to the General Theory of Relativity*. Besides, many specific techniques of this

*For instance, the excellent book *Riemannian Geometry and Tensor Analysis*, written by Raschewski [10].

science, which occupy substantial part of mathematical textbooks, are not used in theoretical physics. For this reason this Chapter gives only a basic introduction into tensor algebra and the analysis, which is necessary for understanding applications of the methods of chronometric invariants to the theory of fields*. For instance, if we now come across an antisymmetric tensor or a differential operator, we do not have to undertake special calculations of their components or physical observable projections, but may rather use templates already obtained in this Chapter.

*For other applications of the mathematical methods of chronometric invariants to numerous problems in the General Theory of Relativity, see [11, 12].

Chapter 2

THE FIELD OF NON-UNIFORMITY OF TIME

This Chapter looks the field of non-uniformities of time coordinates. Equations of motion, expressed through the field tensor, show that particles move along time lines because of rotation of the space itself. Maxwell-like equations of the field display its sources, which are derived from gravitation, rotations, and inhomogeneity of the space. Energy-momentum tensor of the field sets up that it is an inhomogeneous viscous media, which is in the state of ultrarelativistic gas. Waves of the field are transverse, the wave pressure is derived from mainly sub-atom processes — exciting/relaxing atoms produce the positive/negative wave pressure, that lead to testing the whole theory.

§ 2.1 Observable time density, defining the field

As it is well-known [5], $dS = m_0 c ds$ is an elementary action to displace a free mass-bearing particle of the rest-mass m_0 at an interval of four-dimensional distance ds. What is a matter doing this action? To answer this question let us substitute the square of the interval $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$ into the action. As a result we see that

$$dS = m_0 c\, ds = m_0 c \sqrt{g_{\alpha\beta}\, dx^\alpha dx^\beta}\,, \tag{2.1}$$

so the particle moves in space-time along geodesic lines (free motion), because of carrying by the field of the fundamental metric tensor $g_{\alpha\beta}$. At the same time Einstein's equations link the metric tensor $g_{\alpha\beta}$ to the energy-momentum tensor of matter through the four-dimensional curvature of space-time. This implies that gravitational field is linked to the field of the space-time metric in the frames of the General Theory of Relativity. From this reason ones regularly conclude, that the action (2.1) to displace free mass-bearing particles is produced by gravitational field.

Let us find which field will be shown by the action (2.1) as a source of free motion, if the space-time interval ds therein would be written with the terms of physical observable quantities (chronometric invariants). Using chr.inv.-formula for the interval (1.11), we can write down the action (2.1) to displace a free mass-bearing particle in the form

$$dS = m_0 c\sqrt{b_\alpha b_\beta\, dx^\alpha dx^\beta - h_{\alpha\beta}\, dx^\alpha dx^\beta}\,. \tag{2.2}$$

If the particle rests in respect of the observer's reference body, then its observable displacement along his spatial section is $dx^i = 0$, so the chr.inv.-vector of its observable velocity v^i equals zero

$$\mathrm{v}^i = \frac{dx^i}{d\tau} = 0\,. \tag{2.3}$$

Such particle moves along only its own time line. In this case, in the accompanying reference frame we have

$$h_{\alpha\beta}\, dx^\alpha dx^\beta = h_{ik}\, dx^i dx^k = 0\,, \tag{2.4}$$

hence the action is

$$dS = m_0 c\, b_\alpha dx^\alpha\,, \tag{2.5}$$

so the mass-bearing particle moves freely along time lines because of carrying by solely the vector field b^α.

What is physical sense of this field? The monad vector b^α (1.1) is the operator of projection on time lines (non-uniform, in general case) of a real observer, which accompanies to his reference body. This implies that the vector field b^α defines the geometrical structure of the real space-time along time lines. Projecting an interval of four-dimensional coordinates dx^α on the time line of a real observer in his accompanying reference frame, we obtain an interval of real physical time $d\tau$ (1.8) he observes

$$d\tau = \frac{1}{c}\, b_\alpha dx^\alpha = \sqrt{g_{00}}\, dt + \frac{g_{0i}}{c\sqrt{g_{00}}} dx^i = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt - \frac{1}{c^2}\, v_i\, dx^i. \tag{2.6}$$

If the observer get back his measurements in the same spatial point, in other word — along the same time line, then

$$d\tau = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt\,. \tag{2.7}$$

This formula (2.7) and the previous (2.6) lead us to the conclusion that components of the observer's monad vector b^α define a

"density" of physical observable time in his accompanying reference frame. As it is easy to see, the density of observable time depends on gravitational potential and, in the general case (2.6), on rotation of the space. Hence, the vector field b^α in the accompanying reference frame is the field of non-uniformity of observable time references. For this reason, we will refer to the field as the *field of density of observable time*.

In the same way, a field of the tensor $h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta$ projecting four-dimensional quantities on the observer's spatial section is the *field of density of the spatial section*.

From geometric viewpoint, we can illustrate the conclusions in this way. The vector field b^α and the tensor field $h_{\alpha\beta}$ of the accompanying reference frame of an observer, located in a four-dimensional pseudo-Riemannian space, "split" the space into time lines and a spatial section, properties of which (like as inhomogeneity, anisotopy, curvature, etc.) depend on physical properties of the observer's reference body. Being this "splitting" is processed, the field of the fundamental metric tensor $g_{\alpha\beta}$, standing the geometrical structure of this space, "splits" as well (2.2). Its "transverse component" is the field of density of time, a four-dimensional potential of which is the monad vector b^α. The "longitudinal component" of this splitting is the field of density of the spatial section.

In the case, where a free mass-bearing particle is at rest in respect of the observer and his reference body (its observable velocity in the spatial section is $\mathrm{v}^i = 0$), its four-dimensional motion is realized along only time lines. Such particle is moved by only the transverse component b^α of the splitting of $g_{\alpha\beta}$, i. e. because of only the field of time density. Therefore, if a free particle moves along the spatial section ($\mathrm{v}^i \neq 0$) because of only the field of time density, such motion is possible under only some limitations on geometrical properties of this space (see §2.4 below).

§2.2 Introducing the field tensor

Looking at components (1.18) of the four-dimensional vector potential b^α of the field of time density taken in the accompanying reference frame ($b^i = 0$), we see that its chr.inv.-projections equal, respectively

$$\varphi = \frac{b_0}{\sqrt{g_{00}}} = 1\,, \qquad q^i = b^i = 0\,. \tag{2.8}$$

In the same way that ones introduce Maxwell's tensor of electromagnetic fields, we introduce the *tensor of the field of time density* as the rotor of its four-dimensional vector potential

$$F_{\alpha\beta} = \nabla_\alpha\, b_\beta - \nabla_\beta\, b_\alpha = \frac{\partial b_\beta}{\partial x^\alpha} - \frac{\partial b_\alpha}{\partial x^\beta}\,. \tag{2.9}$$

Taking into account that $F_{00} = F^{00} = 0$ just like for any antisymmetric tensor of the 2nd rank, after some algebra we obtain the rest components of the field tensor $F_{\alpha\beta}$

$$F_{0i} = \frac{1}{c^2}\sqrt{g_{00}}\, F_i\,, \qquad F_{ik} = \frac{1}{c}\left(\frac{\partial v_i}{\partial x^k} - \frac{\partial v_k}{\partial x^i}\right), \tag{2.10}$$

$$F_{0\cdot}^{\cdot 0} = -\frac{1}{c^3}\, v_k F^k, \qquad F_{0\cdot}^{\cdot i} = -\frac{1}{c^2}\sqrt{g_{00}}\, F^i, \tag{2.11}$$

$$F_{k\cdot}^{\cdot 0} = -\frac{1}{\sqrt{g_{00}}}\left(\frac{1}{c^2}\, F_k + \frac{2}{c^2}\, v^m A_{mk} - \frac{1}{c^4}\, v_k v_m F^m\right), \tag{2.12}$$

$$F_{k\cdot}^{\cdot i} = \frac{1}{c^3}\, v_k F^i + \frac{2}{c}\, A_{k\cdot}^{\cdot i}\,, \tag{2.13}$$

$$F^{0k} = -\frac{1}{\sqrt{g_{00}}}\left(\frac{1}{c^2}\, F^k + \frac{2}{c^2}\, v_m A^{mk}\right), \qquad F^{ik} = -\frac{2}{c}\, A^{ik}. \tag{2.14}$$

From here we can easy obtain chr.inv.-projections of the field tensor. So forth we assume denotations of the projections just like for chr.inv.-projections of the Maxwell tensor [12] to display their physical sense. We will refer to the time projection

$$E^i = \frac{F_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} = -\frac{1}{c^2}\, F^i, \qquad E_i = h_{ik} E^k = -\frac{1}{c^2}\, F_i \tag{2.15}$$

of the field tensor $F_{\alpha\beta}$ (2.9) as "electric". The spatial projection

$$H^{ik} = F^{ik} = -\frac{2}{c}\, A^{ik}, \qquad H_{ik} = h_{im} h_{kn} F^{mn} = -\frac{2}{c}\, A_{ik} \tag{2.16}$$

of the field tensor will be referred to as "magnetic". So, the "electric" and the "magnetic" observable components of the field of time density display themselves as gravitational inertial force and rotation of the space, respectively. In accordance with the above, two particular cases of the field are possible. These are:

1. If the field of time density has $H_{ik} = 0$ and $E^i \neq 0$, then the field is of strictly "electric" kind. This particular case corresponds to a holonomic (non-rotating) space filled with gravitational force fields;
2. The field of time density is of "magnetic" kind, if therein $E^i = 0$ and $H_{ik} \neq 0$. This is a non-holonomic space, where fields of gravitational inertial forces are homogeneous or absent. This case is possible also if, according to chr.inv.-definition (1.16) of the force

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \qquad \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}\,, \tag{2.17}$$

its "gravitational" part (the first term, which is the force of gravitational attraction) would be fully reduced with the "inertial" part (the second term — centrifugal force of inertia).

This implies that the field of observable time density, deriving from inhomogeneity of coordinates along real time lines, is linked to the presence of the next forces:

- gravitational force — the gradient of gravitational potential, which is the first term in the chr.inv.-definition of gravitational inertial force F_i;
- centrifugal force of inertia — the second term in the F_i;
- other forces linked to the space rotation, defined by the tensor A_{ik} (1.15) — Coriolis' force [1], for instance.

In addition to the field tensor $F_{\alpha\beta}$ (2.9), we introduce the field pseudotensor $F^{*\alpha\beta}$, components of which will be used in equations of the field in §2.5, and also the field invariants. We define the pseudotensor $F^{*\alpha\beta}$ of the field of time density dual to the given field tensor $F_{\alpha\beta}$ (2.9) in the regular way

$$F^{*\alpha\beta} = \frac{1}{2}\, E^{\alpha\beta\mu\nu} F_{\mu\nu}\,, \qquad F_{*\alpha\beta} = \frac{1}{2}\, E_{\alpha\beta\mu\nu} F^{\mu\nu}, \tag{2.18}$$

where the four-dimensional completely antisymmetric discriminant tensors $E^{\alpha\beta\mu\nu} = \frac{e^{\alpha\beta\mu\nu}}{\sqrt{-g}}$ and $E_{\alpha\beta\mu\nu} = e_{\alpha\beta\mu\nu}\sqrt{-g}$ (1.77), transforming regular tensors into pseudotensors in inhomogeneous anisotropic pseudo-Riemannian spaces are not physical observable quantities. The completely antisymmetric unit tensor $e^{\alpha\beta\mu\nu}$, being defined in a Galilean reference frame in the Minkowski space [5], also has not this quality. Therefore we employ the discriminant chr.inv.-tensors $\varepsilon^{\alpha\beta\gamma} = b_\sigma E^{\sigma\alpha\beta\gamma}$ (1.82) and $\varepsilon_{\alpha\beta\gamma} = b^\sigma E_{\sigma\alpha\beta\gamma}$ (1.83), which in the accompanying reference frame are

$$\varepsilon^{ikm} = b_0 E^{0ikm} = \frac{e^{ikm}}{\sqrt{h}}\,, \qquad \varepsilon_{ikm} = b^0 E_{0ikm} = e_{ikm}\sqrt{h}\,. \tag{2.19}$$

Using the components of the field tensor $F_{\alpha\beta}$ we have deduced (2.10–2.14), we obtain chr.inv.-projections of the field pseudotensor $F^{*\alpha\beta}$ (2.18). The chr.inv.-projections are

$$H^{*i} = \frac{F^{*\cdot i}_{0\cdot}}{\sqrt{g_{00}}} = -\frac{1}{c}\,\varepsilon^{ikm} A_{km} = -\frac{2}{c}\,\Omega^{*i}, \tag{2.20}$$

$$E^{*ik} = F^{*ik} = \frac{1}{c^2}\,\varepsilon^{ikm} F_m\,, \tag{2.21}$$

where $\Omega^{*i} = \frac{1}{2}\,\varepsilon^{ikm} A_{km}$ is the chr.inv.-pseudovector of angular velocities of the space rotation. Relations of them to the chr.inv.-projections of the field tensor $F_{\alpha\beta}$ express themselves just like for any chr.inv.-pseudotensors by the formulas

$$H^{*i} = \frac{1}{2}\,\varepsilon^{imn} H_{mn} \qquad H_{*i} = \frac{1}{2}\,\varepsilon_{imn} H^{mn}, \tag{2.22}$$

$$\varepsilon^{ipq} H_{*i} = \frac{1}{2}\,\varepsilon^{ipq}\varepsilon_{imn} H^{mn} = \frac{1}{2}\,(\delta^p_m\delta^q_n - \delta^q_m\delta^p_n)\,H^{mn} = H^{pq}, \tag{2.23}$$

$$\varepsilon_{ikp} H^{*p} = E_{ik}\,, \qquad E^{*ik} = -\varepsilon^{ikm} E_m\,. \tag{2.24}$$

So forth, we introduce invariants $J_1 = F_{\alpha\beta}F^{\alpha\beta}$ and $J_2 = F_{\alpha\beta}F^{*\alpha\beta}$ for the field of time density. Their formulas are

$$J_1 = F_{\alpha\beta}F^{\alpha\beta} = \frac{4}{c^2} A_{ik}A^{ik} - \frac{2}{c^4}\,F_iF^i, \tag{2.25}$$

$$J_2 = F_{\alpha\beta}F^{*\alpha\beta} = -\frac{8}{c^3}\,F_i\Omega^{*i}, \tag{2.26}$$

so the field of time density can be *spatially isotropic* (one of the invariants becomes zero) under the next particular conditions:

- invariant $A_{ik}A^{ik}$ of the field of rotation of the space and invariant F_iF^i of the field of gravitational inertial force are proportional one to another $A_{ik}A^{ik} = \frac{1}{2c^2}F_iF^i$;

- the acting gravitational inertial force F_i is orthogonal to the pseudovector Ω^{*i} of the space rotation (the equality $F_i\Omega^{*i} = 0$ is true);
- both of the conditions are realized together.

§ 2.3 Motion along time lines

Time lines are geodesics by definition. In accordance with the *least action principle*, an action replacing a particle along a geodesic line is the least. Actually, the least action principle implies that geodesic lines are also lines of the least action. This is physical viewpoint.

In the same time a geometric viewpoint is also exist. From this geometric viewpoint, motion of a particle is *parallel transfer* of its four-dimensional impulse vector along its four-dimensional trajectory and tangential to the trajectory at any of its point. Geodesic lines (lines of the least distance) are a particular case of parallel transfer lines. In Riemannian spaces, the length of any transferred vector remains unchanged in its parallel transfer along geodesic lines, so absolute derivative of the vector equals zero. The latest is known as equations of parallel transfer. So, we have a possibility to deduce equations of motion of free particles in two different ways:

1. Using the least action principle, we can take variation of the action a field carrying a free particle spent and then equalize the variation to zero. This way leads to equations of motion derived from the least action principle;
2. We can take absolute derivative of the four-dimensional impulse vector of the free particle with respect to interval along its trajectory, then we equalize the derivative to zero. Those will be equations of motion, deduced employing the parallel transfer method.

However it does not the fact that the resulting equations will be the same, because lines of the least action are only a particular case of parallel transfer lines. For instance, as it had been obtained earlier [12], the equations are different for charged particle in electromagnetic fields. The reason was that its motion is non-geodesic, because electromagnetic fields deviate charged particles from geodesic lines. As soon as the equations had been compared, some additional conditions of the motion had been created. However, because time lines are geodesics by their definition, they are lines of the least action under any conditions. For this reason equations of motion along time lines, deriving from the least action principle or the parallel transfer method must be the same.

In this §2.3 we are going to consider a free mass-bearing particle, which is at rest in respect of an observer and his reference body. Such particle moves along only time lines, so the particle displaces because of action of solely the field of non-uniformity of time coordinates — the field of time density, produced by the observer's reference body.

The action the field of time density spends to displace a free mass-bearing particle of the rest-mass m_0 is $dS = m_0 c\, b_\alpha dx^\alpha$ (2.5). Because of the least action, variation of the integral of the action along geodesic lines equals zero

$$\delta\int_a^b dS = 0\,, \tag{2.27}$$

that, after substituting $dS = m_0 c\, b_\alpha dx^\alpha$, gives

$$\delta\int_a^b dS = m_0 c\,\delta\int_a^b b_\alpha dx^\alpha = m_0 c\int_a^b \delta b_\alpha dx^\alpha + m_0 c\int_a^b b_\alpha d\delta x^\alpha, \tag{2.28}$$

$$\int_a^b b_\alpha d\delta x^\alpha = b_\alpha \delta x^\alpha \Big|_a^b - \int_a^b db_\alpha \delta x^\alpha = -\int_a^b db_\alpha \delta x^\alpha. \tag{2.29}$$

Given that $\delta b_\alpha = \frac{\partial b_\alpha}{\partial x^\beta}\delta x^\beta$ and $db_\alpha = \frac{\partial b_\alpha}{\partial x^\beta}dx^\beta$, we arrive to

$$\delta\int_a^b dS = m_0 c\,\delta\int_a^b b_\alpha dx^\alpha = m_0 c\,\delta\int_a^b \left(\frac{\partial b_\beta}{\partial x^\alpha} - \frac{\partial b_\alpha}{\partial x^\beta}\right) dx^\beta \delta x^\alpha. \tag{2.30}$$

This variation is zero, according to the least action principle. Hence along time lines we have the condition

$$m_0 c\left(\frac{\partial b_\beta}{\partial x^\alpha} - \frac{\partial b_\alpha}{\partial x^\beta}\right) dx^\beta = 0\,. \tag{2.31}$$

This condition, being divided by the interval ds, gives general covariant equations of motion of the particle

$$m_0 c\, F_{\alpha\beta}\, U^\beta = 0\,, \tag{2.32}$$

wherein $F_{\alpha\beta}$ is the tensor of the field of time density and U^β is the four-dimensional velocity of the particle*.

Taking their projections in the accompanying reference frame and multiplying the projections by c^2, we obtain chr.inv.-equations of motion of the particle in the general form

$$m_0 c^3 \frac{F_{0\sigma}U^\sigma}{\sqrt{g_{00}}} = 0\,, \qquad m_0 c^2 F^{i}_{\cdot\sigma} U^\sigma = 0\,, \tag{2.33}$$

where the scalar equation shows a work to displace the particle per second, while the vector equations show an observable acceleration of the particle with its mass.

It is interesting to note, the left part of the general covariant equations, which is the acting force, both in general covariant form and its chr.inv.-projections we have obtained has the same form that Lorentz' force, which displace charged particles in electromagnetic fields [12]. From mathematical viewpoint this fact implies that the field of time density acts mass-bearing particle as well as electromagnetic field moves electric charge.

Taking into account that formula (1.11) gives

$$ds^2 = c^2 d\tau^2\left(1 - \frac{h_{ik}\,dx^i dx^k}{c^2 d\tau^2}\right) = c^2 d\tau^2\left(1 - \frac{\mathrm{v}^2}{c^2}\right), \tag{2.34}$$

we arrive to

$$U^\alpha = \frac{dx^\alpha}{ds} = \frac{1}{c\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\frac{dx^\alpha}{d\tau}\,, \tag{2.35}$$

$$U^0 = \frac{\frac{1}{c^2}\,v_k \mathrm{v}^k + 1}{\sqrt{g_{00}}\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,, \qquad U^i = \frac{1}{c\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,\mathrm{v}^i. \tag{2.36}$$

Using the obtained components of the field tensor $F_{\alpha\beta}$ (2.10–2.14) and taking into account that the observable velocity of the

*Do not mix this vector $U^\alpha = \frac{dx^\alpha}{ds}$ with the monad vector $b^\alpha = \frac{dx^\alpha}{ds}$ (1.1), because they are built on different displacements dx^α. The monad b^α contains displacement of the observer in respect of his reference body, while the vector U^β contains displacement of the particle. However, if the particle moves along only time lines (its spatial displacement in respect of the observer is $dx^i = 0$) and the observer accompanies to his reference body (his spatial displacement is $dx^i = 0$), then the four-dimensional velocity U^α of the particle and the observer's velocity b^α are the same.

particle we are considering is $\mathrm{v}^i = 0$, we transform the chr.inv.-equations of motion (2.33) to the final form. The scalar equation becomes zero. The vector equations of motion becomes very simple

$$m_0 F^i = 0\,, \tag{2.37}$$

or, substituting the "electric" observable component $E^i = -\frac{1}{c^2}F^i$ (2.15) of the field of time density,

$$m_0 c^2 E^i = 0\,. \tag{2.38}$$

The obtained equations lead us to the next conclusions:

1. Becoming zero the scalar equation testifies: if a free mass-bearing particle moves in space-time along only time lines (the particle is at rest in respect of the observer in his spatial section), then the "electric" and the "magnetic" components of the field of time density do not produce a work to displace the particle. Such particle falls freely along its own time line under the field of time density;
2. Looking at the vector equations of motions, we see that $E^i = 0$ there. So the particle falls freely along its own time line, because of carrying by solely the "magnetic" component $H_{ik} = -\frac{2}{c}A_{ik} \neq 0$ of the field of time density;
3. Inhomogeneity of the spatial section (the Christoffel chr.inv.-symbols Δ^i_{jk}) or its deformations (the chr.inv.-tensor of the deformation rate D_{ik}) do not effect on free motion along time lines.

In other word, the "magnetic" component $H_{ik} = -\frac{2}{c}A_{ik}$ of the field of time density as if "screws" particles into time lines (a very relative analogy). No other sources, which could be causes to move particles along time lines. Because observable particles with whole the spatial section move from past into future, hence $H_{ik} \neq 0$ everywhere in our real world. So, our real space is strictly non-holonomic $A_{ik} \neq 0$.

This pure theoretical result bring us into the very important conclusion that a "start" non-holonomity of the real space shall be under any conditions, that is a "primordial non-orthogonality" of the real spatial section to time lines. Additional physical conditions like as three-dimensional rotations of the reference body (or other

bodies) shall be here only an "add-on", intensifying or reducing this invisible start-rotation of the space in dependence on their relative directions*.

The condition $E^i = 0$ (or $F^i = 0$, that is the same) also implies that during the free mass-bearing particle moves along only time lines (free falling in the field of time density) the force of gravitation, acting the particle, is fully reduced by the acting centrifugal force

$$\frac{1}{\sqrt{g_{00}}}\frac{\partial \mathrm{w}}{\partial x^i} = \frac{{}^*\partial v_i}{\partial t}\,, \tag{2.39}$$

so such particle is at the *condition of weightlessness*. As soon as the condition (2.39) has been broken, then the "electric" component of the field of time density becomes non-zero $E^i \neq 0$ and the component begin to move the particle along the spatial section. This case will be subjected in the next §2.4.

§2.4 Conditions of motion along the spatial section

We are going to consider the general case, where a free mass-bearing particle moves freely not only along time lines, but also along the spatial section. Such particle does not accompany to the observer and his reference body. Chr.inv.-equations of motion in this general case had been deduced by Zelmanov [1] as parallel transfer equations, they have the form

$$\begin{gathered} \frac{dE}{d\tau} - mF_i\mathrm{v}^i + mD_{ik}\mathrm{v}^i\mathrm{v}^k = 0\,, \\ \frac{d}{d\tau}\left(m\mathrm{v}^i\right) - mF^i + 2m\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0\,. \end{gathered} \tag{2.40}$$

*Similar conclusion had also been given by astronomer Kozyrev [13], who had a base his studies of interior of stars. In particular, aside the fundamental "start" self-rotation of the space, he had arrived to the conclusion that additional rotations shall produce a non-uniformity of observable time around rotating bulk bodies like as stars or planets. From his consideration, non-uniformity of time can also be a result of a re-distribution of energy or, to the contrary, re-distribution of energy can produce a non-uniformity of time. The consequences should be displayed better in the interaction of the components of bulk double stars [14]. He was also the first who used the term "the field of density of time". It is interesting that his arguments, deriving from pure phenomenology like as an analysis of astronomical observations, did not link to Riemannian geometry and the mathematical apparatus of the General Theory of Relativity. Kozyrev got attempts to ground his phenomenological conclusions by the methods of Classical Mechanics [13].

As a matter of fact that equations like these, being obtained from the least action principle, must be the same, because the particle moves along a geodesic line which is also the least action line.

Let us express the equations through the "electric" and the "magnetic" observable components of the field of time density. Substituting $E^i = -\frac{1}{c^2}F^i$ and $H_{ik} = -\frac{2}{c}A_{ik}$ into the Zelmanov equations (2.40), we obtain

$$\begin{aligned} &\frac{dE}{d\tau} + mc^2 E_i \mathrm{v}^i + mD_{ik}\mathrm{v}^i\mathrm{v}^k = 0\,, \\ &\frac{d}{d\tau}\left(m\mathrm{v}^i\right) + mc^2\left(E^i + \frac{1}{c}\,H^{ik}\mathrm{v}_k\right) + 2mD^i_k\mathrm{v}^k + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0\,. \end{aligned} \tag{2.41}$$

Looking at the equations, we see that the particle moves freely along the spatial section because of two factors:

1. First, the particle is carried with the field of time density by its "electric" $E^i \neq 0$ and "magnetic" $H_{ik} \neq 0$ components;
2. Second, the particle is also moved by forces, produced by the field of "density" of the spatial section that is the field of the observable metric tensor $h_{ik} = -g_{ik} + \frac{1}{c^2}\,v_i v_k$. We have not a formula to express the forces yet. However the obtained vector equations show, these forces must display themselves as an effect of inhomogeneity Δ^i_{nk} and deformations D_{ik} of the spatial section. As we can see from the scalar equation, the field of inhomogeneities of the spatial section does not produce a work to displace free mass-bearing particles, only the field of the spatial deformations produces the work.

In particular, the particle can be moved freely along the spatial section, because of carrying by solely the field of time density. Then, as it easy to see from the obtained equations (2.41), the next conditions are true

$$D_{ik}\mathrm{v}^i\mathrm{v}^k = 0\,, \qquad D^i_k = -\frac{1}{2}\,\Delta^i_{nk}\mathrm{v}^n, \tag{2.42}$$

so it may be possible in the next particular cases:

- if the spatial section does not deformations $D_{ik} = 0$;
- if, aside the absence of the deformations ($D_{ik} = 0$), the spatial section is homogeneous $\Delta^i_{nk} = 0$.*

*However the first condition $D_{ik} = 0$ would be sufficient.

The scalar equations of motion (2.41) also show that, under the particular conditions (2.42), the energy dE to displace the particle at dx^i equals the work

$$dE = -mc^2 E_i dx^i \tag{2.43}$$

the field "electric" component E_i spends for this displacement. The vector equations of motion in this particular case show that the "electric" and the "magnetic" components of the field of time density, accelerate the particle just like external forces*

$$\frac{dp^i}{d\tau} = -mc^2\left(E^i + \frac{1}{c}\,H^{ik}\mathrm{v}_k\right). \tag{2.44}$$

Looking at the right parts of the equations (2.43, 2.44), we see that they have the form identic to the right parts of chr.inv.-equations of motion of charged particle in electromagnetic field [12]. This fact is in accordance with our conclusion we have obtained in the previous §2.3 that the field of time density acts mass-bearing particle as well as electromagnetic field moves electric charge.

§2.5 Equations of the field of time density

As it is well-known, the theory of electromagnetic field, filled in a pseudo-Riemannian space, characterizes the field by a system of equations known also as the *field equations*:

- Lorentz' condition sets that the four-dimensional vector potential A^α of the field remains unchanged just like any four-dimensional vector in a pseudo-Riemannian space

$$\nabla_\sigma A^\sigma = 0\,; \tag{2.45}$$

- the charge conservation law (the continuity equation) shows that the field-inducing charge can not be destroyed, but merely re-distributed in the space

$$\nabla_\sigma j^\sigma = 0\,, \tag{2.46}$$

where j^α is the four-dimensional current vector, its observable projections are the chr.inv.-scalar of the charge density

*Here the chr.inv.-vector $p^i = m\mathrm{v}^i$ is the three-dimensional observable impulse of the particle.

$\rho = \dfrac{1}{c\sqrt{g_{00}}}\, j_0$ and the chr.inv.-vector of the current density j^i, which are sources inducing the field;

- Maxwell's equations show properties of the field, represented by components of the field tensor $F_{\alpha\beta}$ and its dual pseudotensor $F^{*\alpha\beta}$, in their link to the field-inducing sources. The first group of the Maxwell equations contains the field sources ρ and j^i, the second group do not contain the sources

$$\nabla_\sigma F^{\alpha\sigma} = \frac{4\pi}{c}\, j^\alpha, \qquad \nabla_\sigma F^{*\alpha\sigma} = 0\,. \tag{2.47}$$

All the equations had been deduced in chr.inv.-form for electromagnetic field earlier. In particular, the Maxwell chr.inv.-equations

$$\left.\begin{aligned} &{}^*\nabla_i\, E^i - \frac{1}{c}\, H^{ik}A_{ik} = 4\pi\rho \\ &{}^*\nabla_k\, H^{ik} - \frac{1}{c^2} F_k H^{ik} - \frac{1}{c}\left(\frac{{}^*\partial E^i}{\partial t} + E^i D\right) = \frac{4\pi}{c}\, j^i \end{aligned}\right\} \text{ I}, \tag{2.48}$$

$$\left.\begin{aligned} &{}^*\nabla_i\, H^{*i} - \frac{1}{c}\, E^{*ik}A_{ik} = 0 \\ &{}^*\nabla_k\, E^{*ik} - \frac{1}{c^2} F_k E^{*ik} - \frac{1}{c}\left(\frac{{}^*\partial H^{*i}}{\partial t} + H^{*i} D\right) = 0 \end{aligned}\right\} \text{ II}, \tag{2.49}$$

which actually are chr.inv.-projections of the Maxwell general covariant equations (2.47), had first been obtained for an arbitrary field potential by del Prado and Pavlov [15], the Zelmanov research-students, after as Zelmanov asked them to do this. Zelmanov aimed to apply the results to electromagnetic field. But the complete theory of electromagnetic field in the terms of chronometric invariants had been built only in 1990's [12], after as Zelmanov has been gone.

From mathematical viewpoint the general covariant equations of a field is a system of 10 equations in 10 unknowns (the Lorentz condition, the charge conservation law, and two groups of the Maxwell equations), which define the given vector field A^α and the field-inducing sources in a pseudo-Riemannian space. Actually, equations like these should be exist for any four-dimensional vector field, the field of time density included. The difference must be that the field equations shall be filled in some changed form, according to a formula of the specific vector potential. Therefore we are going to deduce similar equations for the vector field b^α we are considering, actually — the *equations of the field of time density*.

After some algebra, taken into account that chr.inv.-projections of the potential of the field of time density b^α equal $\varphi = 1$ and $q^i = 0$, the *Lorentz condition* for the field of time density $\nabla_\sigma b^\sigma = 0$ becomes the equality

$$D = 0\,. \tag{2.50}$$

It should be noted that the spur of the tensor of the spatial deformations becoming zero $D = h^{ik}D_{ik} = D^n_n = 0$ does not imply equality zero of the tensor D_{ik} itself. Naturally, using definition (1.24) of the tensor D_{ik}, we obtain

$$D = h^{ik}D_{ik} = \frac{1}{2}\, g^{ik}\left[\frac{{}^*\partial g_{ik}}{\partial t} - \frac{1}{c^2}\frac{{}^*\partial}{\partial t}\left(v_i v_k\right)\right], \tag{2.51}$$

so the condition $D = 0$ implies merely

$$\frac{{}^*\partial g_{ik}}{\partial t} = \frac{1}{c^2}\left(v_i \frac{{}^*\partial v_k}{\partial t} + v_k \frac{{}^*\partial v_i}{\partial t}\right), \tag{2.52}$$

where the derivative from v_i in respect to time is centrifugal force of inertia, i. e. the "inertial" part of gravitational inertial force F_i (1.16). In general, the tensor D_{ik} is the rate of changes of the observable metric h_{ik} of an elementary area, taken on the wall of an elementary volume of the space we are considering. Its square $D_{ik}D^{ik}$ is the square of the rate. Its spur $D = h^{ik}D_{ik}$ is the rate of expansion of whole the elementary volume, that is not the same that the quantity $D_{ik}D^{ik}$.

To do future calculations simpler, we collect chr.inv.-projections of the tensor of the field of time density $F_{\alpha\beta}$ and of the field pseudotensor $F^{*\alpha\beta}$ together

$$E_i = -\frac{1}{c^2}\, F_i\,, \qquad H^{ik} = -\frac{2}{c}\, A^{ik}, \tag{2.53}$$

$$H^{*i} = -\frac{2}{c}\,\Omega^{*i}, \qquad E^{*ik} = \frac{1}{c^2}\,\varepsilon^{ikm}F_m\,, \tag{2.54}$$

and take Zelmanov's identities for the discriminant chr.inv.-tensors [1] into account

$$\frac{{}^*\partial \varepsilon_{imn}}{\partial t} = \varepsilon_{imn}\, D\,, \qquad \frac{{}^*\partial \varepsilon^{imn}}{\partial t} = -\varepsilon^{imn} D\,, \tag{2.55}$$

$${}^*\nabla_k\, \varepsilon_{imn} = 0\,, \qquad {}^*\nabla_k\, \varepsilon^{imn} = 0\,. \tag{2.56}$$

Substituting the chr.inv.-projections into (2.48, 2.49) and accepting $D=0$ according to the Lorentz condition we have obtained (2.50), we arrive to *Maxwell-like chr.inv.-equations* for the field of time density*. After similar terms collected the equations arrive to the final form

$$\left.\begin{aligned} &\frac{1}{c^2}\,{}^*\nabla_i\,F^i-\frac{2}{c^2}A_{ik}A^{ik}=-\,4\pi\rho\\ &\frac{2}{c}\,{}^*\nabla_k\,A^{ik}-\frac{2}{c^3}F_kA^{ik}-\frac{1}{c^3}\frac{{}^*\partial F^i}{\partial t}=-\frac{4\pi}{c}\,j^i \end{aligned}\right\}\ \text{I}, \tag{2.57}$$

$$\left.\begin{aligned} &{}^*\nabla_i\,\Omega^{*i}+\frac{1}{c^2}F_i\Omega^{*i}=0\\ &{}^*\nabla_k\big(\varepsilon^{ikm}F_m\big)-\frac{1}{c^2}\,\varepsilon^{ikm}F_kF_m+2\,\frac{{}^*\partial\Omega^{*i}}{\partial t}=0 \end{aligned}\right\}\ \text{II}. \tag{2.58}$$

The "*charge*" *conservation law* $\nabla_\sigma\,j^\sigma=0$ (the continuity equation), after substituting chr.inv.-projections $\varphi=c\rho$ and $q^i=j^i$ of the "current" vector j^α, take the chr.inv.-form

$$\begin{aligned} &\frac{1}{c^2}\frac{{}^*\partial}{\partial t}\left(A_{ik}A^{ik}\right)+\frac{1}{c^2}F_i\frac{{}^*\partial A^{ik}}{\partial x^k}-\frac{{}^*\partial^2A^{ik}}{\partial x^i\partial x^k}-\frac{1}{2c^2}F^i\frac{{}^*\partial\Delta^j_{ji}}{\partial t}+\\ &+\left(\frac{1}{c^2}F_i\Delta^j_{jk}+\frac{{}^*\partial\Delta^j_{jk}}{\partial x^i}+\Delta^j_{ji}\Delta^l_{lk}\right)A^{ik}-\frac{1}{c^4}F_iF_kA^{ik}=0\,, \end{aligned} \tag{2.59}$$

because the field-inducing sources ρ and j^i, expressing themselves from the 1st group of the Maxwell-like equations (2.57), are

$$\rho=-\frac{1}{4\pi c^2}\left({}^*\nabla_i\,F^i-2A_{ik}A^{ik}\right), \tag{2.60}$$

$$j^i=-\frac{1}{2\pi}\,{}^*\nabla_k\,A^{ik}-\frac{1}{2\pi c^2}F_kA^{ik}-\frac{1}{4\pi c^2}\frac{{}^*\partial F^i}{\partial t}\,. \tag{2.61}$$

*Really, it would be possible to deduce the Maxwell-like chr.inv.-equations direct, projecting the general covariant equations $\nabla_\sigma F^{\alpha\sigma}=\frac{4\pi}{c}\,j^\alpha$ and $\nabla_\sigma F^{*\alpha\sigma}=0$. However this frontal method does not shortest way. It would be easier to substitute observable components of a field tensor, specific for the field we are considering, into the Maxwell chr.inv.-equations (2.48) and (2.49) because the electromagnetic field potential A^α and the field tensor $F_{\alpha\beta}$ are given there in the generalized form as an arbitrary vector and an arbitrary antisymmetric tensor of the 2nd rank. As a matter of fact that the resulting Maxwell-like equations, being obtained the both ways, will be the same.

In particular, if no "currents" of the field of time density here ($j^i=0$), the continuity equation $\nabla_\sigma\,j^\sigma=0$ becomes the condition

$$\frac{{}^*\partial}{\partial t}\left(A_{ik}A^{ik}\right)-\frac{1}{2}\frac{{}^*\partial^2F^i}{\partial t\,\partial x^i}-\frac{1}{2}F^i\frac{{}^*\partial\Delta^j_{ji}}{\partial t}-\frac{1}{2}\Delta^j_{ji}\frac{{}^*\partial F^i}{\partial t}=0\,, \tag{2.62}$$

which in a non-deforming homogeneous space is

$$\frac{{}^*\partial}{\partial t}\left(A_{ik}A^{ik}\right)-\frac{1}{2}\frac{{}^*\partial^2F^i}{\partial t\,\partial x^i}=0\,. \tag{2.63}$$

So, the Lorentz condition (2.50), the Maxwell-like equations (2.57, 2.58), and the continuity equation (2.59) we have obtained are the *chr.inv.-equations of the field of time density*. As we can see from the equations, the field characterizes itself by the next peculiarities:

1. The Lorentz condition (2.50), becoming zero spur of the tensor of the spatial deformations $D=h^{ik}D_{ik}=0$, implies that a deforming elementary volume, filled with the field of time density, does not expand, because its deformations at different directions reduce one another*. In other word, the value of the elementary volume of the field of time density remains unchanged under its deformations;
2. The 1st group (2.57) of the Maxwell-like equations defines sources ρ and j^i, which induce the field of time density:

   "Charge" ρ displays itself as the difference between inhomogeneity ${}^*\nabla_i\,F^i$ of the field of gravitational inertial force and invariant $A_{ik}A^{ik}$ of the field of the space rotation. If the acting force is $F^i=0$ (it is possible, if the force field is homogeneous ${}^*\nabla_i\,F^i=0$ or is absent $F^i=0$), then the "charge" ρ is produced by only the space rotation;

   "Currents" j^i of the field of time density are produced by inhomogeneity ${}^*\nabla_k\,A^{ik}$ of the field of the space rotation, corrected with higher order terms depending on non-orthogonality of the fields A_{ik} and F_i (the second term F_kA^{ik}), and also on non-stationarity of the acting gravitational inertial force (the third term). If the acting force is $F^i=0$, then the "currents" j^i of the field of time density are produced by solely inhomogeneity of rotations of the space;

*From geometric viewpoint, the spur of the tensor D_{ik} is the rate of expansion of an elementary volume.

3. The Maxwell-like equations of the 2nd group (2.58) show properties of the "magnetic" component $H^{*i} = -\frac{2}{c}\,\Omega^{*i}$ of the field of time density in their link to the space rotation:

   Inhomogeneity ${}^*\nabla_i\,\Omega^{*i}$ of the field of rotations of the space depends on non-orthogonality of the angular velocity Ω^{*i} of the rotations to the acting gravitational inertial force F_i;

   If the acting force is $F_i = 0$, then rotations of the space are homogeneous ${}^*\nabla_i\,\Omega^{*i} = 0$ and stationary $\Omega^{*i} = const$;
4. To analyse the continuity equation (2.59) would be difficult in this general form. In particular case, when the space is homogeneous and also has not "currents" of the field of time density $j^i = 0$, the continuity equation takes the simplified form (2.63), which sets up that the "charge" ρ inducing the field of time density remains unchanged.

§2.6 Waves of the field of time density

Let us turn d'Alembert's equations. Now we we are going to obtain the equations for the field of time density.

D'Alembert's operator $\Box = g^{\alpha\beta}\,\nabla_\alpha\nabla_\beta$, being applied to a field, can be not zero or equal zero. The first case is known as the d'Alembert equations with the field-inducing sources, while the second case is known as the d'Alembert equation without the sources. If sources of a field are absent, then the field is free. This is a wave. So, the d'Alembert equations without the sources are equations of propagation of waves of the field.

From this reason, the d'Alembert equations for the vector potential b^α of the field of time density we are taking without the sources

$$\Box b^\alpha = 0 \tag{2.64}$$

are the equations of propagation of waves of the field of time density. Chr.inv.-projections of the equations are

$$b_\sigma \Box b^\sigma = 0\,, \qquad h^i_\sigma \Box b^\sigma = 0\,. \tag{2.65}$$

So forth, we substitute chr.inv.-projections $\varphi = 1$ and $q^i = 0$ of the field potential b^α into the d'Alembert chr.inv.-equations of an arbitrary vector field (1.177, 1.178). Then, taking into account that

the Lorentz condition for the field b^α is $D = 0$ (2.50), after some algebra we arrive to the d'Alembert chr.inv.-equations without the sources for the field of time density

$$\begin{gathered}\frac{1}{c^2}\,F_iF^i - D_{ik}D^{ik} = 0\,,\\ \frac{1}{c^2}\frac{{}^*\partial F^i}{\partial t} + h^{km}\left\{\frac{{}^*\partial D^i_m}{\partial x^k} + \frac{{}^*\partial A_{m\cdot}^{\cdot i}}{\partial x^k} + \Delta^i_{kn}\left(D^n_m - A_{m\cdot}^{\cdot n}\right) - \right.\\ \left. - \Delta^n_{km}\left(D^i_n - A_{n\cdot}^{\cdot i}\right)\right\} = 0\,.\end{gathered} \tag{2.66}$$

Unfortunately, a term like $\frac{1}{a^2}\frac{\partial^2 q^i}{\partial t^2}$ containing the linear speed a of the waves is not here, because of $q^i = 0$. For this reason we have not a possibility to say something on the speed of waves traveling in the field of time density. In the same time the obtained equations (2.66) display other peculiarities of this wave field:

1. The scalar equation implies that the rate of deformations of a surface element, taken in the field of time density waves, is powered by the value of the acting gravitational inertial force. If $F_i = 0$, then the observable spatial metric h_{ik} is stationary — the area of the surface element remains unchanged;
2. The vector equations are more difficult. Their analysis would be easier, if the space is homogeneous $\Delta^i_{kn} = 0$ and the field of gravitational inertial force is stationary $F_i = const$. In this particular case the vector equations set up that in the space, filled with the field of time density waves, the spatial structure of the field of the space deformations is the same that the field of the space rotation $\frac{{}^*\partial D^i_m}{\partial x^k} = \frac{{}^*\partial A^{i\cdot}_{\cdot m}}{\partial x^k}$.

Aside the above, we can conclude something more on waves of the field of time density, if we equalize the field-inducing sources ρ (2.60) and j^i (2.61) to zero, because a field without its-inducing sources is a field of waves. As a result we obtain the conditions

$${}^*\nabla_i\,F^i = 2\,A_{ik}A^{ik}\,, \tag{2.67}$$

$${}^*\nabla_k\,A^{ik} = -\frac{1}{c^2}\,F_kA^{ik} - \frac{1}{2c^2}\frac{{}^*\partial F^i}{\partial t}\,, \tag{2.68}$$

which, taking a place in the presence of waves traveling in the field of time density, may be formulated as follows:

3. Inhomogeneity of the acting gravitational inertial force ${}^*\nabla_i F^i$ in the wave field increases with the value of the angular velocity A_{ik} of the space rotation;
4. Inhomogeneity of rotations of an element of the space, filled with the wave field, namely — the quantity ${}^*\nabla_k A^{ik}$, is derived from non-orthogonality of the acting force F_i to the field A^{ik} and from non-stationarity of the force F_i.

§2.7 Energy-momentum tensor of the field

Taking general covariant equations of motion along time lines a base, we are going to deduce energy-momentum tensor for the field of time density. It is possible to do in the next way.

The general covariant equations of motion of a free point-mass particle along time lines $m_0 c\, F_{\alpha\beta}\, U^\beta = 0$ (2.32), being taken in contravariant (upper-index) form, are

$$m_0 c F_{\cdot\sigma}^{\alpha\cdot} U^\sigma = 0\,, \tag{2.69}$$

where U^σ is the four-dimensional velocity of the particle. The left part of the equations has the dimension $[\mathrm{gram}/\mathrm{sec}]$ as well as a four-dimensional force. Actually, as it was mentioned in §2.3, such particle moves, because of carrying by solely the field of time density.

If the free-moving particle is not a point-mass, then the particle can be represented a current of the field of time density j^α. On the other hand, the currents are defined by the 1st group $\nabla_\sigma F^{\alpha\sigma} = \frac{4\pi}{c} j^\alpha$ of the Maxwell-like equations of the field. In this case the general covariant equations of motion (2.69), drawing analogy to currents of electromagnetic field, take the form

$$\mu F_{\cdot\sigma}^{\alpha\cdot} j^\sigma = 0\,. \tag{2.70}$$

The numerical coefficient μ here is a new fundamental constant. This new constant having the dimension $[\mathrm{gram}/\mathrm{sec}]$ provides the dimension $[\mathrm{gram}/\mathrm{cm}^2\times\mathrm{sec}^2]$ to the left part of the equations, so it get the left part a current of the acting four-dimensional force (2.69) through $1\,\mathrm{cm}^2$ per 1 second. The numerical value of this constant μ can be found from measurements of the wave pressure of the field of time density (see §2.10 below). However it does not except that

future studies of the problem will have became an analytic formula for μ, linking the constant to other fundamental constants.

Chr.inv.-projections of the equations (2.70)

$$\frac{\mu F_{0\sigma} j^\sigma}{\sqrt{g_{00}}} = 0\,, \qquad \mu F_{\cdot\sigma}^{i\cdot} j^\sigma = 0\,, \tag{2.71}$$

after substituting components of the field tensor $F_{\alpha\beta}$ (2.10–2.14), take the form

$$\begin{aligned} &\mu E_k\, j^k = 0\,,\\ &\mu c\left(\rho E^i - \frac{1}{c}\, H_{\cdot k}^{i\cdot}\, j^k\right) = 0\,, \end{aligned} \tag{2.72}$$

where E^i is the "electric" component and H_{ik} is the "magnetic" component of the field of time density. Sources ρ and j^i inducing the field are defined by the 1st group of the Maxwell-like chr.inv.-equations (2.57).

Actually, the term*

$$f^\alpha = \mu F_{\cdot\sigma}^{\alpha\cdot} j^\sigma \tag{2.73}$$

in the left part of the general covariant equations of motion (2.70) can be transformed with the 1st Maxwell-like group $\nabla_\beta F^{\sigma\beta} = \frac{4\pi}{c} j^\sigma$ to the form

$$f_\alpha = \frac{\mu c}{4\pi} F_{\alpha\sigma} \nabla_\beta F^{\sigma\beta} = \frac{\mu c}{4\pi}\Big[\, \nabla_\beta \left(F_{\alpha\sigma} F^{\sigma\beta}\right) - F^{\sigma\beta}\, \nabla_\beta F_{\alpha\sigma} \Big]\,, \tag{2.74}$$

where the second term equals

$$\begin{aligned} &F^{\sigma\beta}\, \nabla_\beta F_{\alpha\sigma} = \frac{1}{2} F^{\sigma\beta} \left(\nabla_\beta F_{\alpha\sigma} + \nabla_\sigma F_{\beta\alpha}\right) = \\ &= -\frac{1}{2}\, F^{\sigma\beta} (\nabla_\beta F_{\sigma\alpha} + \nabla_\sigma F_{\alpha\beta}) = -\frac{1}{2} F^{\sigma\beta} \nabla_\sigma F_{\alpha\beta} = \frac{1}{2} F^{\sigma\beta} \nabla_\alpha F_{\sigma\beta}\,. \end{aligned} \tag{2.75}$$

Using this formula, we transform the current f^α (2.73) to the form

$$f_\alpha = \frac{\mu c}{4\pi}\, \nabla_\beta \left(-F_{\alpha\sigma} F^{\beta\sigma} + \frac{1}{4}\, \delta_\alpha^\beta F_{pq} F^{pq}\right), \tag{2.76}$$

$$f^\alpha = \frac{\mu c}{4\pi}\, \nabla_\beta \left(-F_{\cdot\sigma}^{\alpha\cdot} F^{\beta\sigma} + \frac{1}{4}\, g^{\alpha\beta} F_{pq} F^{pq}\right). \tag{2.77}$$

*From physical viewpoint, this term is a current of the acting four-dimensional force, produced by the field of time density.

Now we write down the current f^α in the form

$$f^\alpha = \nabla_\beta T^{\alpha\beta} \tag{2.78}$$

just like as the theory of electromagnetic field does it to extract the energy-momentum tensor $T^{\alpha\beta}$. Following this way, we arrive to the *energy-momentum tensor* of the field of time density

$$T^{\alpha\beta} = \frac{\mu c}{4\pi}\left(-F^{\alpha\cdot}_{\cdot\sigma}F^{\beta\sigma} + \frac{1}{4}\,g^{\alpha\beta}F_{pq}F^{pq}\right), \tag{2.79}$$

a form of which is the same that the energy-momentum tensor of electromagnetic field [5, 12] to within the coefficient of its dimension. As it is easy to see, the tensor is symmetric so its spur is zero $T^\sigma_\sigma = g_{\alpha\beta}T^{\alpha\beta} = 0$.

Let us deduce chr.inv.-projections of the energy-momentum tensor of the field of time density

$$q = \frac{T_{00}}{g_{00}}\,, \qquad J^i = \frac{c\,T^i_0}{\sqrt{g_{00}}}\,, \qquad U^{ik} = c^2T^{ik}, \tag{2.80}$$

substituting the necessary components of the field tensor $F_{\alpha\beta}$ (2.10–2.14). After some algebra we obtain

$$q = \frac{\mu}{4\pi c}\left(A_{ik}A^{ik} + \frac{1}{2c^2}F_kF^k\right), \tag{2.81}$$

$$J^i = -\frac{\mu}{2\pi c}F_kA^{ik}, \tag{2.82}$$

$$U^{ik} = -\frac{\mu c}{4\pi}\left(4A^{i\cdot}_{\cdot m}A^{mk} + \frac{1}{c^2}F^iF^k + A_{pq}A^{pq}h^{ik} - \frac{1}{2c^2}F_pF^ph^{ik}\right). \tag{2.83}$$

In accordance with their dimensions, the chr.inv.-projections have the next physical meanings:

- the time observable projection q [gram/cm×sec²] is that energy [gm×cm²/sec²] the field of time density contains in 1 cm³. Actually, the scalar q is the *observable density of the field*;
- the mixed observable projection J^i [gram/sec³] is that energy the field of time density transferred through 1 cm² per second, in other word this is the *observable density of the field momentum*;
- the spatial observable projection U^{ik} [gm×cm/sec⁴] is the tensor of the observable density of momentum flux of the field, in other word — the *field strength tensor*.

§2.8 Physical properties of the field

As it had been proven by Zelmanov [3], the strength chr.inv.-tensor U^{ik} of a field, being taken in covariant (lower-index) form, can be represented as follows

$$U_{ik} = p_0h_{ik} - \alpha_{ik} = ph_{ik} - \beta_{ik}\,, \tag{2.84}$$

where α_{ik} is the tensor of viscous strengthes of the field

$$\alpha_{ik} = \beta_{ik} + \frac{1}{3}\,\alpha h_{ik}\,, \qquad \alpha = h^{ik}\alpha_{ik} = \alpha^n_n \tag{2.85}$$

called the *viscosity of the 2nd kind*. Its anisotropic part β_{ik}, called the *viscosity of the 1st kind*, displays itself as anisotropic deformations of the space. The quantity p_0 is that pressure inside the media, which equalizes its density in the absence of the viscosity, p is the true pressure of the media*. The tensors of viscous strengthes α_{ik} and β_{ik} are chr.inv.-quantities by their definitions (2.84, 2.85).

Extracting the tensors of viscous strengthes α_{ik} and β_{ik} from the strength tensor U_{ik} of the field of time density, we are going to deduce the equation of state of the field.

Transforming the strength tensor of the field of time density U^{ik} (2.83) into covariant form and also keeping the formula for q (2.81) in the mind, we write down

$$U_{ik} = -qc^2h_{ik} - \frac{\mu c}{4\pi}\left(4A_{im}A^{m\cdot}_{\cdot k} + \frac{1}{c^2}F_iF_k - \frac{1}{c^2}F_mF^mh_{ik}\right), \tag{2.86}$$

that, after equalized to the same value $U_{ik} = p_0h_{ik} - \alpha_{ik}$ (2.84), gives the equilibrium pressure in the field

$$p_0 = -qc^2, \tag{2.87}$$

while the *tensor of viscous strengthes* of the field is

$$\alpha_{ik} = \frac{\mu c}{4\pi}\left(4A_{im}A^{m\cdot}_{\cdot k} + \frac{1}{c^2}F_iF_k - \frac{1}{c^2}F_mF^mh_{ik}\right). \tag{2.88}$$

*Equation of state of a distributed media is the relation between the pressure p inside the media and its density q. In a non-viscous media or when viscous strengthes of a media are isotropic, the true pressure p is equal to the equilibrium pressure p_0. For instance, the equation of state of a dust media has the form $p=0$. Ultra-relativistic gases have the equation of state $p=\frac{1}{3}qc^2$. The equation of state of the matter inside of atomic nuclei is $p=qc^2$. Vacuum and μ-vacuum have the equation of state $p=-qc^2$, see Chapter 5 in [12].

Because the spur of this tensor α_{ik}, as it easy to see, is not zero

$$\alpha = h^{ik}\alpha_{ik} = -\frac{\mu c}{\pi}\left(A_{ik}A^{ik} + \frac{1}{2c^2}F_kF^k\right) \neq 0\,, \qquad (2.89)$$

the tensor $\alpha_{ik} = \beta_{ik} + \frac{1}{3}\,\alpha h_{ik}$ has the non-zero anisotropic part

$$\beta_{ik} = \frac{\mu c}{4\pi}\left(4A_{im}A_{\cdot k}^{m\cdot} + \frac{1}{c^2}F_iF_k - \frac{1}{3c^2}F_mF^m h_{ik} + \frac{4}{3}A_{mn}A^{mn}h_{ik}\right), \qquad (2.90)$$

so the viscous strengthes of the field of time density are anisotropic. As it easy to see, the anisotropy increases with the value $A_{pq}A^{pq}$ of the space rotation.

Because the viscous strengthes α_{ik} are anisotropic, the equilibrium pressure $p_0 = -qc^2$ and the true pressure p inside the media are different. The true pressure is

$$p = \frac{\mu c}{12\pi}\left(A_{ik}A^{ik} + \frac{1}{2c^2}F_kF^k\right), \qquad (2.91)$$

that gives the *equation of state* of the field of time density

$$p = \frac{1}{3}\,qc^2. \qquad (2.92)$$

It is interesting that the way we used here is

$$p = p_0 - \frac{1}{3}\,\alpha = -qc^2 + \frac{4}{3}\,qc^2 = \frac{1}{3}\,qc^2, \qquad (2.93)$$

so the main goal into the true pressure is derived from the viscous strengthes α_{ik} of the field. For this reason we conclude that the field of time density is "super-viscous" media.

Finally, we write the strength tensor $U_{ik} = ph_{ik} - \beta_{ik}$ of the field in the form

$$U_{ik} = \frac{1}{3}\,qc^2h_{ik} - \beta_{ik}\,. \qquad (2.94)$$

So, having this analysis of the energy-momentum tensor of the field of time density, we can extract something on physical properties of the field:

1. In general, the field of time density is non-stationary distributed media, because of the field density may be

$$q = \frac{\mu}{4\pi c}\left(A_{ik}A^{ik} + \frac{1}{2c^2}F_kF^k\right) \neq const\,, \qquad (2.95)$$

so the field becomes stationary $q = const$ under stationary rotation $A_{ik} = const$ of the space and stationary gravitational inertial force $F_i = const$;

2. The field of time density bears momentum, because of

$$J^i = -\frac{\mu}{2\pi c}F_kA^{ik} \neq 0\,, \qquad (2.96)$$

in general. So, the field can transfer impulse. The field does not transfer impulse $J^i = 0$, if the space does not rotate $A_{ik} = 0$. The absence of gravitation does not effect that the field can transfer impulse, because the "inertial" part of gravitational inertial force F_i (1.16) remains unchanged even in the absence of gravitational fields;

3. So, the field of time density can be represented as an emitting media $J^i \neq 0$ in a non-holonomic (rotating) space. In a non-rotating (holonomic) space the field has not a possibility to produce radiations;
4. The field of time density is viscous media. This viscosity α_{ik} (2.88), deriving from non-zero rotation of the space or from gravitational inertial force, is anisotropic. The anisotropy β_{ik} increases with the speed of the space rotation A_{ik}. The field is viscous anisotropic anyhow, because its viscous strengthes become $\alpha_{ik} = 0$ and $\beta_{ik} = 0$ if only $A_{ik} = 0$ and $F_i = 0$ together. However in this case the field density is $q = 0$, so the field itself disappears;
5. Therefore the equilibrium pressure p_0 does not possess physical sense for the field of time density, real is only the true pressure $p = p_0 - \frac{1}{3}\alpha$;
6. The equation of state of the field of time density $p = \frac{1}{3}qc^2$ (2.92) sets up that the field is a media, filled in the *state of ultrarelativistic gas*. Actually the obtained equation of state $p = \frac{1}{3}qc^2$ implies that at positive density of the media its inner pressure becomes positive — the media compresses.

§2.9 Action of the field without its sources

Looking at §27 of *The Classical Theory of Fields* [5] we see, that an elementary action for the whole system consisting an electromagnetic field and a single charged particle, which are located in a pseudo-Riemannian space, contains three parts*

$$dS = dS_{\rm m} + dS_{\rm mf} + dS_{\rm f} = m_0 c\,ds + \frac{e}{c}\,\mathcal{A}_\alpha dx^\alpha + a\,\mathcal{F}_{\alpha\beta}\,\mathcal{F}^{\alpha\beta} dV dt\,, \quad (2.97)$$

where $\mathcal{A}^\alpha$ is the four-dimensional electromagnetic field potential, $\mathcal{F}_{\alpha\beta} = \nabla_\alpha \mathcal{A}_\beta - \nabla_\beta \mathcal{A}_\alpha$ is the electromagnetic field tensor, $dV = dx dy dz$ is an elementary thee-dimensional (spatial) volume filled with this field.

The first term $S_{\rm m}$ is "that part of the action which depends only on the properties of the particles, that is, just the action for free particles... The quantity $S_{\rm mf}$ is that part of the action which depends on the interaction between the particles and the field... Finally $S_{\rm f}$ is that part of the action which depends only on the properties of the field itself, that is, $S_{\rm f}$ is the action for a field in the absence of charges. Up to now, because we were interested only in the motion of charges in a given electromagnetic field, the quantity $S_{\rm f}$, which does not depend on the particles, did not concern us, since the term can not act the motion of particles. Nevertheless this term is necessary when we want to find equations determining the field itself".

To find a formula for the action $S_{\rm f}$, note that the action $S_{\rm f}$ we are considering must be depended of only the properties of the field. So the action must be taken over the space volume, filled with the field. The action must be scalar, only the 1st field

*In accordance with the least action principle, the action must have minimum, so integral of the action between a pair of world points and the action itself must be positive. A negative action could makes a quantity with arbitrarily large negative absolute value, which can not have minimum. Taken a pseudo-Riemannian space with the signature $(-+++)$, where time is imaginary, spatial coordinates are real, and three-dimensional coordinate impulse is positive, Landau and Lifshitz in §3 of *The Classical Theory of Fields* wrote "the clock at rest always indicates a greater time interval than the moving one. Thus we arrive at the result that the integral, taken between a given pair of world points, has its maximal value if it is taken along the straight world line joining these two points". Therefore they put "minus" before the action.

To the contrary, we stick to a pseudo-Riemannian space with Zelmanov's signature $(+---)$, where time is real and spatial coordinates are imaginary, because in this case three-dimensional observable impulse is positive. In a space with such signature, a regular observer, moving from past into future, observe his own flow of observable time positive always $d\tau > 0$. Any particle, moving from past into future, has also the positive change of its own time coordinate $dt > 0$ in respect of the observer's clock [11, 12]. Therefore, following Zelmanov, we always have "plus" before the action.

invariant $J_1 = \mathcal{F}_{\alpha\beta}\,\mathcal{F}^{\alpha\beta}$ has this property. The 2nd field invariant $J_2 = \mathcal{F}_{\alpha\beta}\,\mathcal{F}^{*\alpha\beta}$ is pseudoscalar, not scalar, go into the detailed discussion with Landau and Lifshitz.

"The numerical value of a depends on the choice of units for measurement of the field. We note that after the choice of a definite value for a and for the units of measurement of field, the units for measurement of all other electromagnetic quantities are determined.

From now we shall use the Gaussian system of units; in this system a is a dimensionless quantity equal to $\frac{1}{16\pi}$. In addition to the Gaussian system, one also uses the Heaviside system, in which $a = \frac{1}{4}$. In this system of units the field equations have more convenient form (4π does not appear) but on the other hand, π appears in the Coulomb law. Conversely, in the Gaussian system the field equations contain 4π, but the Coulomb law has a simple form".

As a result of the cited §27 of *The Classical Theory of Fields* we have

$$dS_{\rm f} = a\,\mathcal{F}_{\alpha\beta}\,\mathcal{F}^{\alpha\beta} dV dt = \frac{1}{16\pi c}\,\mathcal{F}_{\alpha\beta}\,\mathcal{F}^{\alpha\beta} d\Omega\,, \qquad (2.98)$$

where $d\Omega = c dt dV = c dt dx dy dz$ is an elementary four-dimensional volume. So the action of the whole system of electromagnetic field and a single charged particle located in it (2.97) takes the final form

$$dS = m_0 c\,ds + \frac{e}{c}\,\mathcal{A}_\alpha dx^\alpha + \frac{1}{16\pi c}\,\mathcal{F}_{\alpha\beta}\,\mathcal{F}^{\alpha\beta} d\Omega\,. \qquad (2.99)$$

In accordance with this consideration, we can represent an elementary action for the whole system consisting the field of time density and a single mass-bearing particle, which falls freely along time lines in a pseudo-Riemannian space, as follows

$$\begin{aligned} dS = dS_{\rm m} + dS_{\rm mt} = m_0 c ds + a_{\rm mt}\,F_{\alpha\beta}\,F^{\alpha\beta} d\Omega = \\ = m_0 c\,b_\alpha dx^\alpha + a_{\rm mt}\,F_{\alpha\beta}\,F^{\alpha\beta} d\Omega\,, \end{aligned} \qquad (2.100)$$

where $F_{\alpha\beta}$ is the tensor of the field of time density, $a_{\rm mt}$ is a constant consisting of other fundamental constants.

The first term $S_{\rm m}$ here is that part of the action, which displays the interaction between the particle and the field of time density carrying the particle into free motion along time lines. The second term $S_{\rm mt}$ depending on only the properties of the field itself is the action for the field in absence of its sources.

In the absence of the field of time density the second term S_{mt} is zero, only $S_{\text{m}} = m_0 c ds$ remains here. The field of time density is absent if no the space rotation $A_{ik} = 0$ and the field of gravitational inertial force $F_i = 0$, so if the conditions $g_{0i} = 0$ and $g_{00} = 1$ are true. This situation is possible in a pseudo-Riemannian space with the unit diagonal metric, which is the Minkowski space of the Special Theory of Relativity, where no any gravitation and the rotation. However, if we like to consider the real space, then we are forced to take the field of time density into account. So, we need to consider the both terms S_{m} and S_{mt} together.

The constant a_{mt}, according to the dimensions, is the same that the constant μ in the energy-momentum tensor of the field of time density (see §2.7), taken with the numerical coefficient $a = \frac{1}{16\pi}$, because we used the Gaussian system of units here.

As a result, we obtain the action (2.100) in the final form

$$dS = dS_{\text{m}} + dS_{\text{mt}} = m_0 c\, b_\alpha dx^\alpha + \frac{\mu}{16\pi} F_{\alpha\beta} F^{\alpha\beta} d\Omega \,. \tag{2.101}$$

Because an action for a system is expressed through Lagrange's function L of the system as $dS = Ldt$, we take the action dS_{mt} in the form

$$dS_{\text{mt}} = \frac{\mu c}{16\pi} F_{\alpha\beta} F^{\alpha\beta} dV dt = Ldt \,, \tag{2.102}$$

which show the Lagrangian of an elementary volume $dV = dx\,dy\,dz$ of the field. Following this way, we obtain the *Lagrangian density* in the field of time density

$$\Lambda = \frac{\mu c}{16\pi} F_{\alpha\beta} F^{\alpha\beta} = \frac{\mu}{4\pi c} \left(A_{ik} A^{ik} - \frac{1}{2c^2} F_i F^i \right) . \tag{2.103}$$

The term $A_{ik} A^{ik}$ here, being expressed through the pseudovector of angular velocities of the space rotation Ω^{*i}, is

$$A_{km} A^{km} = \varepsilon_{kmn} \Omega^{*n} A^{km} = 2\Omega_{*n} \Omega^{*n}, \tag{2.104}$$

because of

$$\varepsilon_{nkm} \Omega^{*n} = \frac{1}{2}\, \varepsilon^{npq} \varepsilon_{nkm} A_{pq} = \frac{1}{2} \left(\delta^p_k \delta^q_m - \delta^q_k \delta^p_m \right) A_{pq} = A_{km} \,, \tag{2.105}$$

$$\Omega_{*n} = \frac{1}{2}\, \varepsilon_{nkm} A^{km}, \tag{2.106}$$

so the space rotation plays the first violin, defining the Lagrangian density in the field of time density.

Rotation velocities we observe in macro-processes are incommensurably small in comparison with rotations of atoms and the particles. For instance, in the 1st Bohr orbit in an atom of hydrogen, measuring the value of Λ in the units of the energy-momentum constant μ, we have $\Lambda \simeq 9.1 \times 10^{21} \mu$. On the Earth surface near the equator the value is $\Lambda \simeq 2.8 \times 10^{-20} \mu$, so it is in order of 10^{42} less than in atoms. Therefore, because the Lagrangian of a system is the difference between its kinetic and potential energies, we conclude that the field of time density produces its main energy flux in atom and sub-atom interactions, while the energy flux produced by the field in macro-processes is neglected.

§2.10 Plane waves of the field

In general, because the electric E_i and the magnetic H^{ik} strengthes of the field of time density are

$$E_i = -\frac{1}{c^2} F_i \,, \qquad H^{ik} = -\frac{2}{c} A^{ik}, \tag{2.107}$$

the chr.inv.-vector of its momentum density J^i (2.82) can be written as follows

$$J^i = -\frac{\mu}{2\pi c} F_k A^{ik} = -\frac{\mu c}{4\pi} E_k H^{ik}. \tag{2.108}$$

We are going to consider a particular case, where the field depends on only one coordinate. Waves of such field traveling along the sole direction are known as *plane waves*.

We assume that the field depends on only axis $x^1 = x$, so only component J^1 of the field momentum density J^i is non-zero here. Then a plane wave of the field travels along the axis $x^1 = x$. Assuming that the space rotates in xy plane, we have

$$J^1 = -\frac{\mu}{2\pi c} F_k A^{1k} = -\frac{\mu}{2\pi c} F_2 A^{12}, \tag{2.109}$$

because in this particular case only the components $A^{12} = -A^{21}$ are non-zeroes. Replacing the tensor A^{ik} with the pseudovector Ω_{*m} of the space rotation

$$\varepsilon^{mik} \Omega_{*m} = \frac{1}{2}\, \varepsilon^{mik} \varepsilon_{mpq} A^{pq} = \frac{1}{2} \left(\delta^i_p \delta^k_q - \delta^k_p \delta^i_q \right) A^{pq} = A^{ik}, \tag{2.110}$$

we can re-write formula (2.109) in the form

$$J^1 = -\frac{\mu}{2\pi c}\, F_2\, \varepsilon^{123}\Omega_{*3}\,. \tag{2.111}$$

As it easy to see, while a plane wave of the field travels along the axis $x^1 = x$, the field "electric" and "magnetic" strengthes are directed along the axes $x^2 = y$ and $x^3 = z$, i. e. orthogonal to the direction the wave travels. Therefore waves of the field of time density are *transverse waves*.

§2.11 The wave pressure

As well as Landau and Lifshitz did it in §47 of *The Classical Theory of Fields* [5], we define the *wave pressure* of a field as the total flux of energy-momentum of the field, passing through an unit area of a wall. Actually, the wave pressure $\mathfrak{F}_i$ is the sum

$$\mathfrak{F}_i = T_{ik}n^k + T'_{ik}n^k \tag{2.112}$$

of spatial components of the energy-momentum tensor $T_{\alpha\beta}$ in a wave, falling on the wall, and of the energy-momentum tensor $T'_{\alpha\beta}$ in the reflected wave, projected on the unit spatial vector $\vec{n}_{(k)}$ orthogonal to the wall surface.

In accordance with the general definition for the chr.inv.-tensor of strengthes of a field $U_{ik} = c^2 h_{i\alpha} h_{k\beta} T^{\alpha\beta} = c^2 T_{ik}$ [1], we have

$$\mathfrak{F}_i = \frac{1}{c^2}\left(U_{ik}n^k + U'_{ik}n^k\right), \tag{2.113}$$

where $U_{ik} = c^2 T_{ik}$ and $U'_{ik} = c^2 T'_{ik}$ are the chr.inv.-tensors of the field strengthes in the falling wave and in the reflected wave. So, the three-dimensional vector of wave pressure $\mathfrak{F}_i$ has the property of chronometric invariance.

Employing formulas

$$q = \frac{\mu}{4\pi c}\left(A_{ik}A^{ik} + \frac{1}{2c^2}F_kF^k\right), \tag{2.114}$$

$$U_{ik} = -\frac{\mu c}{4\pi}\left(4A_{im}A_{\cdot k}^{m\cdot} + \frac{1}{c^2}F_iF_k + A_{mn}A^{mn}h_{ik} - \frac{1}{2c^2}F_mF^m h_{ik}\right), \tag{2.115}$$

we have obtained for the field of time density, we are going to find the pressure a wave of the field exerts on a wall.

We consider the problem in a weak gravitational field, assuming its potential w and the attracting force of gravity negligible. We can do it, because the formulas (2.114) and (2.115) contain gravitation in only higher order terms. So the space rotation plays the first violin in the wave pressure $\mathfrak{F}_i$ of the field of time density. In such weak field the gravitational inertial force acts, because of only its inertial part

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right) = -\frac{\partial v_i}{\partial t}\,, \qquad \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2} = 1\,. \tag{2.116}$$

A plane wave travels along a sole spatial direction, we assume axis $x^1 = x$. In this case the chr.inv.-tensor of the field strengthes U_{ik} has the sole non-zero component U_{11}. All the rest components of the strength tensor U_{ik} are zeroes, that simplifies this consideration.

We assume also that the space rotates around the axis $x^3 = z$ (the rotation is in xy plane) at the constant angular velocity Ω

$$A_{12} = -A_{21} = -\Omega\,, \qquad A_{13} = 0\,, \qquad A_{23} = 0\,, \tag{2.117}$$

so the linear velocity of this rotation $v_i = A_{ik}x^k$ has the components

$$v_1 = -\Omega y\,, \qquad v_2 = \Omega x\,, \qquad v_3 = 0\,. \tag{2.118}$$

Then components of the acting gravitational inertial force are

$$F_1 = -\frac{\partial v_1}{\partial t} = \Omega\,\frac{\partial y}{\partial t} = \Omega\, v_2 = \Omega^2 x\,, \tag{2.119}$$

$$F_2 = -\frac{\partial v_2}{\partial t} = \Omega^2 y\,, \qquad F_3 = 0\,. \tag{2.120}$$

Calculating the quantities

$$A_{ik}A^{ik} = 2A_{12}A^{12} = 2\Omega^2, \tag{2.121}$$

$$A_{1m}A_{\cdot 1}^{m\cdot} = A_{1m}A^{mn}h_{1n} = A_{12}A^{21}h_{11} = -\Omega^2 h_{11}\,, \tag{2.122}$$

we arrive to

$$q = \frac{\mu}{4\pi c}\left[2\Omega^2 + \frac{1}{2c^2}\,\Omega^4\left(x^2+y^2\right)\right], \tag{2.123}$$

$$U_{11} = \frac{\mu c}{4\pi}\left[2\Omega^2 h_{11} - \frac{1}{c^2}\,\Omega^4 x^2 + \frac{1}{2c^2}\,\Omega^4\left(x^2+y^2\right)h_{11}\right]. \tag{2.124}$$

We assume a coefficient of the reflection $\Re$ as the ratio between the density of the field energy q' in the reflected wave to the energy density q in the falling wave. Actually, because of $q' = \Re q$, the reflection coefficient $\Re$ is the energy loss of the field after the reflection.

So forth we assume $x = x_0 = 0$ at the reflection point on the surface of the wall. Then we have $U_{11} = qc^2 h_{11}$, that after substituting into (2.113) gives the pressure

$$\mathfrak{F}_1 = (1+\Re)\, q h_{11} n^1 \tag{2.125}$$

a plane wave of the field of time density exerts on the wall.

To bring this formula into final form in a Riemannian space becomes a problem, because coordinate axes in Riemannian spaces are curved and inhomogeneous in general. From this reason we can not define the angles between directions in a Riemannian space itself, the angle of incidence and the reflection angle of a wave for instance. At the same time, to take this problem in the Minkowski space of the Special Theory of Relativity, like as Landau and Lifshitz did it for the pressure of plane electromagnetic waves [5], would be senseless — because in the Minkowski space we have $g_{00} = 1$ and $g_{0i} = 0$, then $F_i = 0$ and $A_{ik} = 0$ by their definitions, so no the field of time density there.

To solve this problem correctly in a Riemannian space, let us introduce a *locally geodesic reference frame* as well as Zelmanov did it. So forth we introduce a locally geodesic reference frame in the point of reflection of a wave on the surface of a wall. Within infinitesimal vicinities of any point of such reference frame the fundamental metric tensor is

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \frac{1}{2}\left(\frac{\partial^2 \tilde{g}_{\alpha\beta}}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu}\right)(\tilde{x}^\mu - x^\mu)(\tilde{x}^\nu - x^\nu) + \ldots\,, \tag{2.126}$$

i. e. its components at a point, located in the vicinities, are different from the those at the point of reflection to within only the higher order terms, values of which can be neglected. Therefore, at any point of a locally geodesic reference frame the fundamental metric tensor can be accepted constant, while the first derivatives of the metric (the Christoffel symbols) are zeroes.

As a matter of fact that within infinitesimal vicinities of any point, located in a Riemannian space, a locally geodesic reference frame can be set. In the same time, at any point of this locally geodesic reference frame a tangential flat Euclidean space can be set so that this reference frame, being a locally geodesic for the Riemannian space, is the global geodesic for that tangential flat space.

The fundamental metric tensor of a flat Euclidean space is constant, so the values of $\tilde{g}_{\mu\nu}$, taken in the vicinities of a point of the Riemannian space, converge to the values of the tensor $g_{\mu\nu}$ in the flat space tangential at this point. Actually, this means that we can build a system of basic vectors $\vec{e}_{(\alpha)}$, located in this flat space, tangential to curved coordinate lines of the Riemannian space.

In general, coordinate lines in Riemannian spaces are curved, inhomogeneous, and are not orthogonal to each other (if the space is non-holonomic). Coordinate lines of the pseudo-Riemannian space of the General Theory of Relativity included. The lengths of the basic vectors are sometimes very different from the unit.

Let us denote $d\vec{r} = (dx^0, dx^1, dx^2, dx^3)$ a four-dimensional vector of infinitesimal displacement. Then

$$d\vec{r} = \vec{e}_{(\alpha)} dx^\alpha, \tag{2.127}$$

where components of the basic vectors $\vec{e}_{(\alpha)}$ tangential to these coordinate lines are

$$\begin{aligned} &\vec{e}_{(0)} = (e^0_{(0)}, 0, 0, 0)\,, \qquad &&\vec{e}_{(1)} = (0, e^1_{(1)}, 0, 0)\,, \\ &\vec{e}_{(2)} = (0, 0, e^2_{(2)}, 0)\,, \qquad &&\vec{e}_{(3)} = (0, 0, 0, e^3_{(3)})\,. \end{aligned} \tag{2.128}$$

Scalar product of the vector $d\vec{r}$ with itself gives $d\vec{r}d\vec{r} = ds^2$. On the other hand, we can write the same as follows $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$. Therefore

$$g_{\alpha\beta} = \vec{e}_{(\alpha)}\vec{e}_{(\beta)} = e_{(\alpha)} e_{(\beta)} \cos(x^\alpha; x^\beta)\,. \tag{2.129}$$

In accordance with this formula, we have

$$g_{00} = e^2_{(0)}\,. \tag{2.130}$$

Gravitational potential is $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$. So, the length of the time basic vector $\vec{e}_{(0)}$ tangential to the time line $x^0 = ct$ is

$$e_{(0)} = \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}\,, \tag{2.131}$$

the value is the lesser than the unit the greater is the gravitational potential w.

In accordance with the same formula (2.129), we have

$$g_{0i} = e_{(0)} e_{(i)} \cos\left(x^0; x^i\right), \tag{2.132}$$

$$g_{ik} = e_{(i)} e_{(k)} \cos\left(x^i; x^k\right), \tag{2.133}$$

so the linear velocity of the space rotation is

$$v_i = -c\, e_{(i)} \cos\left(x^0; x^i\right) \tag{2.134}$$

and the metric chr.inv.-tensor $h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k$ takes the form

$$h_{ik} = e_{(i)} e_{(k)} \Big[\cos\left(x^0; x^i\right) \cos\left(x^0; x^k\right) - \cos\left(x^i; x^k\right)\Big]. \tag{2.135}$$

Now, let us get back to that formula for the pressure $\mathfrak{F}_1$ (2.125), a plane wave of the field of time density traveling along the axis $x^1 = x$ exerts on a wall. As a result, we have

$$\mathfrak{F}_1 = (1 + \Re)\, q \Big[\cos^2\left(x^0; x^1\right) + 1\Big]\, n_{(1)} e^2_{(1)} \cos\left(x^1; n^1\right), \tag{2.136}$$

because the signature conditions $(+---)$ we have assumed imply that the spatial coordinate axes in the pseudo-Riemannian space are directed opposite to the same axes x^i in the tangential flat Euclidean space.

So forth we denote $\cos\left(x^1; n^1\right) = \cos\theta$, where θ is the reflection angle. Assuming $e_{(1)} = 1$, $n_{(1)} = 1$, and $v_{(1)} = v$ we obtain the wave pressure $\mathfrak{F}_{\rm N} = \mathfrak{F}_1 \cos\theta$ normal to the wall surface

$$\mathfrak{F}_{\rm N} = (1 + \Re)\left(1 + \frac{v^2}{c^2}\right) q \cos^2\theta, \tag{2.137}$$

$$q = \frac{\mu}{2\pi c}\, \Omega^2 \left(1 + \frac{v^2}{4c^2}\right), \tag{2.138}$$

that at low rotational velocities gives*

$$\mathfrak{F}_{\rm N} = (1 + \Re)\, q \cos^2\theta, \tag{2.139}$$

*Compare this formula (2.139) with the normal pressure $\mathfrak{F}_{\rm N} = (1 + \Re)\, q \cos^2\theta$ exerted by a plane electromagnetic wave in the Minkowski space, see §47 in *The Classical Theory of Fields* [5]. From this comparison we see that the wave pressure of the field of time density depends on the reflection coefficient $0 \leqslant \Re \leqslant 1$ in the same way that the pressure of electromagnetic waves.

$$q = \frac{\mu}{2\pi c}\, \Omega^2. \tag{2.140}$$

It should be noted that the most of rotations we observe can be placed slow. The reason is that the maximal of the known velocities is taken by an electron in the 1st Bohr orbit, the velocity is $v_{\rm b} = 2.1877 \times 10^8$ cm/sec. Therefore the ratio of a rotational linear velocity to the light velocity $\frac{v^2}{c^2}$, taking its maximal numerical value in the 1st Bohr orbit, reaches only 5.326×10^{-5}.

The presence of wave pressure in the field of time density opens a way to measure a numerical value of the energy-momentum constant μ of the field. For instance a gyroscope, rotating around the axis $x^3 = z$, will be a source of circular waves of the field of time density propagating in xy plane*. In this case the field strengthes chr.inv.-tensor U_{ik} has the non-zero components U_{11}, U_{12}, U_{21}. However, as it easy to calculate, the normal wave pressure of the circular wave will be different from the pressure of the plane wave (2.137) in only higher order terms. The same situation will be for spherical waves of the field. Therefore the normal pressure, the waves exert on a wall orthogonal to the direction $x^1 = x$, shall be equal to the value

$$\mathfrak{F}_{\rm N} = \frac{\mu}{2\pi c}\, (1 + \Re)\, \Omega^2 \tag{2.141}$$

to within of higher order terms withheld. Rotations at 6×10^3 rpm ($\Omega = 100$ rps) are accessible for modern gyroscopes, however rotations in atoms are much more, taking their maximal angular velocity 4.1341×10^{16} rps in the 1st Bohr orbit. A torsion balance can register forces, values of which are about 10^{-5} din. Then in accordance with the formula (2.141), if an experiment will have discovered the wave pressure $\mathfrak{F}_{\rm N} \approx 10^{-5}$ din/cm², derived from atomic transformations, the constant's numerical value will be in order of $\mu \approx 10^{-28}$ gram/sec.

Of course this is a crude supposition, based on only the limits of measurement precision. Anyhow, the exact numerical value of the energy-momentum constant μ will be arrived from special measurements with torsion balance.

*In a real experiment such gyroscope, being a non-absolute thin disk, will be a source of spherical waves of the field of time density which propagate at all spatial directions. Merely the waves will have a maximal amplitude in the gyroscope's rotation plane xy.

§2.12 Physical conditions in atoms

As it have been obtained in §2.7, chr.inv.-projections of the energy-momentum tensor of the field of time density are physical observable characteristics of the field — its density q, its momentum density J^i, and its strength tensor U_{ik}, which are

$$q = \frac{\mu}{4\pi c}\left(A_{ik}A^{ik} + \frac{1}{2c^2}F_k F^k\right), \tag{2.142}$$

$$J^i = -\frac{\mu}{2\pi c}\,F_k A^{ik} = -\frac{\mu}{2\pi c}\,F_k\,\varepsilon^{ikm}\Omega_{*m}\,, \tag{2.143}$$

$$U_{ik} = -\frac{\mu c}{4\pi}\left(4A_{im}A^{m\cdot}_{\cdot k} + \frac{1}{c^2}F_iF_k + A_{mn}A^{mn}h_{ik} - \frac{1}{2c^2}F_mF^m h_{ik}\right). \tag{2.144}$$

The formulas must work everywhere, in atoms included. In the same time, physical conditions in atoms are under control of Bohr's quantum postulates. So looking at an atom from outside, we can represent it as a tiny gyroscope, rotations of which are defined by the quantum rules. The quantified rotations of electrons are sources of the field of time density, which shall be perceptible, because of the super-rapid angular velocities up to the maximal value in the 1st Bohr orbit $\Omega_{\text{b}} = 4.1341\times10^{16}$ rps. This is a way to formulate physical conditions, under which the field of time density is in atoms.

So forth, taking the formulas into account, we formulate the physical conditions with the postulates, which actually are a result of that we have applied the Bohr postulates to the field of time density in atoms.

Postulate I *The field of time density in atoms remains unchanged in the absence of outer effects. An atom radiates or absorbs waves of the field of time density in only transitions of the electrons between their stationary orbits.*

Naturally, when an atom is in a stable state, then all the electrons are located in their orbits. Such stable atom, having a quantum set of the orbital angular velocities, must possess numerous quantum values of the density of time. The values are set up with the second postulate*.

*To introduce the second postulate we assume a reference frame in an atom, where an electron rotates around the nucleus at the angular velocity Ω in xy plane.

Postulate II *The field of time density is quantified in atoms. Its energy density and the momentum density take quantum numerical values which, in accordance with the quantization of electron orbits, in n-th stationary orbit are*

$$q_{\text{n}} = \frac{\mu}{2\pi c}\left(1 + \frac{v_{\text{n}}^2}{4c^2}\right)\frac{v_{\text{n}}^2}{R_{\text{n}}^2}\,, \tag{2.145}$$

$$J_{\text{n}} = \sqrt{(J_iJ^i)_{\text{n}}} = \frac{\mu}{2\pi c}\,\Omega_{\text{n}}^3 R_{\text{n}} = \frac{\mu}{2\pi c}\frac{v_{\text{n}}^3}{R_{\text{n}}^2}\,. \tag{2.146}$$

As a matter of fact that q_{n} and J_{n} take their maximal numerical values in an atom in the 1st Bohr orbit where, measuring the values in the units of the energy-momentum constant μ to within four significant digits, we have

$$q_{\text{n}=1} = 9.074\times10^{21}\,\mu \quad \text{erg/cm}^3, \tag{2.147}$$

$$J_{\text{n}=1} = 1.985\times10^{30}\,\mu \quad \text{erg/cm}^2\times\text{sec}\,. \tag{2.148}$$

Then $A_{12} = -A_{21} = -\Omega$, $A_{13} = 0$, $A_{23} = 0$. So out of all components of Ω^{*i} solely the Ω^{*3} is non-zero, which equals

$$\Omega^{*3} = \frac{1}{2}\,\varepsilon^{3mn}A_{mn} = \frac{1}{2}\left(\varepsilon^{312}A_{12} + \varepsilon^{321}A_{21}\right) = \varepsilon^{312}A_{12} = \frac{e^{312}}{\sqrt{h}}\,A_{12} = -\frac{\Omega}{\sqrt{h}}\,,$$

$$\Omega_{*3} = \frac{1}{2}\,\varepsilon_{3mn}A^{mn} = \varepsilon_{312}A^{12} = e_{312}\sqrt{h}\,A_{12} = -\sqrt{h}\,\Omega\,.$$

Calculating $h = \det\|h_{ik}\|$ here, we note that components of the linear velocity $v_i = A_{ik}x^k$ of the space rotation in this reference frame are $v_1 = -\Omega y$, $v_2 = \Omega x$, $v_3 = 0$. So forth we obtain

$$h_{11} = 1 + \frac{1}{c^2}\Omega^2 y^2, \quad h_{22} = 1 + \frac{1}{c^2}\Omega^2 x^2, \quad h_{12} = -\frac{1}{c^2}\Omega^2 xy\,, \quad h_{33} = 1\,,$$

$$h = \det\|h_{ik}\| = h_{11}h_{22} - \left(h_{12}\right)^2 = 1 + \frac{1}{c^2}\Omega^2\left(x^2 + y^2\right).$$

In the 1st Bohr orbit we have

$$\frac{1}{c^2}\Omega^2\left(x^2 + y^2\right) = \frac{1}{c^2}\Omega^2 R^2 \approx 5.3\times10^{-7},$$

so we can accept $h \approx 1$ to within of higher order terms withheld. Looking back at formulas for Ω^{*3} and Ω_{*3}, we see that the space rotates in atoms at the constant angular velocity

$$\Omega^{*3} = -\Omega\,, \qquad \Omega_{*3} = -\Omega\,,$$

then in the assumed reference frame we have $A_{ik}A^{ik} = 2A_{12}A^{12} = 2\Omega_{*3}\Omega^{*3} = 2\Omega^2$, and also

$$F_1 = -\frac{\partial v_1}{\partial t} = \Omega^2 x\,, \qquad F_2 = -\frac{\partial v_2}{\partial t} = \Omega^2 y\,, \qquad F_3 = 0\,,$$

that is taken into account in the Postulate II.

Calculating the field density in the neighbour quantum levels n and n+1, we take into account that the n-th orbital radius relates to the 1st Bohr radius as $R_{\rm n} = {\rm n}^2 R_{\rm b}$. As a result we obtain

$$\overline{q} = q_{\rm n} - q_{\rm n+1} = \frac{\mu}{2\pi c}\,\Omega_{\rm b}^2 \left\{\left[\frac{1}{{\rm n}^6} - \frac{1}{({\rm n}+1)^6}\right] + \frac{v_{\rm b}^2}{4c^2}\left[\frac{1}{{\rm n}^8} - \frac{1}{({\rm n}+1)^8}\right]\right\}, \qquad (2.149)$$

so the difference between the field density in the neighbour levels is inversely proportional to $\rm n^7$ at $\rm n \gg 1$

$$\overline{q} = q_{\rm n} - q_{\rm n+1} \approx \frac{1}{{\rm n}^7}\frac{3\mu}{\pi c}\,\Omega_{\rm b}^2\,, \qquad (2.150)$$

and $\overline{q} \to 0$ at the quantum numbers $\rm n \to \infty$.

Theoretically, non-zero the field density $q \neq 0$ must result a flux of the field momentum (this flux is characterized by the field strength tensor $U_{ik} = \frac{1}{3}\,qc^2 h_{ik} - \beta_{ik})$. Then an electron, moving in its orbit, should be radiating a momentum flux of the field of time density (waves of the field). Because of the momentum loss for the radiation, the electron would be decreasing its own angular velocity, that contradicts to experimental facts on the stability of atoms in the absence of outer effect. To remove this contradiction the third postulate is.

Postulate III *An atom radiates a quantum portion of momentum flux of the field of time density, when an electron transits from n-th quantum level into (n+1)-th level in the atom. When an electron transits from (n+1)-th level into n-th level, then the atom absorbs the same portion of the momentum flux, which is*

$$\overline{U}_{11} = U_{11}^{\rm n} - U_{11}^{\rm n+1} = \frac{\mu c}{2\pi}\,\Omega_{\rm b}^2 \left\{\left[\frac{1}{{\rm n}^6} - \frac{1}{({\rm n}+1)^6}\right] - \frac{v_{\rm b}^2}{4c^2}\left[\frac{1}{{\rm n}^8} - \frac{1}{({\rm n}+1)^8}\right]\right\}. \qquad (2.151)$$

We assume in this formula that the atom radiates/absorbs a plane wave of the field of time density, which travels along $x^1 = x$ axis. Taking the formula at $\rm n \gg 1$, we have

$$\overline{U}_{11} = U_{11}^{\rm n} - U_{11}^{\rm n+1} \approx \frac{1}{{\rm n}^7}\frac{3\mu c}{\pi}\,\Omega_{\rm b}^2\,, \qquad (2.152)$$

that at the quantum numbers $\rm n \to \infty$ gives $\overline{U}_{11} \to 0$. So, at the high quantum numbers $\rm n \gg 1$ we have the ratio

$$\overline{U}_{11} = \overline{q} c^2. \qquad (2.153)$$

In accordance with the correspondence principle, any result of quantum theory at the high quantum numbers must become the same result, obtained in the frames of classical approach. Following this way we take definitions for q (2.142) and U_{ik} (2.144), obtained using the methods of classical theory, into consideration under that $h \approx 1$ in atoms (see §2.12). As a result we arrive to the formulas

$$q = \frac{\mu}{2\pi c}\,\Omega^2\left(1 + \frac{v^2}{4c^2}\right) \approx \frac{\mu}{2\pi c}\,\Omega^2, \qquad (2.154)$$

$$U_{ik} = \frac{\mu c}{2\pi}\,\Omega^2\left(h_{11} - \frac{v^2}{2c^2} + \frac{v^2}{4c^2}h_{11}\right) \approx \frac{\mu c}{2\pi}\,\Omega^2, \qquad (2.155)$$

which lead to the same relationship

$$U_{11} = qc^2, \qquad (2.156)$$

that quantum theory have given (2.153). So, the correspondence principle is true for the field of time density in atoms.

Postulate III has two consequences, which put wave pressure in the field of time density into dependence on sub-atom processes.

Consequence I *An exciting atom, radiating the momentum flux of the field of time density, produces positive wave pressure in the field.*

So, when an atom is excited, then its electron is displaced from n-th quantum orbit into (n+1)-th orbit and so forth. Such atom radiates a wave of the field of time density, a pressure of which is positive. Calculating this positive pressure, orthogonal to the surface of a wall, at the high quantum numbers $\rm n \gg 1$ we obtain

$$\overline{\mathfrak{F}}_{\rm N} = (1 + \Re)\,\overline{q}\cos^2\theta\,, \qquad (2.157)$$

where θ is the reflection angle, $\Re$ is the reflection coefficient.

Consequence II *When an atom absorbs the momentum flux of the field of time density, then wave pressure in the field near the atom becomes negative.*

As a matter of fact that the negative wave pressure, produced by a relaxing atom in its field of time density, must be

$$\overline{\mathfrak{F}}_{\rm N} = -\,(1 + \Re)\,\overline{q}\cos^2\theta\,. \qquad (2.158)$$

That is, in accordance with the theory, exciting atoms must radiate waves of the field of time density. One of effects derived from the radiation must be positive pressure of the waves. To the contrary, relaxing atoms, absorbing waves of the field of time density, must be sources of negative wave pressure in the field near them. The predicted repulsion/attraction produced by sub-atom processes, being outside actions of electromagnetic or gravitational fields, are peculiarities of only the theory of the field of time density that here. Therefore the given conclusions open wide possibilities to check the whole theory in practice.

In particular case, for instance, if a torsion balance will have registered the repulsing/attracting wave pressure $\overline{\mathfrak{F}}_{\mathrm{N}}$, derived from exciting/relaxing sub-atom processes, then we will have obtain the numerical value of the energy-momentum constant μ. After substituting $\overline{q}$ (2.150) into the wave pressure $\overline{\mathfrak{F}}_{\mathrm{N}}$, assuming $\cos\theta=1$, we arrive to the necessary formula for experimental calculations

$$\mu = \frac{\pi c \mathrm{n}^7}{3\Omega_{\mathrm{b}}^2}\frac{\overline{\mathfrak{F}}_{\mathrm{N}}}{(1+\mathfrak{R})}\,. \tag{2.159}$$

§ 2.13 Conclusions

Let us collect the main results we have obtained in this Chapter together.

Projecting an interval of four-dimensional coordinates dx^{α} on the time line of an observer, who accompanies to his references ($b^i=0$), we obtain an interval of physical time $d\tau=\frac{1}{c}\,b_{\alpha}dx^{\alpha}$ he observes. Observations in the same spatial point give $d\tau=\sqrt{g_{00}}\,dt$, so the operator of projection on time lines b^{α} defines observable non-uniformity of time references in the accompanying reference frame.

So, non-uniformities of time references in the observer's spatial section can be represented as the field of "density" of observable time τ. The projecting operator b^{α} is the field "potential", chr.inv.-projections of which are $\varphi=1$ and $q^i=0$.

The field tensor $F_{\alpha\beta}=\nabla_{\alpha}b_{\beta}-\nabla_{\beta}b_{\alpha}$ was introduced in the way as well as Maxwell's tensor of electromagnetic field. Its chr.inv.-projections $E^i=-\frac{1}{c^2}F^i$ and $H_{ik}=-\frac{2}{c}A_{ik}$ are derived from gravitational inertial force and rotation of the space. We called E^i the "electric" observable component and H_{ik} the "magnetic" observable component of the field of time density. In the same way we introduced the field pseudotensor $F^{*\alpha\beta}$ dual to the tensor $F_{\alpha\beta}$.

The field of time density can be spatially isotropic (one of its invariants $J_1=F_{\alpha\beta}F^{\alpha\beta}$ or $J_2=F_{\alpha\beta}F^{*\alpha\beta}$ becomes zero) under the conditions:

- invariant $A_{ik}A^{ik}$ of the field of the space rotation and invariant F_iF^i of the field of gravitational inertial force reduce one another;
- the acting force F_i is orthogonal to pseudovector Ω^{*i} of the space rotation, i. e. the equality $F_i\Omega^{*i}=0$ is true;
- both of the conditions are realized together.

Equations of motion of a free mass-bearing particle, being expressed through the "electric" E^i and the "magnetic" H_{ik} components of the field of time density, group them into an acting force, a form of which is like Lorentz' force. So, the field of time density acts mass-bearing particle as well as electromagnetic field moves electric charge. In particular, if the particle moves along only time lines (the particle is at rest in respect of the observer in his spatial section), the equations show follows:

1. The "electric" and the "magnetic" components of the field of time density do not produce a work to displace the particle;
2. In this case $E^i=0$, so the particle falls freely along its own time line, because of carrying by solely the "magnetic" component $H_{ik}\neq 0$ of the field of time density;
3. Inhomogeneity Δ^i_{jk} of the spatial section or its deformations D_{ik} do not effect on free motion along time lines.

In other word, the space rotation A_{ik} as if "screws" particles into time lines. Because observable particles with whole the spatial section move from past into future, a "start" non-holonomity $A_{ik}\neq 0$ shall be in our real space, that is a "primordial non-orthogonality" of the real spatial section to time lines. Additional conditions shall be only an "add-on" intensifying or reducing this start-rotation of the space.

In general case, when a free mass-bearing particle moves also along the spatial section, its displacement realizes because of two factors:

1. The particle is carried with the field of time density by its "electric" $E^i\neq 0$ and "magnetic" $H_{ik}\neq 0$ components;

2. The particle is also moved by forces, which display themselves as effects of inhomogeneity Δ^i_{nk} and deformations D_{ik} of the space. From these, only the field of spatial deformations produces work to displace the particle.

A system of equations of the field of time density consists of Lorentz' condition $\nabla_\sigma b^\sigma = 0$, Maxwell-like equations $\nabla_\sigma F^{\alpha\sigma} = \frac{4\pi}{c} j^\alpha$ and $\nabla_\sigma F^{*\alpha\sigma} = 0$, and also the continuity equation $\nabla_\sigma j^\sigma = 0$, which define the main properties of the field:

1. The Lorentz condition becoming zero spur $D = h^{ik} D_{ik} = 0$ of the tensor of the spatial deformations, implies that a deforming elementary volume, filled with the field of time density, does not expand;
2. The 1st group of the Maxwell-like equations defines sources ρ and j^i, inducing the field of time density:

   "Charge" ρ displays itself as the difference between inhomogeneity ${}^*\nabla_i F^i$ of the field of gravitational inertial force and invariant $A_{ik} A^{ik}$ of the field of the space rotation;

   "Currents" j^i of the field of time density are derived from inhomogeneity ${}^*\nabla_k A^{ik}$ of the space rotation, corrected with higher order terms depending on the A_{ik} and F_i;
3. The Maxwell-like equations of the 2nd group show properties of the "magnetic" component $H^{*i} = -\frac{2}{c}\,\Omega^{*i}$ of the field in their link to the space rotation:

   Inhomogeneity ${}^*\nabla_i \Omega^{*i}$ of the space rotation depends on non-orthogonality of its angular velocity Ω^{*i} to the acting force F_i;

   If the acting force is $F_i = 0$, then the space rotation is homogeneous ${}^*\nabla_i \Omega^{*i} = 0$ and stationary $\Omega^{*i} = const$;
4. The continuity equation sets up that in a homogeneous space ($\Delta^i_{nk} = 0$) without the field "currents" ($j^i = 0$) the "charge" ρ inducing the field remains unchanged.

Any field without its-inducing sources is wave. Thus d'Alembert's equations without the sources we have deduced for the field of time density in couple with the field sources equalized to zero define the main properties of waves of the field:

1. The rate of deformations of a surface element in waves of the field of time density is powered by the value of the acting gravitational inertial force F_i. If the force is $F_i = 0$, then the observable spatial metric h_{ik} is stationary;
2. If the space, filled with waves of the field of time density, is homogeneous $\Delta^i_{kn} = 0$ and the acting force field is stationary $F_i = const$ there, then the spatial structure of the space deformations is the same that the space rotation;
3. Inhomogeneity of the acting force ${}^*\nabla_i F^i$ in the wave field increases with the speed of the space rotation A_{ik};
4. Inhomogeneity of rotations ${}^*\nabla_k A^{ik}$ in a space element, filled with the wave field, is derived from non-orthogonality of the acting force F_i to the field A^{ik}, and also from non-stationarity of the force F_i.

Energy-momentum tensor $T^{\alpha\beta}$ we have deduced for the field of time density has the observable projections: chr.inv.-scalar q of the field density; chr.inv.-vector J^i of the field momentum density, and chr.inv.-tensor U^{ik} of the field strengthes. Their specific formulas define physical properties of the field:

1. The field of time density is non-stationary distributed media $q \neq const$, it becomes stationary $q = const$ under stationary rotation $A_{ik} = const$ of the space and stationary gravitational inertial force $F_i = const$;
2. The field bears momentum ($J^i \neq 0$, in general case), so it can transfer impulse. The field does not transfer impulse $J^i = 0$, if the space is holonomic $A_{ik} = 0$. The absence of gravitation does not effect that the field can transfer impulse;
3. In a rotating space $A_{ik} \neq 0$ the field is emitting media $J^i \neq 0$;
4. The field is viscous. This viscosity α_{ik}, deriving from the space rotation or from gravitational inertial force, is anisotropic. The anisotropy β_{ik} of the viscosity increases with the speed of the rotation. The field is viscous anisotropic under any conditions;
5. The equation of state of the field is $p = \frac{1}{3} q c^2$, so the field is filled in the state of ultrarelativistic gas. At positive density of the media its pressure becomes positive — the media compresses.

Having a plane wave of the field considered, we have concluded that waves of the field of time density are transverse. Calculations of the wave pressure showed that the main goal into the pressure is arrived from atom and sub-atom transformations, because of high rotational velocities are there. So forth we have formulated physical conditions, under which the field of time density is in atoms. Being results of Bohr's postulates applied to the field, the physical conditions have two substantial consequences:

CONSEQUENCE I *An exciting atom, radiating the momentum flux of the field of time density, produces positive wave pressure in the field.*

CONSEQUENCE II *When an atom absorbs the momentum flux of the field of time density, then wave pressure in the field near the atom becomes negative.*

Possible experimental tests of the conclusions may be based on that the predicted repulsion/attraction, produced by sub-atom processes, being outside of known effects of electromagnetic or gravitational fields, are peculiarities of only this theory.

So, the results we have obtained in this Chapter imply that even if non-uniformity of time references is a tiny correction to ideal time, a field of this non-uniformity that is the field of time density does more fundamental effect on observable phenomena, than ones presupposed before.

◇

Chapter 3

THE SPEED OF GRAVITATION

This Chapter defines the speed of gravitation as a speed of waves, traveling in the field of gravitational inertial force. This speed, deriving from d'Alembert's equations of the field, is equal to the light velocity corrected with gravitational potential. This approach leads to the new experiment to measure the speed of gravitation, which is not linked to deviation of geodesic lines and quadrupole mass-detectors. Considering vortical gravitational fields, a tensor of which is the rotor of gravitational inertial force, we conclude that regular are traveling waves of the force, while the standing waves are linked to exotic conditions.

§ 3.1 Preliminaries

D'Alembert's operator $\square = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ is the four-dimensional generalization of Laplace's operator $\Delta = -g^{ik}\nabla_i\nabla_k$ in pseudo-Riemannian spaces. We take a pseudo-Riemannian space with the signature $(+---)$, where time is real and spatial coordinates are imaginary, because the projection of four-dimensional impulse on the spatial section of any given observer is positive in this case. In touch to the operators we are considering it implies that the time term in the d'Alembert operator will be positive, while the spatial terms will be negative. For instance, oversimplifying the pseudo-Riemannian space to Minkowski's space (the flat, homogeneous, and isotropic space-time of the Special Theory of Relativity) we have

$$\square = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\,, \qquad \Delta = \frac{\partial^2}{\partial x^1 \partial x^1} + \frac{\partial^2}{\partial x^2 \partial x^2} + \frac{\partial^2}{\partial x^3 \partial x^3}\,. \tag{3.1}$$

Applying the d'Alembert operator $\square = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ to a tensor field (any rank tensor field welcome, zero rank included) we obtain the *d'Alembert equations* of the field. The resulting equations may be equal to zero or not. The non-zeroes are the d'Alembert equations of the field containing its-inducing sources like as "charges" or "currents". If no the field-inducing sources, then the field is free. This is a field of free traveling waves. So, the d'Alembert equations equalized to zero are equations of propagation of waves of the field.

As a matter of fact that in a pseudo-Riemannan space, which is inhomogeneous, anisotropic, and curved in general case, the d'Alembert operator have more compound form than it is in the Minkowski space (3.1). So, more additional terms and functional coefficients are there, which take the space inhomogeneity, the anisotropy, and the curvature into account. However the second derivatives in respect to time and spatial coordinates remain, they are possible with functional coefficients, which can reach different numerical values, zero included.

In general case the time term of the d'Alembert equations can be represented in the form $\frac{1}{a^2}\frac{\partial^2}{\partial t^2}$, which includes the linear velocity a of waves traveling in the field. For this reason, considering the d'Alembert equations without the sources, we can conclude something on the speed of waves of the field.

So, being applied to gravitational fields, the d'Alembert equations can give a possibility to calculate the speed of propagation of gravitational attraction (the *speed of gravitation*) and to propose an experiment measuring the speed in practice. In the same time the result may be different in the dependence of a way we define the speed as the velocity of waves of the metric or something else. Accordingly, different in principle will be the final "experimentum crucis" to measure the speed of gravitation.

A regular approach set forth the speed of propagation of gravitational attraction as follows [5, 16]. One considers the space-time metric $g_{\alpha\beta}=g^{(0)}_{\alpha\beta}+\zeta_{\alpha\beta}$, composed of a Galilean metric $g^{(0)}_{\alpha\beta}$ (wherein $g^{(0)}_{00}=1$, $g^{(0)}_{0i}=0$, $g^{(0)}_{ik}=-\delta_{ik}$) plus tiny corrections $\zeta_{\alpha\beta}$ defining a weak gravitational field. Because $\zeta_{\alpha\beta}$ are tiny, we can lift and lower indices with the Galilean metric tensor $g^{(0)}_{\alpha\beta}$. The contravariant tensor $\zeta^{\alpha\beta}$ is defined with the main peculiarity of the fundamental metric tensor $g_{\alpha\sigma}g^{\sigma\beta}=\delta^{\beta}_{\alpha}$ applied to this case as $(g^{(0)}_{\alpha\sigma}+\zeta_{\alpha\sigma})\,g^{\sigma\beta}=\delta^{\beta}_{\alpha}$. Besides the approach defines $g^{\alpha\beta}$ and $g=\det\|g_{\alpha\beta}\|$ to within higher order terms as $g^{\alpha\beta}=g^{(0)\alpha\beta}-\zeta^{\alpha\beta}$ and $g=g^{(0)}(1+\zeta)$, where $\zeta=\zeta^{\sigma}_{\sigma}$. As it was shown forth [16, 5], because $\zeta_{\alpha\beta}$ are tiny we have a possibility to take Ricci's tensor $R_{\alpha\beta}=R^{\cdots\sigma}_{\alpha\sigma\beta}$ (the Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$ contracted by two indices) to within a first approximation of higher order terms withheld. As a result the Ricci tensor for the metric $g_{\alpha\beta}=g^{(0)}_{\alpha\beta}+\zeta_{\alpha\beta}$ arrives to the form

$$R_{\alpha\beta}=\frac{1}{2}\,g^{(0)\mu\nu}\frac{\partial^2\zeta_{\alpha\beta}}{\partial x^{\mu}\partial x^{\nu}}=\frac{1}{2}\Box\,\zeta_{\alpha\beta}\,, \tag{3.2}$$

that simplifies Einstein's equations $R_{\alpha\beta}-\frac{1}{2}\,g_{\alpha\beta}\,R=-\kappa\,T_{\alpha\beta}+\lambda g_{\alpha\beta}$, wherein this case means $R=g^{(0)\mu\nu}R_{\mu\nu}$. In the absence of substance and λ-fields ($T_{\alpha\beta}=0$, $\lambda=0$), that is in emptiness, the Einstein equations for the metric $g_{\alpha\beta}=g^{(0)}_{\alpha\beta}+\zeta_{\alpha\beta}$ become to the equations

$$\Box\,\zeta^{\beta}_{\alpha}=0\,, \tag{3.3}$$

go into details of the speculations with Eddington [16] or with Landau and Lishitz [5].

From geometric viewpoint these are the d'Alembert equations of the corrections $\zeta_{\alpha\beta}$ defining a weak gravitational field in the metric $g_{\alpha\beta}=g^{(0)}_{\alpha\beta}+\zeta_{\alpha\beta}$. So, the equations (3.3) are equations for waves, which propagate in the field of weak corrections of metric. Actually, the waves are *weak waves of the metric*. Because the second derivatives of the metric (the space-time curvature) are not zero herein, then the 1st derivatives defining deformation of space shall be non-zeroes as well. From this reason ones concluded that waves of the corrections $\zeta_{\alpha\beta}$ of the metric are also waves of deformation of space. Considered a flat wave of the field $\zeta_{\alpha\beta}$ that propagates at a sole spatial direction $x^1=x$, we see

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)\zeta^{\beta}_{\alpha}=0\,, \tag{3.4}$$

so weak waves of the metric $\zeta_{\alpha\beta}$ travel with the light velocity in an empty space.

This approach leads to an experiment, based on that geodesic lines of two infinitely close test-particles deviate in a field of waves of the metric. A system of two real particles connected through a spring (that is a quadrupole mass-detector) shall react to the waves. This problem collected much literature, looking it from different viewpoints. In particular, a solution of the problem with the mathematical methods of chronometric invariants had been given in [17, 18]. The most of attempts to register waves of the metric in deviating test-particles were linked with Weber's detector since 1968. Those experiments have not arrived to a result yet, because of problems with measuring precision [19] and some other theoretical problems, mentioned in [18]. So, the velocity of waves of the metric is not measured until now.

Actual is the next question. Is the approach that above the best? Really, the resulting d'Alembert equations (3.3) are derived from the formula of the Ricci tensor (3.2), which was obtained under the substantial simplifications of higher order terms withheld. Eddington in his *The Mathematical Theory of Relativity* [16] wrote that a source of this approximation is a specific reference frame which differences from Galilean reference frames with the tiny corrections $\zeta_{\alpha\beta}$, an origin of which could be very different, not only gravitation. If we take the corrections $\zeta_{\alpha\beta}$ under the substantial simplifications that the above (a geometrical sense of those is not clear), then waves of the "add-on" $\zeta_{\alpha\beta}$ to the metric $g_{\alpha\beta}$ will propagate with the light velocity. If the "add-on" would be chosen otherwise, then the speed of the waves would be another, not the velocity of light. So, the result of the approach depends on our choice of a specific reference frame. This is a "vicious circle", Eddington wrote.

As an alternative to this approach, Eddington considered waves of invariant $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ of the metric $g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \zeta_{\alpha\beta}$. Following this way, after some algebra, he had arrived to that this case is depended on those simplifications that the above, so he had arrived to the same d'Alembert equations $\Box\zeta^{\beta}_{\alpha} = 0$.

So, we can conclude that the regular approach has got its own drawbacks as follows:

1. This approach gives a possibility to obtain the Ricci tensor and hence the d'Alembert equations of the metric to within only a first approximation of higher order terms withheld. From this reason the velocity of waves of the metric, being calculated this way, does not finally exact theoretical result;
2. Besides, as Eddington noted it, a source of this approximation is a specific reference frame which differences from Galilean reference frames with tiny corrections, an origin of which may be very different, not only gravitation. In general, a nature of this approximation is not clear;
3. Third, two bodies attract one another because of gravitational forces, so from physical viewpoint a speed to transfer gravitational interaction is the velocity of propagation of gravitational force. A wave traveling in the field of gravitational force is not the same that a wave of the metric. These are different tensor fields in a four-dimensional pseudo-Riemannian space*. For instance, waves of the metric are possible in homogeneous gravitational fields, however waves of gravitational force field as well as gravitational attraction itself are absent there, because the gradient of the field potential w is zero. When a quadrupole mass-detector registers a signal, then the detector reacts a wave of the metric in accordance with this experiment theory. Therefore it is possible that quadrupole mass-detectors would be good to discover waves of the metric, however the experiment is only oblique to measure the speed of gravitation.

The mentioned reasons lead us to consider gravitational waves as waves traveling in the field of gravitational force, that provides two important advantages:

1. The mathematical methods of chronometric invariants define gravitational inertial force F_i (1.16) without the Riemann-Christoffel curvature tensor that is without the second derivatives of the metric. As a result, using the mathematical methods we have a possibility to obtain the exact d'Alembert equations for the field of gravitational inertial force, hence we can to found an exact formula defining the velocity of the waves traveling in the field;
2. Experiments to register waves of the field of gravitational inertial force can take a base other in principle "detectors" like as planets or their satellites, that does not link to the quadrupole mass-detector experiment and its own specific technical problems. So, this approach opens a way to set up other in principle experiments to measure the speed of gravitation.

This new approach in details and the results the approach has found will be a subject of the next Paragraphs.

§ 3.2 The new approach

Four-dimensional generalizations of the chr.inv.-quantities F_i, A_{ik}, D_{ik} had been obtained by Zelmanov in 1960's [20]

$$F_{\alpha} = -2c^2 b^{\beta} a_{\beta\alpha}\,, \quad A_{\alpha\beta} = ch^{\mu}_{\alpha} h^{\nu}_{\beta} a_{\mu\nu}\,, \quad D_{\alpha\beta} = ch^{\mu}_{\alpha} h^{\nu}_{\beta} d_{\mu\nu}\,, \qquad (3.5)$$

*This does not except that the velocities of these waves may be equal.

where the auxiliary quantities $a_{\alpha\beta}$ and $d_{\alpha\beta}$ have the form

$$a_{\alpha\beta}=\frac{1}{2}\left(\nabla_\alpha b_\beta-\nabla_\beta b_\alpha\right)=\frac{1}{2}\left(\frac{\partial b_\beta}{\partial x^\alpha}-\frac{\partial b_\alpha}{\partial x^\beta}\right), \tag{3.6}$$

$$d_{\alpha\beta}=\frac{1}{2}\left(\nabla_\alpha b_\beta+\nabla_\beta b_\alpha\right). \tag{3.7}$$

Note, as it have been shown in the previous Chapter, the vortex $F_{\alpha\beta}=2a_{\alpha\beta}$ defines the field of non-uniformity of time coordinates in pseudo-Riemannian spaces.

So, in this Chapter we are going to consider a field of the gravitational inertial force

$$F_\alpha=-2c^2b^\beta a_{\beta\alpha}=-\,c^2b^\beta\left(\frac{\partial b_\alpha}{\partial x^\beta}-\frac{\partial b_\beta}{\partial x^\alpha}\right). \tag{3.8}$$

As it is not difficult to see, chr.inv.-projections of this four-dimensional vector F_α are

$$\varphi=\frac{F_0}{\sqrt{g_{00}}}=0\,,\qquad q^i=F^i=\frac{1}{\sqrt{g_{00}}}\,h^{ik}\left(\frac{\partial \mathrm{w}}{\partial x^k}-\frac{\partial v_k}{\partial t}\right), \tag{3.9}$$

hence its covariant (lower-index) chr.inv.-component is $F_i=h_{ik}F^k$ (1.16). The d'Alembert equations of the vector field $F^\alpha=-2c^2a_{\sigma\cdot}^{\cdot\alpha}b^\sigma$ without its-inducing sources

$$\Box\,F^\alpha=0 \tag{3.10}$$

are the equations of propagation of waves traveling in the field. Chr.inv.-projections of the equations are

$$b_\sigma\,\Box\,F^\sigma=0\,,\qquad h^i_\sigma\,\Box\,F^\sigma=0\,. \tag{3.11}$$

Actually, to obtain the projections in detailed form is the same that to express

$$b_\sigma\,g^{\alpha\beta}\,\nabla_\alpha\nabla_\beta\,F^\sigma=0\,,\qquad h^i_\sigma\,g^{\alpha\beta}\,\nabla_\alpha\nabla_\beta\,F^\sigma=0 \tag{3.12}$$

through chr.inv.-quantities. It should be noted that this is not very easy process, so long introduction into chronometric invariants was given in Chapter 1 to support someone who would like to pass this way independently. I think, those should be enough to approve the calculations. A help here is that the chr.inv.-projection of the vector F^α on time line is zero (3.9). Everyone can check the resulting

equations by substituting $\varphi=0$ and $q^i=F^i$ into the d'Alembert chr.inv.-equations for an arbitrary vector field (see §1.6).

Following this way after some algebra we obtain to the d'Alembert chr.inv.-equations for the field $F^\alpha=-2c^2a_{\sigma\cdot}^{\cdot\alpha}b^\sigma$ without its-inducing sources (3.11) in the final form

$$\begin{gathered}\frac{1}{c^2}\frac{^*\partial}{\partial t}\left(F_kF^k\right)+\frac{1}{c^2}F_i\frac{^*\partial F^i}{\partial t}+D^k_m\frac{^*\partial F^m}{\partial x^k}+\\ +h^{ik}\frac{^*\partial}{\partial x^i}\left[(D_{kn}+A_{kn})F^n\right]-\frac{2}{c^2}A_{ik}F^iF^k+\frac{1}{c^2}F_mF^mD+\\ +\Delta^m_{kn}D^k_mF^n-h^{ik}\Delta^m_{ik}\left(D_{mn}+A_{mn}\right)F^n=0\,,\end{gathered} \tag{3.13}$$

$$\begin{gathered}\frac{1}{c^2}\frac{^*\partial^2F^i}{\partial t^2}-h^{km}\frac{^*\partial^2F^i}{\partial x^k\partial x^m}+\frac{1}{c^2}\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\frac{^*\partial F^k}{\partial t}+\\ +\frac{1}{c^2}\frac{^*\partial}{\partial t}\left[\left(D^i_k+A^{\cdot i}_{k\cdot}\right)F^k\right]+\frac{1}{c^2}D\frac{^*\partial F^i}{\partial t}+\frac{1}{c^2}F^k\frac{^*\partial F^i}{\partial x^k}+\\ +\frac{1}{c^2}\left(D^i_n+A^{\cdot i}_{n\cdot}\right)F^nD+\frac{1}{c^2}\Delta^i_{km}F^kF^m+\frac{1}{c^4}F_kF^kF^i-\\ -h^{km}\left\{\frac{^*\partial}{\partial x^k}\left(\Delta^i_{mn}F^n\right)+\left(\Delta^i_{kn}\Delta^n_{mp}-\Delta^n_{km}\Delta^i_{np}\right)F^p+\right.\\ \left.+\Delta^i_{kn}\frac{^*\partial F^n}{\partial x^m}-\Delta^n_{km}\frac{^*\partial F^i}{\partial x^n}\right\}=0\,,\end{gathered} \tag{3.14}$$

where we select the d'Alembert chr.inv.-operator and the Laplace chr.inv.-operator according to Zelmanov's definitions of those [1]

$$^*\Box=\frac{1}{c^2}\frac{^*\partial^2}{\partial t^2}-h^{ik}\frac{^*\partial^2}{\partial x^i\partial x^k}\,,\qquad {^*\Delta}=-g^{ik}\,{^*\nabla_i}{^*\nabla_k}=h^{ik}\frac{^*\partial^2}{\partial x^i\partial x^k}\,. \tag{3.15}$$

Called on the chr.inv.-derivatives (1.13), we transform the first term in the vector d'Alembert equations (3.14) to the form

$$\frac{1}{c^2}\frac{^*\partial^2F^i}{\partial t^2}=\frac{1}{c^2g_{00}}\frac{\partial^2F^i}{\partial t^2}+\frac{1}{c^4\sqrt{g_{00}}}\frac{^*\partial\mathrm{w}}{\partial t}\frac{^*\partial F^i}{\partial t}\,,\qquad \sqrt{g_{00}}=1-\frac{\mathrm{w}}{c^2}\,, \tag{3.16}$$

which show that waves of the field of gravitational inertial force travel with a velocity u^k, a modulus of which is

$$u=\sqrt{u_ku^k}=c\left(1-\frac{\mathrm{w}}{c^2}\right). \tag{3.17}$$

Because waves of the field of gravitational inertial force transfer gravitational interaction, the waves speed (3.17) is the *speed of* *gravitation* as well. The speed depends on the potential w of the field itself*, that lead us to the next conclusions:

1. In a weak gravitational field, a potential of which is neglected $\mathrm{w}\to 0$ but its gradient is non-zero $F_i\neq 0$, the speed of gravitation is equal to the velocity of light;
2. The speed of gravitation shall be lesser than the light velocity near bulk bodies like as stars or planets, where gravitational potential is perceptible. On the Earth surface the gravitational potential is $\mathrm{w}\simeq 6.27\times 10^{11}\ \mathrm{cm^2/sec^2}$, so slowing gravitational waves down in an Earth laboratory shall be about $\frac{\mathrm{w}}{c^2}\simeq 7\times 10^{-10}$ that is nothing but $21\ \mathrm{cm/sec}$ less than the light velocity. The solar field is stronger, its potential at the radius of the Sun equals $\mathrm{w}\simeq 1.9\times 10^{15}\ \mathrm{cm^2/sec^2}$. Hence, slowing gravitation down near the Sun shall be $\frac{\mathrm{w}}{c^2}\simeq 2.1\times 10^{-6}$ and the speed of gravitation shall be $6.3\times 10^{4}\ \mathrm{cm/sec}$ slow than light;
3. Under the particular condition $\mathrm{w}=c^2$, when gravitational collapse occurs, the speed of propagation of gravitational inertial force becomes zero.

Let us consider a method to measure the speed of gravitation, representing gravitational waves as the waves of gravitational force. The idea to measure the speed of gravitation as a speed to transfer the attracting force between space bodies had been proposed by mathematician Dombrowski in his conversation with me a decade before. The Weber detectors are inapplicable to put this idea into experiment, because they react to deviating geodesic lines of close test-particles in the wave field of the metric. The idea did not move experiment that time, however those friendly conversation stimulated me to make the theoretical study that here. As a result we have the mathematical representation of the speed of gravitation as a velocity of waves traveling in the field of gravitational inertial force we have calculated from the obtained d'Alembert equations.

*It is interesting, formula (1.16) defining the chr.inv.-vector F_i consists of two term, only the first of which is gravitational force. The second term is centrifugal inertial force, caused by rotation of the observer's space. The d'Alembert equations we have obtained contain the force F_i in its general form. If a gravitational field is homogeneous or absent, then the "gravitational" term of the force F_i becomes zero. Only the "inertial" term remains. In this case the obtained d'Alembert equations will be the same for centrifugal forces of inertia, so waves of the inertial force field will propagate under the same conditions as well as gravitation.

So, we have a possibility to compose a specific experiment to measure the speed of the waves in practice. An essence of the propounded experiment is as follows.

The Moon attracts the Earth surface looking to her stronger than the opposite. As a result, the flow "hump" in the ocean surface follows the moving Moon that produce ebbs and flows. Analogous "hump" follows the Sun, however its height is more lesser. A satellite in an Earth orbit have the same ebb and flow oscillations, its orbit lowers and lifts for a little following the Moon and the Sun as well. A satellite being it flies in airless space does not any friction to the contrary of viscous water in the ocean. A satellite is a perfect system without inertia, the system reacts to flow carrying of the Moon or the Sun instantly.

If the speed to transfer gravitation would be infinite (as Classical Mechanics predicts it), then the maximal rise of the lunar flow wave in a satellite orbit that is the satellite's maximal rise above the Earth should be coincided with the moment of the lunar upper culmination at the same Earth point under the satellite. If the speed of gravitation is limited, in this case the moment of the satellite's maximal flow rise should be late from the lunar culmination with the time that waves of gravitational force field traveled from the Moon to the satellite. Similar lateness shall be exist also on the flow carrying the satellite by the Sun.

The Earth gravitational field is not absolute symmetric, because of the imperfect terrestrial globe. A real satellite reacts to defects of the Earth gravitational field during its orbital flight around the Earth. Because of those defects, the height of a satellite's orbit oscillates about decades of centimeters that would be substantial noise in the propounded experiment. From this reason a geostationary satellite would be the best. Such satellite having an equatorial orbit requiring an angular velocity the same as that of the Earth, so the satellite's position in an orbit is fixed in respect of the Earth. As a result, the height of a geostationary satellite above the Earth does not depend on defects of the Earth gravitational field. The height could be measured by a laser range-finder with high precision almost without interruption, that give a possibility to register the moment of the maximal flow rise of the satellite perfectly.

In accordance with our approach, gravitational attraction is transferred with waves of the field of gravitational inertial force. The potential of the Earth gravitational field on the Earth surface is so little $w \simeq 6.27 \times 10^{11}$ cm²/sec², so the speed of gravitation that is the formula (3.17) we have obtained equals the light velocity minus a tiny correction of 21 cm/sec nearly the Earth, that is within measurement precision. Hence forth we can put the speed of gravitation in an Earth laboratory equalized to the velocity of light. Keeping this in mind, we conclude that the maximum of the lunar flow wave in a satellite orbit shall be about 1 second late from the lunar upper culmination. The lateness of the flow wave of the Sun shall be about 500 second after the upper transit of the Sun. Astronomical methods give culmination moments of the Moon and the Sun with the necessary high precision. A question is how much precisely could be registered the moment of the maximal flow rise of a satellite in its orbit, because the real maximum can be "fuzzy" in time. Anyhow this problem focuses on merely to choice the lunar flow effect or the solar flow effect to measure the speed of gravitation in the frames of the new approach.

§ 3.3 Effect of the curvature

We are going to consider the obtained d'Alembert chr.inv.-equations of the field of gravitational inertial force. Let us take the equations in a space, which is homogeneous $\Delta^i_{km} = 0$ and its metric is stationary $h_{ik} = const$ (the latest is true if the space does not deformations $D_{ik} = 0$). In this particular case the d'Alembert chr.inv.-equations (3.13, 3.14) take the form

$$\frac{1}{c^2}\frac{^*\partial}{\partial t}\left(F_k F^k\right) + \frac{1}{c^2}F_i\frac{^*\partial F^i}{\partial t} + h^{ik}\frac{^*\partial}{\partial x^i}\left(A_{kn}F^n\right) - \frac{2}{c^2}A_{ik}F^i F^k = 0\,, \tag{3.18}$$

$$\begin{gathered}\frac{1}{c^2}\frac{^*\partial^2 F^i}{\partial t^2} - h^{km}\frac{^*\partial^2 F^i}{\partial x^k \partial x^m} - \frac{2}{c^2}A^{in}\frac{^*\partial F_n}{\partial t} - \frac{1}{c^2}F_n\frac{^*\partial A^{in}}{\partial t} + \\ + \frac{1}{c^2}F^k\frac{^*\partial F^i}{\partial x^k} + \frac{1}{c^4}F_k F^k F^i = 0\,.\end{gathered} \tag{3.19}$$

This implies that waves of gravitational inertial force can be even in a non-deforming homogeneous space. Waves of the metric are linked to the space-time curvature deriving from the Riemann-Christoffel curvature tensor. If the first derivatives of the metric (that means deformation of the space) are zeroes, then its second derivatives (the space-time curvature) are zeroes as well. Therefore no any waves of the metric in a non-deforming space, while waves of gravitational inertial force are possible therein.

Under the particular conditions plus the space does not rotate, the d'Alembert equations become still more simpler

$$\frac{1}{c^2}\frac{^*\partial}{\partial t}\left(F_k F^k\right) + \frac{1}{c^2}F_i\frac{^*\partial F^i}{\partial t} = 0\,, \tag{3.20}$$

$$\frac{1}{c^2}\frac{^*\partial^2 F^i}{\partial t^2} - h^{km}\frac{^*\partial^2 F^i}{\partial x^k \partial x^m} + \frac{1}{c^2}F^k\frac{^*\partial F^i}{\partial x^k} + \frac{1}{c^4}F_k F^k F^i = 0\,, \tag{3.21}$$

so waves of gravitational inertial force are possible even in a non-deforming non-rotating homogeneous space (a metric of which is diagonal, because all the mixed components are zeroes $g_{0i} = 0$).

In connection with this fact the next question would be interesting. How much acts the curvature on waves of gravitational inertial force?

To answer the question, let us remind that Zelmanov had built a chr.inv.-tensor of the curvature [1]. This tensor, describing the curvature of a three-dimensional space of an observer (the observable curvature of his spatial section), possesses all properties of the Riemann-Christoffel curvature tensor in the three-dimensional space and in the same time has the property of chronometric invariance. It was made in the same way that the Riemann-Christoffel tensor was built, deriving from the non-commutativity of the second derivatives from an arbitrary vector taken in the given space. Taken the second chr.inv.-derivatives from an arbitrary vector

$$^*\nabla_i {}^*\nabla_k Q_l - {}^*\nabla_k {}^*\nabla_i Q_l = \frac{2A_{ik}}{c^2}\frac{^*\partial Q_l}{\partial t} + H_{lki\cdot}^{\cdots j}Q_j\,, \tag{3.22}$$

he obtained the chr.inv.-tensor

$$H_{lki\cdot}^{\cdots j} = \frac{^*\partial\Delta^j_{il}}{\partial x^k} - \frac{^*\partial\Delta^j_{kl}}{\partial x^i} + \Delta^m_{il}\Delta^j_{km} - \Delta^m_{kl}\Delta^j_{im}\,, \tag{3.23}$$

which is like Schouten's tensor from the theory of non-holonomic manifolds [21]. The tensor $H_{lki\cdot}^{\cdots j}$ algebraically differences from the Riemann-Christoffel tensor, because of the presence of rotation of the space A_{ik} in the formula (3.22). Nevertheless its generalization gives the chr.inv.-tensor

$$C_{lkij} = \frac{1}{4}\left(H_{lkij} - H_{jkil} + H_{klji} - H_{iljk}\right), \tag{3.24}$$

which has all algebraical properties of the Riemann-Christoffel tensor in this three-dimensional space. Therefore Zelmanov called C_{iklj} the *curvature chr.inv.-tensor*, which actually is the tensor of the observable curvature of the observer's spatial section. Its contraction step-by-step gives

$$C_{kj} = C_{kij\cdot}^{\cdots i} = h^{im} C_{kimj}\,, \qquad C = C_j^j = h^{lj} C_{lj}\,, \tag{3.25}$$

where the chr.inv.-scalar C is the *scalar observable curvature of the spatial section*.

The Riemann-Christoffel tensor has 256 components, while only 20 of those are significant*. Its chr.inv.-projections

$$X^{ik} = -c^2 \frac{R_{\cdot 0 \cdot 0 \cdot}^{\cdot i \cdot k}}{g_{00}}\,, \qquad Y^{ijk} = -c\,\frac{R_{0\cdots}^{\cdot ijk}}{\sqrt{g_{00}}}\,, \qquad Z^{ijkl} = c^2 R^{ijkl} \tag{3.26}$$

had been deduced by Zelmanov in lower-indices-form [1], they are

$$X_{ij} = \frac{{}^*\partial D_{ij}}{\partial t} - \left(D_i^l + A_{i\cdot}^{\cdot l}\right)\left(D_{jl} + A_{jl}\right) + \frac{1}{2}\left({}^*\nabla_i F_j + {}^*\nabla_j F_i\right) - \frac{1}{c^2} F_i F_j\,, \tag{3.27}$$

$$Y_{ijk} = {}^*\nabla_i \left(D_{jk} + A_{jk}\right) - {}^*\nabla_j \left(D_{ik} + A_{ik}\right) + \frac{2}{c^2} A_{ij} F_k\,, \tag{3.28}$$

$$Z_{iklj} = D_{ik} D_{lj} - D_{il} D_{kj} + A_{ik} A_{lj} - A_{il} A_{kj} + 2 A_{ij} A_{kl} - c^2 C_{iklj}\,, \tag{3.29}$$

where we have $Y_{(ijk)} = Y_{ijk} + Y_{jki} + Y_{kij} = 0$ like it is in the Riemann-Christoffel tensor. Contraction of the spatial observable projection Z_{iklj} step-by-step gives

$$Z_{il} = D_{ik} D_l^k - D_{il} D + A_{ik} A_{l\cdot}^{\cdot k} + 2 A_{ik} A_{\cdot l}^{k\cdot} - c^2 C_{il}\,, \tag{3.30}$$

$$Z = h^{il} Z_{il} = D_{ik} D^{ik} - D^2 - A_{ik} A^{ik} - c^2 C\,. \tag{3.31}$$

As it was shown in Synge's book [22], in a pseudo-Riemannian space of a constant four-dimensional curvature $K = const$ the next correlations take a place

$$R_{\alpha\beta\gamma\delta} = K \left(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}\right), \quad R_{\alpha\beta} = -3 K g_{\alpha\beta}\,, \quad R = -12 K\,. \tag{3.32}$$

*Because of the Riemann-Christoffel tensor is symmetric by each pair of indices $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$, antisymmetric in respect to transformation inside the each pair $R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}$, and has the peculiarity $R_{\alpha(\beta\gamma\delta)} = 0$ that stands for transpositions by the "inner" three indices.

Having the formulas in the mind, after some algebra we obtain

$$R_{0i0k} = -K h_{ik}\, g_{00}\,, \tag{3.33}$$

$$R_{0ijk} = \frac{K}{c} \sqrt{g_{00}} \left(v_j h_{ik} - v_k h_{ij}\right), \tag{3.34}$$

$$R_{ijkl} = K \Big[h_{ik} h_{jl} - h_{il} h_{kj} + \frac{1}{c^2} v_i \left(v_l h_{kj} - v_k h_{jl}\right) + \frac{1}{c^2} v_j \left(v_k h_{il} - v_l h_{ik}\right) \Big], \tag{3.35}$$

so the observable projections X^{ik}, Y^{ijk}, and Z^{ijkl} of the Riemann-Christoffel tensor in a constant curvature space are

$$X^{ik} = c^2 K h^{ik}, \quad Y^{ijk} = 0\,, \quad Z^{ijkl} = c^2 K \left(h^{ik} h^{jl} - h^{il} h^{jk}\right). \tag{3.36}$$

So forth we take the covariant component Z_{ijkl} contracted step-by-step in a constant curvature space

$$Z_{ijkl} = c^2 K \left(h_{ik} h_{jl} - h_{il} h_{jk}\right), \qquad Z_{jl} = Z_{\cdot jil}^{i\cdots} = 2 c^2 K h_{jl}\,, \tag{3.37}$$

$$Z = Z_j^j = 6 c^2 K\,, \tag{3.38}$$

and then equalize the Z (3.38) to the same quantity in an arbitrary curvature space (3.31). As a result, we obtain a correlation between the four-dimensional curvature K and the observable spatial curvature C in the constant curvature space. The correlation is

$$6 c^2 K = D_{ik} D^{ik} - D^2 - A_{ik} A^{ik} - c^2 C\,. \tag{3.39}$$

Let us consider this formula in relation to our study on gravitational waves. If the four-dimensional curvature of a constant curvature space is zero $K = 0$, and also the space does not deformations $D_{ik} = 0$, then no waves of the metric there exactly. In such space the observable three-dimensional curvature

$$C = -\frac{1}{c^2} A_{ik} A^{ik} \tag{3.40}$$

is non-zero $C \neq 0$, if the space rotates $A_{ik} \neq 0$. In the absence of deformations and rotation of the space, its three-dimensional observable curvature becomes zero $C = 0$. However even in this simply case, the obtained d'Alembert equations (3.20, 3.21) show the presence of waves of gravitational inertial force.

As a matter of fact that gravitational attraction is an everyday reality in our world, so waves of gravitational inertial force transferring the attraction shall be incontrovertible. This fact forces us to choice one of the alternatives:

1. Waves of gravitational inertial force depend on the space curvature, then the real space-time is not constant curvature space;
2. Either waves of gravitational inertial force do not depend on the curvature.

§ 3.4 Vortical gravitational fields

As a matter of fact that the attracting force of gravity is absent in homogeneous gravitational fields, because the gradient of gravitational potential w is zero everywhere there. So waves of gravitational inertial force F^α also take not a place in homogeneous gravitational fields. Therefore it is reasoning to consider the field of the vector potential F^α as a media transferring gravitational attraction via waves of the force.

So forth we introduce the tensor of the field as a rotor of its four-dimensional vector potential F^α as well as Maxwell's tensor of electromagnetic fields, namely

$$F_{\alpha\beta} = \nabla_\alpha F_\beta - \nabla_\beta F_\alpha = \frac{\partial F_\beta}{\partial x^\alpha} - \frac{\partial F_\alpha}{\partial x^\beta}\,. \tag{3.41}$$

We will refer to $F_{\alpha\beta}$ (3.41) as the *tensor of vortical gravitational field*, because actually this is a four-dimensional vortex of the acting gravitational inertial force F^α.

Taking into account that chr.inv.-projections of the field potential $F^\alpha = g^{\alpha\beta}F_\beta = -2c^2 a_{\sigma\cdot}^{\cdot\alpha}b^\sigma$ (3.5) are

$$\varphi = \frac{F_0}{\sqrt{g_{00}}} = 0\,, \qquad q^i = F^i = h^{ik}F_k\,, \tag{3.42}$$

after some algebra we arrive to components of the field tensor $F_{\alpha\beta}$ (3.41). The components are

$$F_{00} = F^{00} = 0\,, \qquad F_{0i} = -\frac{1}{c}\sqrt{g_{00}}\,\frac{{}^*\partial F_i}{\partial t}\,, \tag{3.43}$$

$$F_{ik} = \frac{{}^*\partial F_i}{\partial x^k} - \frac{{}^*\partial F_k}{\partial x^i} + \frac{1}{c^2}\left(v_i\frac{{}^*\partial F_k}{\partial t} - v_k\frac{{}^*\partial F_i}{\partial t}\right), \tag{3.44}$$

$$F_{0\cdot}^{\cdot 0} = \frac{1}{c^2}\,v^k\frac{{}^*\partial F_k}{\partial t}\,, \qquad F_{0\cdot}^{\cdot i} = \frac{1}{c}\sqrt{g_{00}}\,h^{ik}\frac{{}^*\partial F_k}{\partial t}\,, \tag{3.45}$$

$$F_{k\cdot}^{\cdot 0} = \frac{1}{\sqrt{g_{00}}}\left[\frac{1}{c}\frac{{}^*\partial F_k}{\partial t} - \frac{1}{c^3}\,v_k v^m\frac{{}^*\partial F_m}{\partial t} + \frac{1}{c}\,v^m\left(\frac{{}^*\partial F_m}{\partial x^k} - \frac{{}^*\partial F_k}{\partial x^m}\right)\right], \tag{3.46}$$

$$F_{k\cdot}^{\cdot i} = h^{im}\left(\frac{{}^*\partial F_m}{\partial x^k} - \frac{{}^*\partial F_k}{\partial x^m}\right) - \frac{1}{c^2}\,h^{im}v_k\frac{{}^*\partial F_m}{\partial t}\,, \tag{3.47}$$

$$F^{0k} = \frac{1}{\sqrt{g_{00}}}\left[\frac{1}{c}\,h^{km}\frac{{}^*\partial F_m}{\partial t} + \frac{1}{c}\,v^n h^{mk}\left(\frac{{}^*\partial F_n}{\partial x^m} - \frac{{}^*\partial F_m}{\partial x^n}\right)\right], \tag{3.48}$$

$$F^{ik} = h^{im}h^{kn}\left(\frac{{}^*\partial F_m}{\partial x^n} - \frac{{}^*\partial F_n}{\partial x^m}\right). \tag{3.49}$$

Using the formulas, we obtain chr.inv.-projections of the field tensor $F_{\alpha\beta}$. We will refer to its time projection

$$E^i = \frac{F_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} = \frac{1}{c}\,h^{ik}\frac{{}^*\partial F_k}{\partial t}\,, \qquad E_i = h_{ik}E^k = \frac{1}{c}\frac{{}^*\partial F_i}{\partial t} \tag{3.50}$$

as the "electric" observable component of the vortical gravitational field. Its spatial projection will be called the "magnetic" observable component of the field

$$H^{ik} = F^{ik} = h^{im}h^{kn}\left(\frac{{}^*\partial F_m}{\partial x^n} - \frac{{}^*\partial F_n}{\partial x^m}\right), \tag{3.51}$$

$$H_{ik} = h_{im}h_{kn}H^{mn} = \frac{{}^*\partial F_i}{\partial x^k} - \frac{{}^*\partial F_k}{\partial x^i}\,, \tag{3.52}$$

which, after using the Zelmanov 1st identity* (1.22), takes the form

$$H^{ik} = 2h^{im}h^{kn}\frac{{}^*\partial A_{mn}}{\partial t}\,, \qquad H_{ik} = 2\frac{{}^*\partial A_{ik}}{\partial t}\,. \tag{3.53}$$

The "electric" observable component E^i (3.50) of a vortical gravitational field display itself as non-stationarity of the acting gravitational inertial force F^i. The "magnetic" observable component H_{ik} (3.52) display itself as the presence of spatial vortexes of the force F^i or, that is the same, as non-stationarity of the space rotation A_{ik} (see formula 3.53). So, two different kinds of the vortical fields are possible, namely:

*The identity links spatial vortexes of gravitational inertial force to non-stationarity of rotation of the observer's space.

1. Vortical gravitational fields of "electric" kind, in which $H_{ik}=0$ and $E^i \neq 0$. In this case vortexes of the acting gravitational inertial force F^i are absent that is the same as if the space rotation is stationary. So, a vortical field of this kind is derived from only its own "electric" component E^i (3.50) that is non-stationarity of the force F^i. Note, vortical gravitational fields of "electric" kind are possible both in a non-holonomic (rotating) space if its rotation is stationary, and also in a holonomic space because the zero rotation can be assumed the ultimate case of stationary rotations;
2. The "magnetic" kind of vortical gravitational fields characterizes with $E^i=0$ and $H_{ik}\neq 0$. Such vortical field is derived from only its own "magnetic" component H_{ik} that is spatial vortexes of the acting force F^i and non-stationary rotation of the space. Therefore vortical gravitational fields of "magnetic" kind are possible if only the space is non-holonomic. Because the obtained d'Alembert equations (3.13, 3.14) under the condition $E^i=0$ do not depend on time, "magnetic" vortical gravitational fields are a media of *standing waves* of gravitational inertial force.

In addition, we introduce the pseudotensor $F^{*\alpha\beta}$ of the field dual to the field tensor

$$F^{*\alpha\beta} = \frac{1}{2}\, E^{\alpha\beta\mu\nu} F_{\mu\nu}\,, \qquad F_{*\alpha\beta} = \frac{1}{2}\, E_{\alpha\beta\mu\nu} F^{\mu\nu}, \tag{3.54}$$

where the four-dimensional completely antisymmetric discriminant tensors $E^{\alpha\beta\mu\nu}$ and $E_{\alpha\beta\mu\nu}$ (1.77) transform regular tensors into pseudotensors in inhomogeneous anisotropic pseudo-Riemannian spaces.

Using the components of the tensor $F_{\alpha\beta}$ (3.43–3.49), we obtain chr.inv.-projections of the field pseudotensor $F^{*\alpha\beta}$, which are

$$H^{*i} = \frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}} = \frac{1}{2}\, \varepsilon^{ikm}\left(\frac{^*\partial F_k}{\partial x^m} - \frac{^*\partial F_m}{\partial x^k}\right), \tag{3.55}$$

$$E^{*ik} = F^{*ik} = -\frac{1}{c}\, \varepsilon^{ikm}\frac{^*\partial F_m}{\partial t}\,, \tag{3.56}$$

where ε^{ikm} (1.82) and ε_{ikm} (1.83) are the discriminant chr.inv.-tensors. Taking the Zelmanov 1st identity (1.22) and the formula to

differentiate ε^{ikm} (2.55) into account, we can write the "magnetic" component H^{*i} (3.55) down as follows

$$H^{*i} = \varepsilon^{ikm}\frac{^*\partial A_{km}}{\partial t} = 2\left(\frac{^*\partial \Omega^{*i}}{\partial t} + \Omega^{*i} D\right), \tag{3.57}$$

where $\Omega^{*i} = \frac{1}{2}\,\varepsilon^{ikm} A_{km}$ is the chr.inv.-pseudovector of the angular velocity of the space rotation, the spur $D = h^{ik} D_{ik} = D_n^n$ of the tensor D_{ik} (1.24) is the rate of relative expansion of an elementary volume filled with the field.

So forth, calculating invariants $J_1 = F_{\alpha\beta}F^{\alpha\beta}$ and $J_2 = F_{\alpha\beta}F^{*\alpha\beta}$ for a vortical gravitational field, we arrive to the formulas

$$J_1 = h^{im}h^{kn}\left(\frac{^*\partial F_i}{\partial x^k} - \frac{^*\partial F_k}{\partial x^i}\right)\left(\frac{^*\partial F_m}{\partial x^n} - \frac{^*\partial F_n}{\partial x^m}\right) - \frac{2}{c^2}\, h^{ik}\frac{^*\partial F_i}{\partial t}\frac{^*\partial F_k}{\partial t}, \tag{3.58}$$

$$J_2 = -\frac{2}{c}\,\varepsilon^{imn}\left(\frac{^*\partial F_m}{\partial x^n} - \frac{^*\partial F_n}{\partial x^m}\right)\frac{^*\partial F_i}{\partial t}\,, \tag{3.59}$$

which, having the Zelmanov 1st identity (1.22) substituted, are

$$J_1 = 4h^{im}h^{kn}\frac{^*\partial A_{ik}}{\partial t}\frac{^*\partial A_{mn}}{\partial t} - \frac{2}{c^2}\, h^{ik}\frac{^*\partial F_i}{\partial t}\frac{^*\partial F_k}{\partial t}\,, \tag{3.60}$$

$$J_2 = -\frac{4}{c}\,\varepsilon^{imn}\frac{^*\partial A_{mn}}{\partial t}\frac{^*\partial F_i}{\partial t} = -\frac{8}{c}\left(\frac{^*\partial \Omega^{*i}}{\partial t} + \Omega^{*i} D\right)\frac{^*\partial F_i}{\partial t}\,. \tag{3.61}$$

The formulas we have obtained for the field invariants imply, that a vortical gravitational field is *spatially isotropic* (one of the invariants becomes zero) in the next cases:

- if the field is of strictly "electric" kind ($E^i \neq 0$, $H_{ik}=0$), so spatial vortexes of gravitational inertial force are absent and the space rotation (if it rotates) is stationary, then the invariants are
$$J_1 = -\frac{2}{c^2}\, h^{ik}\frac{^*\partial F_i}{\partial t}\frac{^*\partial F_k}{\partial t}\,, \qquad J_2 = 0\,, \tag{3.62}$$
- if the field is of strictly "magnetic" kind ($H_{ik}\neq 0$, $E^i=0$), so it is a non-stationary rotating space filled with the spatial vortexes F_{ik} of gravitational inertial force which does not depend on time, then
$$J_1 = F_{\cdot k}^{m\cdot} F_{m\cdot}^{\cdot k} = 4h^{im}h^{kn}\frac{^*\partial A_{ik}}{\partial t}\frac{^*\partial A_{mn}}{\partial t}, \qquad J_2 = 0\,, \tag{3.63}$$

- if $J_1 = 0$, that could be if the next condition is true (however its geometrical sense is not clear)

$$h^{im}h^{kn}\frac{{}^*\partial A_{ik}}{\partial t}\frac{{}^*\partial A_{mn}}{\partial t} = \frac{1}{2c^2}\,h^{ik}\frac{{}^*\partial F_i}{\partial t}\frac{{}^*\partial F_k}{\partial t}\,. \tag{3.64}$$

Thus anisotropic can be only a mixed vortical gravitational field bearing both the "electric" and the "magnetic" components as well. Strictly "electric" or "magnetic" vortical gravitational fields are spatially isotropic always.

Taking this conclusion with the peculiarity that gravitational inertial force does not depend on time in vortical gravitational fields of "magnetic" kind (see above), we arrive to the necessary and sufficient conditions to exist *standing gravitational waves*:

1. A media of the standing waves is vortical gravitational fields of strictly "magnetic" kind — given spatial vortexes of gravitational inertial force, the force does not depend on time;
2. The media of the standing waves is spatially isotropic — vortexes of gravitational inertial force are distributed equally at all spatial directions. Hence the field of standing gravitational waves is isotropic as well;
3. Standing gravitational waves are possible in only a strictly non-holonomic space, a rotation of which is non-stationary.

As soon as one of the conditions has been broken, then the acting gravitational inertial force begins to change its own value and the direction, so standing waves of the force transform themselves to the traveling waves.

§ 3.5 Equations of the vortical force field

As it was mentioned in §2.5, equations of a field is a system consisting of 10 equations in 10 unknowns, which are:

- Lorentz's condition $\nabla_\sigma A^\sigma = 0$ sets up that the four-dimensional potential A^α of the field remains unchanged;
- the charge conservation law $\nabla_\sigma j^\sigma = 0$ (the continuity equation), which sets up that the field-inducing sources like as "charges" or "currents" can not be destroyed but merely redistributed in the space;

- the 1st group $\nabla_\sigma F^{\alpha\sigma} = \frac{4\pi}{c}\,j^\alpha$ and the 2nd group $\nabla_\sigma F^{*\alpha\sigma} = 0$ of Maxwell's equations, where the 1st group defines the "charge" and the "current" as components of the four-dimensional current vector j^α of the field.

This system defines a vector field A^α and its sources in a pseudo-Riemannian space. We are going to deduce the equations for a vortical gravitational field as the field of the four-dimensional potential $F^\alpha = -2c^2 a_{\sigma\cdot}^{\cdot\alpha} b^\sigma$ (3.5).

Taking divergence $\nabla_\sigma F^\sigma$ (1.122) from the field potential F^α, after substituting the chr.inv.-projections $\varphi = 0$ and $q^i = F^i = h^{ik}F_k$ (3.42) of the potential F^α, we obtain the *Lorentz condition* for the vortical gravitational field

$$\nabla_\sigma F^\sigma = 0 \tag{3.65}$$

in chr.inv.-form, namely

$$\frac{{}^*\partial F^i}{\partial x^i} + F^i\Delta^j_{ji} - \frac{1}{c^2}F_iF^i = 0\,. \tag{3.66}$$

Looking forward to deduce Maxwell-like equations for the vortical gravitational field, let us collect chr.inv.-projections of the field tensor $F_{\alpha\beta}$ and the field pseudotensor $F^{*\alpha\beta}$ together.

After expressing the necessary projections with the tensor of the rate of the space deformation D^{ik} (1.24) to delete free h^{ik} terms from there, we have

$$E^i = \frac{1}{c}\,h^{ik}\frac{{}^*\partial F_k}{\partial t} = \frac{1}{c}\frac{{}^*\partial F^i}{\partial t} + \frac{2}{c}\,F_k D^{ik}, \tag{3.67}$$

$$H^{ik} = 2h^{im}h^{kn}\frac{{}^*\partial A_{mn}}{\partial t} = 2\frac{{}^*\partial A^{ik}}{\partial t} + 4\left(A^{i\cdot}_{\cdot n}D^{kn} - A^{k\cdot}_{\cdot m}D^{im}\right), \tag{3.68}$$

$$H^{*i} = \varepsilon^{imn}\frac{{}^*\partial A_{mn}}{\partial t} = 2\frac{{}^*\partial\Omega^{*i}}{\partial t} + 2\Omega^{*i}D\,, \tag{3.69}$$

$$E^{*ik} = -\frac{1}{c}\,\varepsilon^{ikm}\frac{{}^*\partial F_m}{\partial t}\,. \tag{3.70}$$

Substituting the components into the Maxwell equations taken in their generalized form (2.48, 2.49), which had been deduced for a field of an arbitrary four-dimensional vector potential, after some algebra we obtain *Maxwell-like chr.inv.-equations* for the vortical gravitational field

$$\left.\begin{aligned} &\frac{1}{c}\frac{{}^*\partial^2 F^i}{\partial x^i\partial t}+\frac{2}{c}\frac{{}^*\partial}{\partial x^i}\left(F_kD^{ik}\right)+\frac{1}{c}\left(\frac{{}^*\partial F^i}{\partial t}+2F_kD^{ik}\right)\Delta^j_{ji}-\\ &\qquad\qquad -\frac{2}{c}A_{ik}\left(\frac{{}^*\partial A^{ik}}{\partial t}+A^{i\cdot}_{\cdot n}D^{kn}\right)=4\pi\rho \\ &2\frac{{}^*\partial^2 A^{ik}}{\partial x^k\partial t}-\frac{1}{c^2}\frac{{}^*\partial^2F^i}{\partial t^2}+4\frac{{}^*\partial}{\partial x^k}\left(A^{i\cdot}_{\cdot n}D^{kn}-A^{k\cdot}_{\cdot m}D^{im}\right)+\\ &+2\left(\Delta^j_{jk}-\frac{1}{c^2}F_k\right)\left\{\frac{{}^*\partial A^{ik}}{\partial t}+2\left(A^{i\cdot}_{\cdot n}D^{kn}-A^{k\cdot}_{\cdot m}D^{im}\right)\right\}-\\ &-\frac{2}{c^2}\frac{{}^*\partial}{\partial t}\left(F_kD^{ik}\right)-\frac{1}{c^2}\left(\frac{{}^*\partial F^i}{\partial t}+2F_kD^{ik}\right)D=\frac{4\pi}{c}\,j^i\end{aligned}\right\}\ \text{I},\quad (3.71)$$

$$\left.\begin{aligned} &\frac{{}^*\partial^2\Omega^{*i}}{\partial x^i\partial t}+\frac{{}^*\partial}{\partial x^i}\left(\Omega^{*i}D\right)+\frac{1}{c^2}\,\Omega^{*m}\frac{{}^*\partial F_m}{\partial t}+\\ &\qquad\qquad+\left(\frac{{}^*\partial\Omega^{*i}}{\partial t}+\Omega^{*i}D\right)\Delta^j_{ji}=0 \\ &\varepsilon^{ikm}\frac{{}^*\partial^2F_m}{\partial x^k\partial t}+\varepsilon^{ikm}\left(\Delta^j_{jk}-\frac{1}{c^2}F_k\right)\frac{{}^*\partial F_m}{\partial t}+2\frac{{}^*\partial^2\Omega^{*i}}{\partial t^2}+\\ &\qquad\qquad+4D\frac{{}^*\partial\Omega^{*i}}{\partial t}+2\left(\frac{{}^*\partial D}{\partial t}+D^2\right)\Omega^{*i}=0\end{aligned}\right\}\ \text{II}.\quad (3.72)$$

So forth the *continuity equation* $\nabla_\sigma j^\sigma=0$ for the field, deriving from the 1st group of the Maxwell-like equations (3.71), is

$$\begin{aligned} &\frac{{}^*\partial^2}{\partial x^i\partial x^k}\left(\frac{{}^*\partial A^{ik}}{\partial t}\right)-\frac{1}{c^2}\left(\frac{{}^*\partial A^{ik}}{\partial t}+A^{i\cdot}_{\cdot n}D^{kn}\right)\left(A_{ik}D+\frac{{}^*\partial A_{ik}}{\partial t}\right)-\\ &-\frac{1}{c^2}\left[\frac{{}^*\partial^2A^{ik}}{\partial t^2}+\frac{{}^*\partial}{\partial t}\left(A^{i\cdot}_{\cdot n}D^{nk}\right)\right]A_{ik}+\frac{1}{2c^2}\left(\frac{{}^*\partial F^i}{\partial t}+2F_kD^{ik}\right)\times\\ &\times\left(\frac{{}^*\partial\Delta^j_{ji}}{\partial t}+\frac{D}{c^2}F_i-\frac{{}^*\partial D}{\partial x^i}\right)+2\frac{{}^*\partial^2}{\partial x^i\partial x^k}\left(A^{i\cdot}_{\cdot n}D^{kn}-A^{k\cdot}_{\cdot m}D^{im}\right)+\\ &+\left[\frac{{}^*\partial A^{ik}}{\partial t}+2\left(A^{i\cdot}_{\cdot n}D^{kn}-A^{k\cdot}_{\cdot m}D^{im}\right)\right]\left[\frac{{}^*\partial}{\partial x^i}\left(\Delta^j_{jk}-\frac{1}{c^2}F_k\right)+\right.\\ &\qquad\qquad\left.+\left(\Delta^j_{ji}-\frac{1}{c^2}F_i\right)\left(\Delta^l_{lk}-\frac{1}{c^2}F_k\right)\right]=0\,. \end{aligned}\qquad (3.73)$$

To see a sense of the obtained equations would be simpler in a homogeneous space $\Delta^i_{km}=0$ which does not deformation $D_{ik}=0$, because no waves of the metric there. Only waves of gravitational inertial force are possible in such space, that simplifies our task to consider the waves.

In a non-deforming homogeneous space the obtained Maxwell-like equations (3.71, 3.72) take the simplified form

$$\left.\begin{aligned} &\frac{1}{c}\frac{{}^*\partial^2F^i}{\partial x^i\partial t}-\frac{2}{c}A_{ik}\frac{{}^*\partial A^{ik}}{\partial t}=4\pi\rho \\ &2\frac{{}^*\partial^2A^{ik}}{\partial x^k\partial t}-\frac{2}{c^2}F_k\frac{{}^*\partial A^{ik}}{\partial t}-\frac{1}{c^2}\frac{{}^*\partial^2F^i}{\partial t^2}=\frac{4\pi}{c}\,j^i\end{aligned}\right\}\ \text{I},\qquad (3.74)$$

$$\left.\begin{aligned} &\frac{{}^*\partial^2\Omega^{*i}}{\partial x^i\partial t}+\frac{1}{c^2}\,\Omega^{*m}\frac{{}^*\partial F_m}{\partial t}=0 \\ &\varepsilon^{ikm}\frac{{}^*\partial^2F_m}{\partial x^k\partial t}-\frac{1}{c^2}\,\varepsilon^{ikm}F_k\frac{{}^*\partial F_m}{\partial t}+2\frac{{}^*\partial^2\Omega^{*i}}{\partial t^2}=0\end{aligned}\right\}\ \text{II},\qquad (3.75)$$

while the field-inducing sources in this case are

$$\rho=\frac{1}{4\pi c}\left(\frac{{}^*\partial^2F^i}{\partial x^i\partial t}-2A_{ik}\frac{{}^*\partial A^{ik}}{\partial t}\right),\qquad (3.76)$$

$$j^i=\frac{c}{2\pi}\left(\frac{{}^*\partial^2A^{ik}}{\partial x^k\partial t}-\frac{1}{c^2}F_k\frac{{}^*\partial A^{ik}}{\partial t}-\frac{1}{2c^2}\frac{{}^*\partial^2F^i}{\partial t^2}\right),\qquad (3.77)$$

so the continuity equation (3.73) takes the form

$$\begin{aligned} &\frac{{}^*\partial^2}{\partial x^i\partial x^k}\left(\frac{{}^*\partial A^{ik}}{\partial t}\right)-\frac{1}{c^2}A_{ik}\frac{{}^*\partial^2A^{ik}}{\partial t^2}-\frac{1}{c^2}\frac{{}^*\partial A_{ik}}{\partial t}\frac{{}^*\partial A^{ik}}{\partial t}-\\ &\qquad\qquad-\frac{1}{c^2}\left(\frac{{}^*\partial F_k}{\partial x^i}-\frac{1}{c^2}F_iF_k\right)\frac{{}^*\partial A^{ik}}{\partial t}=0\,. \end{aligned}\qquad (3.78)$$

The field equations we have obtained show the main peculiarities of vortical gravitational fields:

1. The Lorentz condition (3.66) shows that inhomogeneity of a vortical gravitational field depends on the value of the acting gravitational inertial force F^i and on the space inhomogeneity Δ^j_{ji} at the direction the force F^i acts;
2. The 1st group of the Maxwell-like equations (3.71) displays the origin of the field-inducing sources called "charges" ρ and "currents" j^i (a relative analogy to electromagnetic field):

"Charge" ρ inducing the vortical gravitational field is derived from inhomogeneity of oscillations of the acting gravitational inertial force F^i and from non-stationarity of the space rotation, taken with corrections defined by the space inhomogeneity and the deformations;

"Currents" j^i of the field are derived from non-stationarity of the space rotation, inhomogeneity of this non-stationarity in the space, and also non-stationarity of oscillations of the acting force F^i, corrected with the space inhomogeneity and the deformations;

3. The 2nd group of the Maxwell-like equations (3.72) shows properties of the field magnetic component H^{*i} (this component is derived from non-stationarity of the space rotation):

   Oscillations of the acting gravitational inertial force F^i are the main factor, which produce inhomogeneous distribution of the "magnetic" component H^{*i} of the field;

   If the acting force is $F^i=0$ and the space does not deformation $D_{ik}=0$, then the "magnetic" component H^{*i} of the field is stationary;

4. The continuity equation (3.78) shows that "charges" and "currents" inducing a vortical gravitational field, located in a non-deforming homogeneous space, remain unchanged if the space rotation is stationary. So, the sources inducing vortical gravitational fields of "electric" kind ($H_{ik}=0$, $E_i\neq 0$) conserve themselves in only a non-deforming homogeneous space. The sources inducing vortical gravitational fields of "magnetic" kind ($E_i=0$, $H_{ik}\neq 0$) remain unchanged, because of the effects of the space inhomogeneity and the deformations.

Aside these, we can deduce some additional peculiarities of vortical gravitational fields, using that a free field (that is a field without the sources) is wave. Having inhomogeneity and deformations of the space excluded, we exclude waves of the metric. Thus we have a possibility to consider conditions in which solely waves of gravitational inertial force exist — waves of a vortical gravitational field. So, equalizing formulas for the field-inducing sources φ (3.76) and j^i (3.77) to zero in a non-deforming homogeneous space, we obtain

$$\frac{{}^*\partial^2 F^i}{\partial x^i\partial t} = 2A_{ik}\frac{{}^*\partial A^{ik}}{\partial t}\,, \tag{3.79}$$

$$\frac{{}^*\partial^2 A^{ik}}{\partial x^k\partial t} = \frac{1}{c^2}F_k\frac{{}^*\partial A^{ik}}{\partial t} + \frac{1}{2c^2}\frac{{}^*\partial^2 F^i}{\partial t^2}\,, \tag{3.80}$$

that lead us to the next conclusions:

1. Inhomogeneity of oscillations of the gravitational inertial force F^i, acting in a free vortical gravitational field, is mainly derived from non-stationarity of the space rotation (3.79);
2. Inhomogeneity of non-stationary rotations of a space, filled with a free vortical gravitational field, is mainly defined by the acting gravitational inertial force F^i and also non-stationarity of oscillations of the force (3.80).

The results we have obtained here evident that numerous properties of vortical gravitational fields display themselves only if the fields are of strictly "electric" or of strictly "magnetic" kind. This fact lead us to consider these two kinds of vortical gravitational fields separately. We will focus our power on the tasks in the next §3.6 and §3.7.

§3.6 The vortical field of "electric" kind

Now we are going to consider a vortical gravitational field of strictly "electric" kind, which characterizes itself as follows

$$H_{ik} = \frac{{}^*\partial F_i}{\partial x^k} - \frac{{}^*\partial F_k}{\partial x^i} = 2\frac{{}^*\partial A_{ik}}{\partial t} = 0\,, \tag{3.81}$$

$$H^{ik} = 2h^{im}h^{kn}\frac{{}^*\partial A_{mn}}{\partial t} = 0\,, \tag{3.82}$$

$$E_i = \frac{1}{c}\frac{{}^*\partial F_i}{\partial t} \neq 0\,, \tag{3.83}$$

$$E^i = \frac{1}{c}h^{ik}\frac{{}^*\partial F_k}{\partial t} = \frac{1}{c}\frac{{}^*\partial F^i}{\partial t} + \frac{2}{c}F_k D^{ik} \neq 0\,, \tag{3.84}$$

$$H^{*i} = \varepsilon^{imn}\frac{{}^*\partial A_{mn}}{\partial t} = 2\frac{{}^*\partial \Omega^{*i}}{\partial t} + 2\Omega^{*i}D = 0\,, \tag{3.85}$$

$$E^{*ik} = -\frac{1}{c}\varepsilon^{ikm}\frac{{}^*\partial F_m}{\partial t} \neq 0\,. \tag{3.86}$$

Actually, we are considering a stationary rotating space (if the space rotates) filled with the field of a non-stationary gravitational inertial force without spatial vortexes of the force. This is the main kind of vortical gravitational fields, because we observe non-stationary rotations of space bodies very rarely (see the field of "magnetic" kind in the next §3.7).

In this case the Lorentz condition (the four-dimensional vector potential of the field remains unchanged) does not change itself in comparison to the general formula (3.66), because the condition does not contain components of the field tensor $F_{\alpha\beta}$.

The field invariants (3.62) in this case are

$$J_1 = -\frac{2}{c^2}\, h^{ik} \frac{{}^*\partial F_i}{\partial t} \frac{{}^*\partial F_k}{\partial t}\,, \qquad J_2 = 0\,. \tag{3.87}$$

Maxwell-like chr.inv.-equations for the vortical gravitational field of strictly "electric" kind, having the general form

$$\left.\begin{aligned} &{}^*\nabla_i E^i = 4\pi\rho \\ &\frac{1}{c}\left(\frac{{}^*\partial E^i}{\partial t} + E^i D\right) = -\frac{4\pi}{c}\, j^i \end{aligned}\right\} \text{I}, \tag{3.88}$$

$$\left.\begin{aligned} &E^{*ik} A_{ik} = 0 \\ &{}^*\nabla_k E^{*ik} - \frac{1}{c^2} F_k E^{*ik} = 0 \end{aligned}\right\} \text{II}, \tag{3.89}$$

after substituting E^i and E^{*ik} take the final form

$$\left.\begin{aligned} &\frac{1}{c}\frac{{}^*\partial^2 F^i}{\partial x^i \partial t} + \frac{1}{c}\left(\frac{{}^*\partial F^i}{\partial t} + 2F_k D^{ik}\right)\Delta^j_{ji} + \frac{2}{c}\frac{{}^*\partial}{\partial x^i}\left(F_k D^{ik}\right) = 4\pi\rho \\ &\frac{1}{c^2}\frac{{}^*\partial^2 F^i}{\partial t^2} + \frac{2}{c^2}\frac{{}^*\partial}{\partial t}\left(F_k D^{ik}\right) + \frac{1}{c^2}\left(\frac{{}^*\partial F^i}{\partial t} + 2F_k D^{ik}\right) D = -\frac{4\pi}{c}\, j^i \end{aligned}\right\} \text{I}, \tag{3.90}$$

$$\left.\begin{aligned} &\frac{2}{c^2}\Omega^{*m}\frac{{}^*\partial F_m}{\partial t} = 0 \\ &\varepsilon^{ikm}\frac{{}^*\partial^2 F_m}{\partial x^k \partial t} + \varepsilon^{ikm}\left(\Delta^j_{jk} - \frac{1}{c^2}F_k\right)\frac{{}^*\partial F_m}{\partial t} = 0 \end{aligned}\right\} \text{II}. \tag{3.91}$$

The continuity equation for the field, in general case of a deforming inhomogeneous space, takes the form

$$\left(\frac{{}^*\partial F^i}{\partial t} + 2F_k D^{ik}\right)\left(\frac{{}^*\partial \Delta^j_{ji}}{\partial t} - \frac{{}^*\partial D}{\partial x^i} + \frac{D}{c^2}F_i\right) = 0\,, \tag{3.92}$$

this equation becomes the identity "zero equalized to zero" in the absence of the space inhomogeneity and the deformations. Actually, the continuity equation we have obtained implies that one of the next conditions or the both conditions together

$$\frac{{}^*\partial F^i}{\partial t} = -2F_k D^{ik}, \qquad \frac{{}^*\partial \Delta^j_{ji}}{\partial t} = \frac{{}^*\partial D}{\partial x^i} - \frac{D}{c^2}F_i \tag{3.93}$$

are true in vortical gravitational fields of strictly "electric" kind.

The Maxwell-like equations (3.90, 3.91) in a non-deforming homogeneous space become very simple

$$\left.\begin{aligned} &\frac{1}{c}\frac{{}^*\partial^2 F^i}{\partial x^i \partial t} = 4\pi\rho \\ &\frac{1}{c^2}\frac{{}^*\partial^2 F^i}{\partial t^2} = -\frac{4\pi}{c}\, j^i \end{aligned}\right\} \text{I}, \tag{3.94}$$

$$\left.\begin{aligned} &\frac{1}{c^2}\Omega^{*m}\frac{{}^*\partial F_m}{\partial t} = 0 \\ &\varepsilon^{ikm}\frac{{}^*\partial^2 F_m}{\partial x^k \partial t} - \frac{1}{c^2}\varepsilon^{ikm}F_k\frac{{}^*\partial F_m}{\partial t} = 0 \end{aligned}\right\} \text{II}. \tag{3.95}$$

The obtained equations display peculiarities, specific for vortical gravitational fields of strictly "electric" kind:

1. Because one of the field invariants is zero everywhere (3.87), vortical gravitational fields of "electric" kind are *spatially isotropic* under any conditions, independent of inhomogeneity Δ^i_{jk} or deformations D_{ik} of the space;
2. Sources ρ and j^i inducing such fields in a non-deforming homogeneous space (3.94) are derived from inhomogeneous distribution of oscillations of the acting gravitational inertial force F^i (the "charge" ρ) and from non-stationarity of the oscillations (the "currents" j^i);
3. Because vortical gravitational fields of "electric" kind characterize themselves by $E_i = \frac{1}{c}\frac{{}^*\partial F_i}{\partial t} \neq 0$, such fields are possible in a rotating space $\Omega^{*i} \neq 0$, if the space is inhomogeneous $\Delta^j_{ji} \neq 0$ and deforming $D_{ik} \neq 0$ (see the 1st equation in the 2nd Maxwell-like group — formula 3.91). Such fields are possible in a non-deforming homogeneous space, if the space is holonomic $\Omega^{*i} = 0$ (see the equation in simplified form — formula 3.95);

4. Such fields permit waves of the acting gravitational inertial force F^i (no the field sources there $\rho=0$, $j^i=0$), if oscillations of the force are homogeneous and stable (see the 1st Maxwell-like group — formula 3.94);
5. The continuity equation (3.92) brings us into the conclusion that sources inducing such fields in a non-deforming homogeneous space ($\Delta^i_{jk}=0$, $D_{ik}=0$) remain unchanged under any conditions.

§ 3.7 The vortical field of “magnetic” kind

A vortical gravitational field of strictly “magnetic” kind characterizes itself by its own observable components

$$H_{ik}=\frac{{}^*\partial F_i}{\partial x^k}-\frac{{}^*\partial F_k}{\partial x^i}=2\,\frac{{}^*\partial A_{ik}}{\partial t}\neq 0\,, \tag{3.96}$$

$$H^{ik}=2h^{im}h^{kn}\frac{{}^*\partial A_{mn}}{\partial t}\neq 0\,, \tag{3.97}$$

$$E_i=\frac{1}{c}\frac{{}^*\partial F_i}{\partial t}=0\,, \tag{3.98}$$

$$E^i=\frac{1}{c}\,h^{ik}\frac{{}^*\partial F_k}{\partial t}=\frac{1}{c}\frac{{}^*\partial F^i}{\partial t}+\frac{2}{c}\,F_k\,D^{ik}=0\,, \tag{3.99}$$

$$H^{*i}=\varepsilon^{imn}\frac{{}^*\partial A_{mn}}{\partial t}=2\,\frac{{}^*\partial \Omega^{*i}}{\partial t}+2\Omega^{*i}D\neq 0\,, \tag{3.100}$$

$$E^{*ik}=-\frac{1}{c}\,\varepsilon^{ikm}\frac{{}^*\partial F_m}{\partial t}=0\,. \tag{3.101}$$

Actually, in this case we have a non-stationary rotating (strictly non-holonomic) space filled with spatial vortexes of a stationary gravitational inertial force $F_i=const$. This is a more exotic case of vortical gravitational fields than the “electric”, because non-stationary rotations of bulk space bodies, being possible generators of such fields, are very infrequent phenomena in the Universe.

The Lorentz condition for such field does not change its general formula (3.66) as well as for the fields of “electric” kind, because the condition does not contain components of the field tensor $F_{\alpha\beta}$.

The field invariants (3.63) in the case we are considering are

$$J_1=4h^{im}h^{kn}\frac{{}^*\partial A_{ik}}{\partial t}\frac{{}^*\partial A_{mn}}{\partial t}\,,\qquad J_2=0\,. \tag{3.102}$$

Maxwell-like equations for the vortical gravitational field of strictly “magnetic” kind are

$$\left.\begin{aligned}&\frac{1}{c}\,H^{ik}A_{ik}=-4\pi\rho\\ &{}^*\nabla_k\,H^{ik}-\frac{1}{c^2}F_kH^{ik}=\frac{4\pi}{c}\,j^i\end{aligned}\right\}\ \text{I}, \tag{3.103}$$

$$\left.\begin{aligned}&{}^*\nabla_i\,H^{*i}=0\\ &\frac{{}^*\partial H^{*i}}{\partial t}+H^{*i}D=0\end{aligned}\right\}\ \text{II}, \tag{3.104}$$

and after substituting H^{ik} and H^{*i} take the final form

$$\left.\begin{aligned}&\frac{2}{c}\,A^{ik}\frac{{}^*\partial A_{ik}}{\partial t}=-4\pi\rho\\ &2\,\frac{{}^*\partial^2A^{ik}}{\partial x^k\partial t}+4\,\frac{{}^*\partial}{\partial x^k}\left(A^{i\cdot}_{\cdot n}D^{kn}-A^{k\cdot}_{\cdot m}D^{im}\right)+2\left(\Delta^j_{jk}-\frac{1}{c^2}F_k\right)\times\\ &\qquad\times\left\{\frac{{}^*\partial A^{ik}}{\partial t}+2\left(A^{i\cdot}_{\cdot n}D^{kn}-A^{k\cdot}_{\cdot m}D^{im}\right)\right\}=\frac{4\pi}{c}\,j^i\end{aligned}\right\}\ \text{I}, \tag{3.105}$$

$$\left.\begin{aligned}&\frac{{}^*\partial^2\Omega^{*i}}{\partial x^i\partial t}+\frac{{}^*\partial}{\partial x^i}\left(\Omega^{*i}D\right)+\left(\frac{{}^*\partial\Omega^{*i}}{\partial t}+\Omega^{*i}D\right)\Delta^j_{ji}=0\\ &\frac{{}^*\partial^2\Omega^{*i}}{\partial t^2}+\frac{{}^*\partial}{\partial t}\left(\Omega^{*i}D\right)+\left(\frac{{}^*\partial\Omega^{*i}}{\partial t}+\Omega^{*i}D\right)D=0\end{aligned}\right\}\ \text{II}. \tag{3.106}$$

The continuity equation for the field of strictly “magnetic” kind in a deforming inhomogeneous space takes the form

$$\begin{aligned}&\frac{{}^*\partial^2}{\partial x^i\partial x^k}\left(\frac{{}^*\partial A^{ik}}{\partial t}\right)-\frac{1}{c^2}A^{ik}\frac{{}^*\partial^2A_{ik}}{\partial t^2}-\frac{1}{c^2}\left(\frac{{}^*\partial A^{ik}}{\partial t}+A^{ik}D\right)\times\\ &\times\frac{{}^*\partial A_{ik}}{\partial t}+2\,\frac{{}^*\partial^2}{\partial x^i\partial x^k}\left(A^{i\cdot}_{\cdot n}D^{kn}-A^{k\cdot}_{\cdot m}D^{im}\right)+\left\{\frac{{}^*\partial A^{ik}}{\partial t}+\right.\\ &\left.+2\left(A^{i\cdot}_{\cdot n}D^{kn}-A^{k\cdot}_{\cdot m}D^{im}\right)\right\}\left\{\left(\frac{{}^*\partial\Delta^j_{jk}}{\partial x^i}-\frac{1}{c^2}\frac{{}^*\partial F_k}{\partial x^i}+\right.\right.\\ &\left.\left.+\left(\Delta^j_{jk}-\frac{1}{c^2}F_k\right)\left(\Delta^l_{li}-\frac{1}{c^2}F_i\right)\right\}=0\,.\end{aligned} \tag{3.107}$$

In the absence of inhomogeneity and deformations of the space, this equation is

$$\frac{{}^*\partial^2}{\partial x^i \partial x^k}\left(\frac{{}^*\partial A^{ik}}{\partial t}\right) - \frac{1}{c^2} A^{ik} \frac{{}^*\partial^2 A_{ik}}{\partial t^2} - \\ - \frac{1}{c^2}\left(\frac{{}^*\partial A_{ik}}{\partial t} + \frac{{}^*\partial F_k}{\partial x^i} - \frac{1}{c^2} F_i F_k\right) \frac{{}^*\partial A^{ik}}{\partial t} = 0\,. \tag{3.108}$$

The Maxwell-like equations (3.105, 3.106) in a non-deforming homogeneous space take the form

$$\left.\begin{aligned} &\frac{2}{c} A^{ik} \frac{{}^*\partial A_{ik}}{\partial t} = -4\pi\rho \\ &2\frac{{}^*\partial^2 A^{ik}}{\partial x^k \partial t} - \frac{2}{c^2} F_k \frac{{}^*\partial A^{ik}}{\partial t} = \frac{4\pi}{c}\, j^i \end{aligned}\right\}\ \text{I}, \tag{3.109}$$

$$\left.\begin{aligned} &\frac{{}^*\partial^2 \Omega^{*i}}{\partial x^i \partial t} = 0 \\ &\frac{{}^*\partial^2 \Omega^{*i}}{\partial t^2} = 0 \end{aligned}\right\}\ \text{II}. \tag{3.110}$$

So, the obtained equations characterizing vortical gravitational fields of strictly “magnetic” kind display their specific peculiarities as follows:

1. The field invariants (3.102) show that such fields are *spatially isotropic* anyhow, that is independent of inhomogeneity Δ^i_{jk} or deformations D_{ik} of the space;
2. In a non-deforming homogeneous space the field-inducing “charge” is derived from non-stationarity of the space rotation, the field-inducing “currents” are derived from this non-stationarity and spatial inhomogeneity of the non-stationarity (see the 1st Maxwell-like group — formula 3.109);
3. Because $\frac{{}^*\partial \Omega^{*i}}{\partial t} \neq 0$ in vortical gravitational fields of this kind by their definition, such fields with the sources ($\rho \neq 0$, $j^i \neq 0$) are possible in a non-deforming homogeneous space, if the space rotates homogeneously at a constant acceleration (see the 2nd Maxwell-like group — formula 3.110);
4. Such fields without the sources ($\rho = 0$, $j^i = 0$) would be theoretically possible in a non-deforming homogeneous space, if the space rotation would be stationary (see the 1st Maxwell-like group — 3.109). However in this case the “magnetic” component of such field becomes zero $H_{ik} = 2\frac{{}^*\partial A_{ik}}{\partial t} = 0$ as well as its “electric” component $E_i = 0$, so the field disappears. This implies that waves of vortical gravitational fields of “magnetic” kind, i. e. standing gravitational waves (see §3.4), are possible in only an inhomogeneous deforming space;
5. Looking at the continuity equation (3.108), we see that sources inducing such fields remain unchanged in a non-deforming homogeneous space under only the presence of gravitational inertial force $F^i \neq 0$.

§3.8 Conclusions

Finishing this Chapter, let us make a survey of the main results we have obtained here.

So, a regular approach takes gravitational waves as traveling waves of weak corrections $\zeta_{\alpha\beta}$ to a Galilean metric $g^{(0)}_{\alpha\beta}$. Actually these are waves of the metric $g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \zeta_{\alpha\beta}$. This approach is not the best, because of its own drawbacks:

1. The approach gives the Ricci tensor and the d’Alembert equations of the metric to within higher order terms withheld, so the velocity of waves of the metric calculated from the equations is not finally exact theoretical result;
2. A source of this approximation are the tiny corrections $\zeta_{\alpha\beta}$ to a Galilean metric, an origin of which may be very different, not only gravitation;
3. Two bodies attract one another, because of the transfer of gravitational force. A wave traveling in the field of gravitational force is not the same that a wave of the metric — these are different tensor fields.

The reasons lead us to consider gravitational waves as waves of the field of gravitational force, that provided two important advantages:

1. The mathematical methods of chronometric invariants define gravitational inertial force F_i without the Riemann-Christoffel curvature tensor. Thus we can deduce the exact d’Alembert

equations for the force field, which give an exact formula for the velocity of waves of the force;

2. Experiments to register waves of the force field, having "detectors" like as planets or their satellites a base, does not link to the quadrupole mass-detector and its specific technical problems.

So forth we have deduced the exact d'Alembert equations for the field of gravitational inertial force. In accordance with the equations, waves of the field travel with the velocity, a modulus of which is $u=\sqrt{u_k u^k}=c\left(1-\frac{\mathrm{w}}{c^2}\right)$ that is the speed of gravitation:

1. In a weak gravitational field, a potential w of which is neglected but its gradient F_i is non-zero, the speed of gravitation equals the light velocity;
2. In accordance with this formula, the speed of gravitation in an Earth laboratory shall be 21 cm/sec less than the light velocity. Gravitational waves near the Sun shall be about 6.3×10^4 cm/sec slow than light;
3. Under gravitational collapse ($\mathrm{w}=c^2$) the speed of gravitation becomes zero.

The new approach becomes an experiment to measure the speed of gravitation as a speed to transfer the attracting force between space bodies. An essence of the experiment is to measure a lag time of the lunar (or the solar) flow wave in an orbit of a geostationary satellite in respect of the upper transition of the Moon (or the Sun). In accordance with the obtained formula, the maximum of the lunar flow wave in a satellite orbit shall be about 1 second late from the upper culmination of the Moon. The lateness of the solar flow wave shall be about 500 second after the upper transit of the Sun.

After deducing chr.inv.-projections of the Riemann-Christoffel curvature tensor, we have concluded that waves of gravitational inertial force does not depend on the curvature in constant curvature spaces.

Considering a field of a four-dimensional vector potential F^α, which is gravitational inertial force, we introduced the field tensor $F_{\alpha\beta}=\nabla_\beta F_\alpha-\nabla_\alpha F_\beta$ as well as Maxwell's tensor of electromagnetic fields. The field tensor $F_{\alpha\beta}$ and its dual pseudotensor $F^{*\alpha\beta}$ characterize vortical fields of the force F^α — vortical gravitational fields.

Quantities $E_i=\frac{1}{c}\frac{{}^*\partial F_i}{\partial t}$ and $H_{ik}=\frac{{}^*\partial F_i}{\partial x^k}-\frac{{}^*\partial F_k}{\partial x^i}=2\frac{{}^*\partial A_{ik}}{\partial t}$ are chr.inv.-projections of the field tensor $F_{\alpha\beta}$. We called E_i the "electric" and H_{ik} the "magnetic" observable components of vortical gravitational field.

The field invariants $J_1=F_{\alpha\beta}F^{\alpha\beta}$ and $J_2=F_{\alpha\beta}F^{*\alpha\beta}$ show that strictly "electric" or "magnetic" vortical gravitational fields are spatially isotropic.

The d'Alembert's equations for the field does not depend on time under the condition $E_i=0$, so we conclude:

1. Vortical gravitational fields of strictly "magnetic" kind are the media of standing gravitational waves;
2. This media is spatially isotropic, so the field of the standing waves is isotropic as well;
3. Standing gravitational waves are possible in only a space, which rotation is non-stationary.

A system of the field equations (Lorentz's condition, Maxwell-like equations, and the continuity equation) shows the main peculiarities of vortical gravitational fields:

1. The Lorentz condition shows, inhomogeneity of the field depends on the acting gravitational inertial force F^i and on the space inhomogeneity Δ^j_{ji} at the direction the force F^i acts;
2. The 1st group of the Maxwell-like equations displays a nature of the field-inducing sources:

   "Charge" ρ is derived from inhomogeneity of oscillations of the acting force F^i and from non-stationarity of the space rotation (to within the space inhomogeneity and the deformations withheld);

   "Currents" j^i are derived from non-stationarity of the space rotation, spatial inhomogeneity of the non-stationarity, and non-stationarity of oscillations of the force F^i (to within the same approximation);
3. The 2nd Maxwell-like group shows properties of the field magnetic component H^{*i}:

   Oscillations of the acting force F^i are the main factor, which get the field "magnetic" component H^{*i} inhomogeneous;

   If the acting force is $F^i=0$ and the space does not deformation

$D_{ik}=0$, then the "magnetic" component H^{*i} of the field is stationary;

4. The continuity equation shows that the field-inducing "charges" and "currents", being located in a non-deforming homogeneous space, remain unchanged if the space rotation is stationary.

Properties of waves traveling in the field of gravitational inertial force are derived from equalizing the sources ρ and j^i to zero in the field equations, because a free field (without the sources) is wave:

1. Inhomogeneity of oscillations of the gravitational inertial force F^i, acting in a free vortical gravitational field, is mainly derived from non-stationarity of the space rotation;
2. Inhomogeneity of non-stationary rotations of a space, filled with the free field, is mainly defined by the acting force F^i and also non-stationarity of its oscillations.

Vortical gravitational fields of strictly "electric" kind are fields of a non-stationary gravitational inertial force F^i without the spatial vortexes, located in a stationary rotating space (if the space rotates). Their specific peculiarities are as follows:

1. Such fields are spatially isotropic under any conditions;
2. Their-inducing sources ρ and j^i are mainly derived from inhomogeneity of oscillations of the acting gravitational inertial force F^i (the "charges" ρ) and from non-stationarity of the oscillations (the "currents" j^i);
3. Such fields are possible in a self-rotating space $\Omega^{*i}\neq 0$, if the space is inhomogeneous $\Delta^i_{kn}\neq 0$ and deforming $D_{ik}\neq 0$. The fields can be possible in a non-deforming homogeneous space, if the space is holonomic $\Omega^{*i}=0$.
4. Waves of the acting force F^i traveling in such fields are permitted, if oscillations of the force F^i are homogeneous and stable;
5. Sources ρ and j^i inducing such fields remain unchanged under any condition in a non-deforming homogeneous space.

Vortical gravitational fields of strictly "magnetic" kind display themselves as spatial vortexes of a stationary gravitational inertial force $F_i = const$, filled in a non-stationary rotating space. Their peculiarities are listed below:

1. Such fields are spatially isotropic under any conditions;
2. The field-inducing "charges" are mainly derived from non-stationarity of the space rotation, the field "currents" are mainly derived from this non-stationarity and its spatial inhomogeneity;
3. Such fields are possible in a non-deforming homogeneous space, if the space rotates homogeneously at a constant acceleration;
4. Waves in such fields are standing gravitational waves, they are possible in only an inhomogeneous deforming space;
5. Sources inducing such fields remain unchanged in a non-deforming homogeneous space under the condition $F^i\neq 0$.

Assuming the above we conclude that the main kind of vortical gravitational fields is "electric". Such fields are a media for traveling gravitational waves. Standing gravitational waves and their media, which are vortical gravitational fields of "magnetic" kind, are more exotic — non-stationary rotations of bulk space bodies, which can generate the fields, are very infrequent phenomena in the Universe. As a matter of fact that gravitational attraction is an everyday reality, so traveling waves of gravitational inertial force transferring the attraction shall be incontrovertible. A problem to register the waves is that their amplitudes in laboratory experiments should be tiny. In the same time I think that the satellite experiment, propounded in §3.2, will have solved this problem, because amplitudes of the lunar or the solar flow waves must be perceptible.

Notation

$dA^{\alpha} = \dfrac{\partial A^{\alpha}}{\partial x^{\sigma}}\,dx^{\sigma}$	Ordinary differential of a vector
$\mathrm{D}A^{\alpha} = \nabla_{\beta}\,A^{\alpha}dx^{\beta} = dA^{\alpha} + \Gamma^{\alpha}_{\beta\mu}\,A^{\mu}dx^{\beta}$	Absolute differential of a contravariant vector
$\mathrm{D}A_{\alpha} = \nabla_{\beta}\,A_{\alpha}dx^{\beta} = dA_{\alpha} - \Gamma^{\mu}_{\alpha\beta}\,A_{\mu}dx^{\beta}$	Absolute differential of a covariant vector
$\nabla_{\beta}\,A^{\alpha} = \dfrac{\partial A^{\alpha}}{\partial x^{\beta}} + \Gamma^{\alpha}_{\beta\mu}\,A^{\mu}$	Absolute derivative of a contravariant vector
$\nabla_{\beta}\,A_{\alpha} = \dfrac{\partial A_{\alpha}}{\partial x^{\beta}} - \Gamma^{\mu}_{\alpha\beta}\,A_{\mu}$	Absolute derivative of a covariant vector
$\nabla_{\beta}\,F^{\sigma\alpha} = \dfrac{\partial F^{\sigma\alpha}}{\partial x^{\beta}} + \Gamma^{\alpha}_{\beta\mu}\,F^{\sigma\mu} + \Gamma^{\sigma}_{\beta\mu}\,F^{\alpha\mu}$	Absolute derivative of a contravariant 2nd rank tensor
$\nabla_{\beta}\,F_{\sigma\alpha} = \dfrac{\partial F_{\sigma\alpha}}{\partial x^{\beta}} - \Gamma^{\mu}_{\alpha\beta}\,F_{\sigma\mu} - \Gamma^{\mu}_{\sigma\beta}\,F_{\alpha\mu}$	Absolute derivative of a covariant 2nd rank tensor
$\nabla_{\alpha}\,A^{\alpha} = \dfrac{\partial A^{\alpha}}{\partial x^{\alpha}} + \Gamma^{\alpha}_{\alpha\sigma}\,A^{\sigma}$	Absolute divergence of a vector
${}^*\nabla_i\,q^i = \dfrac{{}^*\partial q^i}{\partial x^i} + q^i\Delta^j_{ji}$	Chr.inv.-divergence of a vector
${}^*\widetilde{\nabla}_i\,q^i = {}^*\nabla_i\,q^i - \dfrac{1}{c^2}\,F_i\,q^i$	Physical chr.inv.-divergence
$\square = g^{\alpha\beta}\,\nabla_{\alpha}\nabla_{\beta}$	D'Alembert's general covariant operator
$\Delta = -\,g^{ik}\,\nabla_i\nabla_k$	Laplace's ordinary operator
${}^*\Delta = h^{ik}\,{}^*\nabla_i\,{}^*\nabla_k$	The Laplace chr.inv.-operator
$\dfrac{{}^*\partial}{\partial t} = \dfrac{1}{\sqrt{g_{00}}}\,\dfrac{\partial}{\partial t}$	Chr.inv.-derivative with respect to time

$\dfrac{{}^*\partial}{\partial x^i} = \dfrac{\partial}{\partial x^i} + \dfrac{1}{c^2}\,v_i\,\dfrac{{}^*\partial}{\partial t}$	Chr.inv.-derivative with respect to spatial coordinates
$\mathrm{v}^2 = \mathrm{v}_i\mathrm{v}^i = h_{ik}\mathrm{v}^i\mathrm{v}^k$	The square of the physical observable velocity
$v^i = -cg^{0i}\sqrt{g_{00}}\,, \qquad v_i = h_{ik}\,v^k$	The linear velocity of the space rotations
$v^2 = h_{ik}\,v^i v^k$	The square of v_i. Here is the proof. Because of $g_{\alpha\sigma}g^{\sigma\beta} = g^{\beta}_{\alpha}$, then under $\alpha = \beta = 0$ we have $g_{0\sigma}g^{\sigma 0} = \delta^0_0 = 1$, hence we become $v^2 = c^2(1 - g_{00}g^{00})$
$\sqrt{-g} = \sqrt{h}\,\sqrt{g_{00}}$	The relation between the determinants of the metric tensors $g_{\alpha\beta}$ and $h_{\alpha\beta}$
$\dfrac{d}{d\tau} = \dfrac{{}^*\partial}{\partial t} + \mathrm{v}^k\,\dfrac{{}^*\partial}{\partial x^k}$	Derivative with respect to physical observable time
$\dfrac{d}{ds} = \dfrac{1}{c\sqrt{1 - \dfrac{\mathrm{v}^2}{c^2}}}\,\dfrac{d}{d\tau}$	The 1st derivative with respect to the space-time interval
$\left.\begin{aligned} &\dfrac{d^2}{ds^2} = \dfrac{1}{c^2 - \mathrm{v}^2}\,\dfrac{d^2}{d\tau^2} + \dfrac{1}{(c^2 - \mathrm{v}^2)^2}\times \\ &\times\left(D_{ik}\mathrm{v}^i\mathrm{v}^k + \mathrm{v}_i\,\dfrac{d\mathrm{v}^i}{d\tau} + \dfrac{1}{2}\,\dfrac{{}^*\partial h_{ik}}{\partial x^m}\,\mathrm{v}^i\mathrm{v}^k\mathrm{v}^m\right)\dfrac{d}{d\tau} \end{aligned}\right\}$	The 2nd derivative with respect to the space-time interval
$\left.\begin{aligned} &h_{ik} = -g_{ik} + \dfrac{1}{c^2}\,v_i\,v_k\,, \\ &h^{ik} = -g^{ik}, \qquad h^k_i = \delta^k_i \end{aligned}\right\}$	The metric chr.inv.-tensor
$\left.\begin{aligned} &g^{i\alpha}g^{k\beta}\Gamma^m_{\alpha\beta} = h^{iq}h^{ks}\Delta^m_{qs}\,, \\ &D^i_k + A^{\cdot i}_{k\cdot} = \dfrac{c}{\sqrt{g_{00}}}\left(\Gamma^i_{0k} - \dfrac{g_{0k}\Gamma^i_{00}}{g_{00}}\right), \\ &F^k = -\dfrac{c^2\Gamma^k_{00}}{g_{00}} \end{aligned}\right\}$	Zelmanov's relations between the Christoffel regular symbols and chr.inv.-characteristics of the reference space
$\dfrac{{}^*\partial A_{ik}}{\partial t} + \dfrac{1}{2}\left(\dfrac{{}^*\partial F_k}{\partial x^i} - \dfrac{{}^*\partial F_i}{\partial x^k}\right) = 0$	Zelmanov's 1st identity

$$\left.\begin{array}{l}\dfrac{{}^*\partial A_{km}}{\partial x^i}+\dfrac{{}^*\partial A_{mi}}{\partial x^k}+\dfrac{{}^*\partial A_{ik}}{\partial x^m}+\\ +\dfrac{1}{2}\left(F_iA_{km}+F_kA_{mi}+F_mA_{ik}\right)=0\end{array}\right\}$$ Zelmanov's 2nd identity

$$\left.\begin{array}{l}\dfrac{d}{d\tau}\mathrm{v}^2=\dfrac{d}{d\tau}\left(h_{ik}\mathrm{v}^i\mathrm{v}^k\right)=2D_{ik}\mathrm{v}^i\mathrm{v}^k+\\ +\dfrac{{}^*\partial h_{ik}}{\partial x^m}\mathrm{v}^i\mathrm{v}^k\mathrm{v}^m+2\mathrm{v}_k\dfrac{d\mathrm{v}^k}{d\tau}\end{array}\right\}$$ Derivative from v^2 with respect to physical observable time

$$\left.\begin{array}{l}\varepsilon^{ikm}=\sqrt{g_{00}}\,E^{0ikm}=\dfrac{e^{0ikm}}{\sqrt{h}}\,,\\ \varepsilon_{ikm}=\dfrac{E_{0ikm}}{\sqrt{g_{00}}}=e_{0ikm}\sqrt{h}\end{array}\right\}$$ The completely antisymmetric chr.inv.-tensor

$$\left.\begin{array}{l}\dfrac{{}^*\partial\varepsilon_{imn}}{\partial t}=\varepsilon_{imn}\,D\,,\\ \dfrac{{}^*\partial\varepsilon^{imn}}{\partial t}=-\varepsilon^{imn}D\,,\\ {}^*\nabla_k\,\varepsilon_{imn}=0\,,\quad {}^*\nabla_k\,\varepsilon^{imn}=0\end{array}\right\}$$ Its chr.inv.-derivatives

Bibliography

1. **Zelmanov A. L.** Chronometric invariants. Dissertation, 1944. First published: CERN, EXT-2004-117, 236 pages.
2. **Zelmanov A. L.** Chronometric invariants and co-moving coordinates in the general relativity theory. *Doklady Acad. Nauk USSR*, 1956, v. 107 (6), 815–818.
3. **Zelmanov A. L.** To relativistic theory of anisotropic inhomogeneous Universe. *Proceedings of the 6th Soviet Conference on Cosmogony*, Nauka, Moscow, 1959, 144–174 (*in Russian*).
4. **Zelmanov A. L. and Agakov V. G.** Elements of the General Theory of Relativity. Nauka, Moscow, 1989, 240 pages (*in Russian*).
5. **Landau L. D. and Lifshitz E. M.** The classical theory of fields. GITTL, Moscow, 1939 (referred with the 4th final expanded edition, Butterworth–Heinemann, 1980, 428 pages).
6. **Cattaneo C.** General Relativity: Relative standard mass, momentum, energy, and gravitational field in a general system of reference. *Il Nuovo Cimento*, 1958, v. 10 (2), 318–337.
7. **Cattaneo C.** On the energy equation for a gravitating test particle. *Il Nuovo Cimento*, 1959, v. 11 (5), 733–735.
8. **Cattaneo C.** Conservation laws in General Relativity. *Il Nuovo Cimento*, 1959, v. 13 (1), 237–240.
9. **Cattaneo C.** Problèmes d'interprétation en Relativité Générale. *Colloques Internationaux du Centre National de la Recherche Scientifique*, No. 170, "Fluides et champ gravitationel en Relativité Générale", Éditions du Centre National de la Recherche Scientifique, Paris, 1969, 227–235.
10. **Raschewski P. K.** Riemannian geometry and tensor analysis. Nauka, Moscow, 1964, 664 pages (*in Russian*).
11. **Rabounski D. D. and Borissova L. B.** Particles here and beyond the Mirror. Editorial URSS, Moscow, 2001, 84 pages (e-print ed.: arXiv, gr-qc/0304018).
12. **Borissova L. B. and Rabounski D. D.** Fields, vacuum, and the mirror Universe. Editorial URSS, Moscow, 2001, 272 pages (the 2nd revised ed.: CERN, EXT-2003-025).
13. **Kozyrev N. A.** On the possibility of experimental investigation of the properties of time. *Time in Science and Philosophy*, *Proceedings of the Conference*, Academia, Prague, 1971, 111–132.

14. **Kozyrev N. A.** Physical peculiarities of the components of double stars. *Colloque "On the evolution of double stars"*, *Comptes rendus*, Communications du Observatoire Royal de Belgique, ser. B, no. 17, Bruxelles, 1967, 197–202.
15. **Del Prado J. and Pavlov N. V.** Private reports to A. L. Zelmanov, Moscow, 1968–1969.
16. **Eddington A. S.** The mathematical theory of relativity. Cambridge University Press, Cambridge, 1924 (referred with the 3rd expanded edition, GTTI, Moscow, 1934, 508 pages).
17. **Borissova L. B.** Relative oscillations of test-particles in accompanying reference frames. *Doklady Acad. Nauk USSR*, 1975, v. 225 (4), 786–789.
18. **Borissova L. B.** Quadrupole mass-detector in a field of weak flat gravitational waves. *Izvestia VUZov*, *Physica*, 1978 (10), 109–114.
19. **Rudenko V. N.** Relativistic experiments in gravitational field. *Uspekhi Fizicheskikh Nauk*, 1978, v. 126 (3), 361–401.
20. **Zelmanov A. L.** Orthometric form of monad formalism and its relations to chronometric and kinemetric invariants. *Doklady Acad. Nauk USSR*, 1976, v. 227 (1), 78–81.
21. **Schouten J. A. und Struik D. J.** Einführung in die neuren Methoden der Differentialgeometrie. Bd. I: Schouten J. A. Algebra und Übertragungslehre. Bd. II: Struik D. J. Geometrie. *Zentralblatt für Mathematik*, 1935, Bd. 11 und Bd. 19 (referred with the edition: Introduction into the new methods of differential geometry. Vol. I: Schouten J. A. Algebra and parallel transfer. Vol. II: Struik D. J. Geometry. Foreign Literature, Moscow–Leningrad, 1939).
22. **Synge J. L.** Relativity: the General Theory. North Holland, Amsterdam, 1960 (referred with the 2nd expanded edition, Foreign Literature, Moscow, 1963, 432 pages).