Quaternionic Relativity. II Non-Inertial Motions*.

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In the framework of six-dimensional quaternionic theory of relativity (a short review is given) non-inertial frames are reasonably described: *uniformly accelerated observer* on rectilinear trajectory and arbitrary accelerated observer on circular orbit. The results are used to derive exact Thomas precession formula and calculate change of position of Jupiter's satellite observed from Earth, an integral cinematic effect for frames with variable relative velocity.

Section 1. Introduction.

As is shown in previous paper [1] relative motion of particles and frames can be non-contradictorily described within framework of six-dimensional non-Abelian scheme based upon fundamental properties of quaternionic (Q) algebra. The key point of the scheme is Q-multiplication rule for one "real" unit, 1, and three "imaginary" units $\mathbf{q}_k$ ($k = 1,2,3$)

$$1\mathbf{q}_k = \mathbf{q}_k 1 = \mathbf{q}_k, \ \mathbf{q}_k \mathbf{q}_j = -\delta_{kj} 1 + \varepsilon_{kjn} \mathbf{q}_n \tag{1}$$

where δ_{kj} and ε_{kjn} are Kroneker and Levi-Civita symbols, summation rule is valid. The non-Abelian units $\mathbf{q}_k$ geometrically can be treated as unit vectors of an orthonormal triad. Such a triad admits ordinary R-rotations with real parameters Φ_ξ (e.g. the angles of subsequent rotations $\Phi_1 = \alpha$, $\Phi_2 = \beta$, $\Phi_3 = \gamma$): $R(\Phi_\xi) \in SO(3,R)$. In this case any real Q-vector $\mathbf{a} \equiv a_k \mathbf{q}_k$ is a SO(3,R)-invariant $a_k \mathbf{q}_k = a_{k'} \mathbf{q}_{k'}$ with $\mathbf{q}_{k'} \equiv R_{k'j}(\Phi_\xi)\mathbf{q}_j$.

Parameters of transformation leaving the multiplication rule (1) intact can be complex, and in particular pure imaginary: $\Phi_\xi \to i\Psi_\xi$. Then real rotations convert into hyperbolic ones $R(\Phi_\xi) \to H(\Psi_\xi)$ while the respective invariance is sought for a vector biquaternion

$$\mathbf{u} \equiv (a_k + ib_k)\mathbf{q}_k .$$

The latter can be rewritten as

$$\mathbf{u} = a_k \mathbf{q}_k + b_k \mathbf{p}_k$$

where $\mathbf{p}_k \equiv i\mathbf{q}_k$ are three unit vectors obeying the Pauli-matrices multiplication rule. This new triad is rigidly attached to $\mathbf{q}_k$, but defines scales and directions in a three-space imaginary with respect to initial one. Necessary condition for the Q-vector $\mathbf{u}$ to be invariant under H-rotations is the orthogonality of its vector parts $a_k b_k = 0$. All such biquaternions posses real norm (zero included). The orthogonality condition allows representing $\mathbf{u}$ in the simplest form

$$\mathbf{u} = a\mathbf{q}_1 + b\mathbf{p}_2 ,$$

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where a, b are lengths of the respective vectors. The complete group of transformations preserving $\mathbf{u}$ invariant is $SO(1,2) \subset SO(3,C)$, the latter being the most general group of the Q-multiplication rule (1) invariance. It means that provided one $\mathbf{q}_k$ (e.g. $\mathbf{q}_1$ or $\mathbf{p}_1$) is chosen to perform about it R-rotations, the other two ($\mathbf{q}_2, \mathbf{q}_3$) can only serve as axes of H-rotations.

Q-relativity arises when $\mathbf{u}$-like vector

$$d\mathbf{z} = dt_k \mathbf{p}_k + dx_k \mathbf{q}_k \tag{2}$$

is considered as specific space-time interval with dx_k being displacement and dt_k respective change-of-time (both vectors! $dx_k dt_k = 0$) of a particle observed from a frame of reference $\Sigma \equiv \{\mathbf{p}_k, \mathbf{q}_k\}$. Fundamental velocity is taken for a unity. SO(1,2)-invariance of the "interval" (2) under finite R- and H-rotations (transfer from one inertial frame to another) leads to cinematic relations all equivalent to those of the Einstein's Special Relativity [1].

A most simple constant frame Σ is represented e.g. by Pauli-type matrices

$$\mathbf{q}_1 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \mathbf{q}_2 = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \mathbf{q}_3 = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

The triad may be realized by a platform with three gyroscopes immobile relatively to "distant stars": the observer in Σ "feels" no acceleration.

On the other hand the most general Q-frames may be functions of complex parameters $Z_\xi = \Phi_\xi + \Psi_\xi$:

$$\mathbf{q}_{k'}(Z_\xi) = O(Z_\xi)_{k'j}\mathbf{q}_j. \tag{4}$$

There is no evident obstacle for parameters of the transformation $O(Z_\xi) \in SO(1,2)$ to be localized; natural is a frame's dependence upon its proper time:

$$\Sigma'(t') = \{\mathbf{p}_{k'}[Z_\xi(t')], \mathbf{q}_{k'}[Z_\xi(t')]\}. \tag{5}$$

Q-frames of the type (5) are non-inertial ones, some of them having very complicated behaviour. Before considering the situation in general it seems reasonable to analyze relativistic motion in simple non-inertial cases.

In Section 2 rectilinear *uniformly accelerated motion* is investigated in detail. Section 3 is devoted to *accelerated circular motion*. In Section 3 classical example of Thomas precession is treated from quaternionic relativity viewpoint. Computation of an integral relativistic effect for two frames with variable relative velocity is suggested in Section 5. Discussion and perspectives are found in Section 6.

Section 2. Uniformly accelerated rectilinear motion.

This simplest case of accelerated observer is known as hyperbolic motion [2], [3]. The motion is usually treated from SR-positions but with the help of necessary additional *assumptions (time-dependence of four-velocity*, demand of Fermi-Walker transport of the observer's tetrad) appropriate rather for GR. The Q-relativity approach allows treating the motion without loss of the theory's logic.

If Σ' is a frame uniformly accelerated along its $\mathbf{q}_{2'}$, then observer in Σ' must feel acceleration $\varepsilon = const$ as in Einstein's elevator. This implies specific dependence of Σ' on its proper time t'; the dependence is found out of following considerations.

Let Σ' move relatively to inertial frame Σ (3) along its $\mathbf{q}_2$. If Σ' is observed then the simplest form of interval (2) in this case is

$$d\mathbf{z}' \equiv dt'\mathbf{p}_{1'} = dt\mathbf{p}_1 + dr\mathbf{q}_2, \tag{6}$$

what is equivalent to the H-rotation

$$\mathbf{q}' = H_3^{\psi}\mathbf{q},$$

or in explicit form

$$\begin{pmatrix} \mathbf{q}_{1'} \\ \mathbf{q}_{2'} \\ \mathbf{q}_{3'} \end{pmatrix} = \begin{pmatrix} \cosh\psi & -i\sinh\psi & 0 \\ i\sinh\psi & \cosh\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \end{pmatrix}. \tag{7}$$

with

$$\psi(t') = ar\tanh\frac{dr}{dt}. \tag{8}$$

Analogously to what was done in [1] for Newtonian non-inertial motion one may compute cinematic Q-vectors of Σ': its proper Q-velocity:

$$\mathbf{v}' \equiv \frac{d\mathbf{z}'}{dt'} = \mathbf{p}_{1'} \tag{9}$$

naturally containing the only unit time-like component, and Q-acceleration:

$$\mathbf{a}' \equiv \frac{d\mathbf{v}'}{dt'} = \frac{d\mathbf{p}_{1'}}{dt'}. \tag{10}$$

Computation of Q-acceleration (10) involves notion of quaternionic connection [1, 4]. For a triad obtained from a constant one as in Eq.(4) the derivative of $\mathbf{q}_{k'}$ in the group space is expressed through coefficients of antisymmetric connection $\omega_{\xi k'n'} = -\omega_{\xi n'k'}$:

$$\frac{d\mathbf{q}_{k'}}{dZ_\xi} = \omega_{\xi k'n'}\mathbf{q}_{n'}. \tag{11}$$

From Eqs.(11), (4) the connection components are found as

$$\omega_{\xi k'n'} = \frac{dO_{k'm}}{dZ_\xi} O_{n'm}. \tag{12}$$

If group parameters depend on observer's time then the time derivative is defined

$$\frac{d\mathbf{q}_{k'}}{dt} = \frac{dZ_\xi}{dt}\omega_{\xi k'n'}\mathbf{q}_{n'} \equiv \omega_{k'n'}\mathbf{q}_{n'} \tag{13}$$

with

$$\omega_{k'n'} \equiv \frac{dZ_\xi}{dt}\omega_{\xi k'n'}. \tag{14}$$

Using Eqs.(12), (14) and (7) computation of the Q-acceleration (10) is straightforward

$$\mathbf{a}' = i\frac{d\mathbf{q}_{1'}}{dt'} = i\omega_{1'2'}\mathbf{q}_{2'} + i\omega_{1'3'}\mathbf{q}_{3'} = \frac{d\psi}{dt'}\mathbf{q}_{2'}. \tag{15}$$

Eq.(15) states that the only component of $\mathbf{a}'$ is the acceleration of Σ' "felt" by its own observer, hence

$$\frac{d\psi}{dt'} = \varepsilon,$$

or

$$\psi = \varepsilon t', \tag{16}$$

$$\psi_{t'=0} = 0.$$

Now general cinematic problem (i.e. functions of time, coordinate, velocity and acceleration of the Q-frames) is easily solved.

<u>Case (a). Frame Σ' is observed</u>; interval $d\mathbf{z}'$ has the form (6) therefore

$$dt = dt'\cosh(\varepsilon t').$$

After integration one obtains time-correlation equations (integration constant is assumed zero)

$$t = \frac{1}{\varepsilon}\sinh(\varepsilon t')$$

or

$$t' = \frac{1}{\varepsilon} ar\sinh(\varepsilon t) = \frac{1}{\varepsilon}\ln[\varepsilon t + \sqrt{1+(\varepsilon t)^2}\,]. \tag{17}$$

Velocity dependence on *t* is found from $v = \tanh(\varepsilon t')$ with *t* substituted by t' from (17)

$$v(t) = \tanh[ar\sinh(\varepsilon t)] = \frac{\varepsilon t}{\sqrt{1+(\varepsilon t)^2}}. \tag{18}$$

Integration of (18) yields the Σ'-motion law

$$r(t) = \frac{1}{\varepsilon}\sqrt{1+(\varepsilon t)^2} - \frac{1}{\varepsilon}, \tag{19}$$

(integration constant is $-1/\varepsilon$), while differentiation of (18) with respect to *t* gives *acceleration* of Σ' seen from Σ

$$a(t) = \frac{\varepsilon}{[1+(\varepsilon t)^2]^{3/2}}. \tag{20}$$

Cinematic problem is solved; the results precisely repeat those of [2,3]. For small *t*: $t' \to t$, $a(t) \to \varepsilon = const$, $v(t) \to \varepsilon t$ and $r(t) \to \varepsilon t^2/2$ as it must be for non-relativistic uniformly accelerated motion. If $t \to \infty$ then $t' \to \frac{1}{\varepsilon}\ln(2\varepsilon t) \to \infty$, $a \to \varepsilon^{-2} t^{-3} \to 0$, $v \to 1$ and $r \to t \to \infty$ as is natural from SR viewpoint.

Case (b). Frame Σ is observed; the interval takes the form

$$d\mathbf{z} \equiv dt'\mathbf{p}_{1'} - dr'\mathbf{q}_{2'} = dt\mathbf{p}_1 \tag{21}$$

where $-dr'$ is apparent displacement of the origin of Σ in a time dt' measured in Σ'. H-rotation parameter is given by Eq.(16), inertial Σ-observer obviously feels no acceleration

$$\mathbf{a} \equiv \frac{d\mathbf{z}'}{dt} = 0.$$

The cinematic problem is solved analogously

$$dt = \frac{dt'}{\cosh(\varepsilon t')}, \tag{22}$$

$$t(t') = \frac{1}{\varepsilon}\arctan[\sinh(\varepsilon t')] \equiv \frac{1}{\varepsilon}\arcsin[\tanh(\varepsilon t')], \tag{23}$$

$$v'(t') = \tanh(\varepsilon t'), \tag{24}$$

$$r'(t') = \frac{1}{\varepsilon}\ln[\cosh(\varepsilon t')], \tag{25}$$

$$a'(t') = \frac{\varepsilon}{\cosh^2(\varepsilon t')}. \tag{26}$$

Behaviour of the quantities in characteristic points of time-ray is the following. If $t' \to 0$ then $t \to t'$, $a'(t') \to \varepsilon = const$, $v' \to \varepsilon t'$ and $r'(t') \to \varepsilon t'^2/2$ as is normal for non-relativistic uniformly accelerated motion. If $t' \to \infty$ then $a' \to 2\varepsilon e^{-2\varepsilon t'} \to 0$, $v' \to 1$, $r' \to t' \to \infty$ which agrees with notions of SR. Asymptotic behaviour of time is specific. Eq.(23) implies that observer in Σ' finds the clock of Σ more and more slow; at infinite time $t' \to \infty$ the Σ-clock tends to stop on the value $\frac{\pi c}{2\varepsilon}$ (*c* is the fundamental velocity).

An important feature of time measurement must be emphasized here. The matter is that Eqs.(17-20) and (23-26) give such values of cinematic quantities as if frames Σ' and Σ

exchange information instantly. Actually in vacuum the signal travels with velocity $c = 1$, so the information about physical status of the object reaches distant observer at a later time.

In the case (a) this time is

$$t_r = t + r(t) . \tag{27}$$

Substitution of Eq.(19) into (27) allows to express t as function of t_r

$$t = \frac{1}{2} t_r \left(1 + \frac{1}{1 + \varepsilon t_r} \right). \tag{28}$$

Now instant time t can be replaced in Eqs. (17-20) by retarded time t_r that the observer reads from his clock. Thus recalculated velocity of Σ' is

$$v(t_r) = \frac{\varepsilon t_r \left(1 + \frac{1}{1 + \varepsilon t_r} \right)}{\sqrt{4 + \left[\varepsilon t_r \left(1 + \frac{1}{1 + \varepsilon t_r} \right) \right]^2}} . \tag{29}$$

Closely to zero and asymptotically t and t_r are similar: if $t \to 0$, then $t_r \to t \to 0$; if $t \to \infty$, then $t_r \to 2t \to \infty$.

In the case (b) the retarded time is

$$t'_r = t' + r'(t') . \tag{30}$$

Time-recalculation formula follows from Eqs.(25) and (30)

$$t' = \frac{1}{2\varepsilon} \ln(2e^{\varepsilon t'_r} - 1) . \tag{31}$$

Eq.(31) helps to introduce time t'_r into Eqs.(23-26); e.g. "observed" velocity of Σ for Σ' clock is

$$v'(t'_r) = \tanh\left[\frac{1}{2} \ln(2e^{\varepsilon t'_r} - 1) \right]. \tag{32}$$

For $t' \to 0$, $t' \to \infty$ behaviour of t' and t'_r is similar.

The *retarded time problem* within framework of non-inertial relativity is much wider. It comprises experimental aspects: since position and velocity of an object are really measured in retarded time, there might be a need for knowledge of "instant" values. Establishing of mathematical ties between retarded quantities also will be helpful to see the whole picture on the base of available data. Detailed analysis of these aspects will be given in a separate paper.

Section 3. Circular motion.

The simplest curvilinear accelerated motion is circular motion. In the Q-relativity framework it needs at least two types of group parameters: a real (rotational) one and an imaginary (hyperbolic) one.

Let the origin of Σ lie in the centre and vectors $\mathbf{q}_2, \mathbf{q}_3$ in the plane of circular orbit (with radius R) of the frame Σ' (Fig.1). There are two steps in building cinematic relativistic model.

The first step is non-relativistic construction of the object's displacement. In the base Σ the coordinates of Σ' are

$$x_2 = R\cos\alpha(t), \; x_3 = R\sin\alpha(t), \; x_3 = 0 .$$

Make one of the triad vectors, say the third, be always parallel to the Σ' velocity

$$\mathbf{q}_{\bar{3}} \equiv \frac{\dot{x}_k}{\sqrt{\dot{x}_j \dot{x}_j}} \mathbf{q}_k = -\sin\alpha \mathbf{q}_2 + \cos\alpha \mathbf{q}_3 ;$$

this as a part of the simple R-rotation

$$\bar{\Sigma} = R_1^{\alpha(t)} \Sigma ,$$

or in explicate form

$$\begin{pmatrix} \mathbf{q}_{\bar{1}} \\ \mathbf{q}_{\bar{2}} \\ \mathbf{q}_{\bar{3}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \end{pmatrix} . \qquad (33)$$

From non-relativistic viewpoint interpretation of $\bar{\Sigma}$ is two-folded. On one hand it can be treated as a frame rotating in the orbit's centre with angular velocity $\omega(t) \equiv \dot{\alpha}$; its $\mathbf{q}_{\bar{2}}$ is constantly chasing the origin of the frame on orbit. On the other hand $\bar{\Sigma}$ represents a frame revolving on the orbit with speed $v = \omega R$ its displacement being $dr = \omega R dt$.

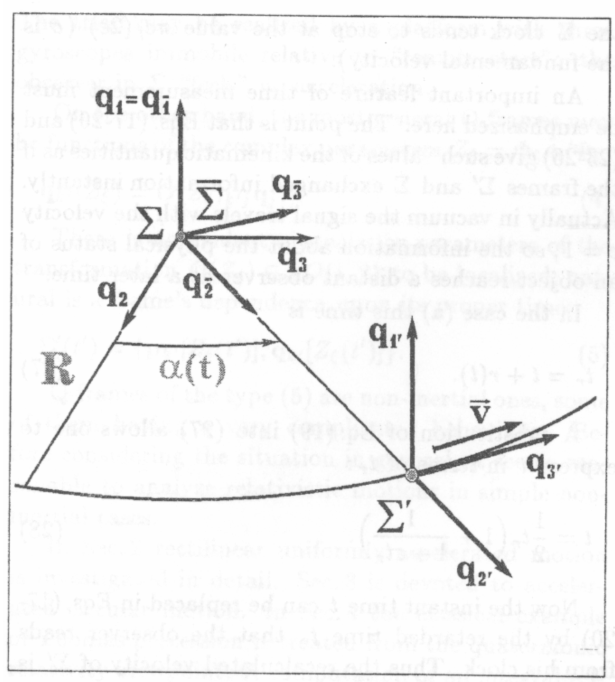

Figure 1: Circular motion

The second step is to "switch" relativity, i.e. to "H-rotate" $\bar{\Sigma}$ at "angle" $\psi = ar\tanh(\omega R)$ about $\mathbf{q}_{\bar{2}}$, so that change-of-time vector becomes aligned with $\mathbf{q}_1 = \mathbf{q}_{\bar{1}}$ not involved into description of space cinematic quantities

$$\Sigma' = H_{\bar{2}}^{\psi} \bar{\Sigma} ,$$

or

$$\begin{pmatrix} \mathbf{q}_{1'} \\ \mathbf{q}_{2'} \\ \mathbf{q}_{3'} \end{pmatrix} = \begin{pmatrix} \cosh\psi & 0 & -i\sinh\psi \\ 0 & 1 & 0 \\ i\sinh\psi & 0 & \cosh\psi \end{pmatrix} \begin{pmatrix} \mathbf{q}_{\bar{1}} \\ \mathbf{q}_{\bar{2}} \\ \mathbf{q}_{\bar{3}} \end{pmatrix} . \qquad (34)$$

Altogether Σ' is combination of two subsequent rotations (20) and (21) subject to SO(1,2) symmetry

$$\Sigma' = H_{\bar{2}}^{\psi} R_1^{\alpha} \Sigma . \qquad (35)$$

Q-acceleration felt by Σ'-observer

$$\mathbf{a}' = i \frac{d\mathbf{q}_{1'}}{dt'} = i\omega_{1'2'} \mathbf{q}_{2'} + i\omega_{1'3'} \mathbf{q}_{3'}$$

is found from (35) with the help of Eqs.(12, 14)

$$\mathbf{a}' = -\frac{d\alpha}{dt'}\sinh\psi\mathbf{q}_{2'} + \frac{d\psi}{dt'}\mathbf{q}_{3'}.$$

Here are normal (centripetal)

$$a_{norm} = -\frac{d\alpha}{dt'}\sinh\psi$$

and tangent (angular)

$$a_{\tan} = \frac{d\psi}{dt'}$$

components of the acceleration.

For simple cases the components are readily found. Uniform motion implies $\psi = ar\tanh\omega' R' = const$ (ω', R' are angular velocity, and the orbit's radius for Σ'-observer), therefore $a_{\tan} = 0$, $\alpha = \omega' t'$, $a_{norm} = \omega'^2 R' \cosh\psi = const$. For uniformly accelerated motion $\psi = ar\tanh\omega' R' = \lambda t'$ with $\lambda = const$, then $a_{\tan} = \lambda$, $a_{norm} = \omega'^2(t')R'\cosh(\lambda t')$. These are quite expected results.

Farther analysis of circular motion is made for general form of the hyperbolic parameter $\psi = \psi(t')$ that is assumed given.

Case (aa). Frame Σ' is observed.

The interval expression is read from the first row of matrix Eq.(34) for rotating (non-inertial) base $\overline{\Sigma}$, or from the first row of Eq.(35) for the inertial base Σ

$$d\mathbf{z}' \equiv idt'\mathbf{q}_{1'} = idt\mathbf{q}_{\overline{1}} + dtR\omega\mathbf{q}_{\overline{3}} = idt\mathbf{q}_1 + dtR\omega(-\sin\alpha\mathbf{q}_2 + \cos\alpha\mathbf{q}_3). \tag{36}$$

Procedure of solving the cinematic problem is analogous to that of Section 2. From Eq.(36) it follows

$$dt = dt'\cosh\psi(t'), \tag{37}$$

$$t = \int dt'\cosh\psi(t'), \tag{38}$$

and the inverse function $t' = t'(t)$ is possibly found. The latter is used in expression for angular velocity

$$\omega(t) = \frac{1}{R}\tanh\psi[t'(t)]. \tag{39}$$

Eq.(38) yields Σ-time dependence of rotation angle

$$\alpha(t) = \int \omega(t)dt \tag{40}$$

and tangent acceleration

$$a_{\tan}(t) = R\frac{d\omega(t)}{dt} = \frac{1}{\cosh^3\psi}\frac{d\psi}{dt'}. \tag{41}$$

These are all quantities available for $\overline{\Sigma}$-observer. Observer in Σ additionally makes conclusion about normal part of the acceleration

$$a_{norm}(t) = R\left(\frac{d\alpha(t)}{dt}\right)^2. \tag{42}$$

Eqs.(38-42) represent solution of the cinematic problem.

An essential note must be made here. An arc segment collinear to relative velocity is relativistically contracted as in SR

$$dl = dl'\cosh\psi,$$

while the orbit's radius perpendicular to velocity and not involved into transformations remains the same for Σ and Σ'

$$R' = R. \tag{43}$$

Hence in the case (a) the angle measures are related as

$$d\alpha = \frac{dl}{R} = \frac{dl'}{R'}\cosh\psi = d\alpha'\cosh\psi \,. \tag{44}$$

The last ratio together with Eqs.(43, 37) gives unique numerical value of the frames' angular velocities for respective observers

$$\omega(\Sigma) = \frac{d\alpha}{dt} = \frac{d\alpha'}{dt'} = \omega'(\Sigma') \,. \tag{45}$$

This result agrees with the axiomatic fact that for Σ and Σ' value of relative velocity (or hyperbolic parameter) is the same
$v = \tanh\psi = \omega R = \omega' R'$.

<u>Case (bb). Frame Σ is observed.</u>

The interval for Σ (and $\overline{\Sigma}$) is equivalent to the first row of matrix equation

$$\Sigma = R_{\bar{1}}^{-\alpha} H_{2'}^{-\psi} \Sigma'$$

inverse to Eq.(35)

$$d\mathbf{z} = dt\mathbf{p}_1 = dt\mathbf{p}_{\bar{1}} = dt'\mathbf{p}_{1'} - R'\omega' dt'\mathbf{q}_{3'} \tag{46}$$

with

$$\omega' R' = \omega R = \tanh\psi(t') \,.$$

Q-acceleration of Σ vanish

$$\mathbf{a} \equiv \frac{d\mathbf{z}}{dt} = 0$$

since Σ -observer is considered genuinely immobile. In this case

$$dt' = dt\cosh\psi$$

$$t = \int \frac{dt'}{\cosh\psi(t')} \,. \tag{47}$$

Σ' -observer is able to measure apparent cinematic quantities: velocity, angle and tangent acceleration

$$v = \tanh\psi(t') \,, \tag{48}$$

$$\alpha(t') = \int \frac{\tanh\psi(t')}{R} dt' \,, \tag{49}$$

$$a_{\tan} = \frac{1}{\cosh^2\psi}\frac{d\psi}{dt'} \,. \tag{50}$$

Eqs.(47-50) give solution of cinematic problem in the case (bb). Reasonable behaviour of the quantities is readily verified for simple types of circular motion in non-relativistic and ultrarelativistic limits.

In both cases (aa) and (bb) observers receive in fact retarded signals. But contrary to situation for rectilinear motion, here constant time delay does not influence noticeably values of cinematic quantities.

Section 4. Thomas precession.

Apparent rotation of genuinely constant "spin" vector of the top relativistically revolving about the origin of an inertial frame (Thomas precession) is regarded in SR either when circular trajectory is *approximated* by straight line segments [5] or when Fermi-Walker transport of vectors is postulated [3]. Quaternionic relativity suggests a shorter and more consistent way (from logical viewpoint) to describe the phenomenon.

Let Σ be an immobile Q-frame (3) in the centre of the circular orbit of another Q-frame, Σ'' uniformly revolving about Σ. Respective observers find space vectors of their frames constantly oriented. Notice that Σ'' can be obtained from Σ' determined by Eq.(35) with the help of inverse R-rotation at appropriate angle $-\alpha'$

$$\Sigma'' = R_{1'}^{-\alpha'} H_{\frac{\psi}{2}} R_1^{\alpha} \Sigma , \tag{51}$$

or in explicit form

$$\begin{pmatrix} \mathbf{q}_{1''} \\ \mathbf{q}_{2''} \\ \mathbf{q}_{3''} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha' & -\sin\alpha' \\ 0 & \sin\alpha' & \cos\alpha' \end{pmatrix} \begin{pmatrix} \cosh\psi & 0 & -i\sinh\psi \\ 0 & 1 & 0 \\ i\sinh\psi & 0 & \cosh\psi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \end{pmatrix} . \tag{52}$$

Suppose that angles of rotation are calculated in terms of laboratory time t: $\alpha = \omega t$, $\alpha' = \omega'(\Sigma)t$. For the base Σ' proper (real) period $T'(\Sigma')$ of its retrograde (second) rotation measured in Σ' due to Eq.(47) is related to apparent period $T'(\Sigma)$ measured in Σ as

$$T'(\Sigma') = T'(\Sigma)\cosh\psi .$$

Then proper cyclic frequency of the rotation (measured in Σ') is

$$\omega'(\Sigma') = \frac{2\pi}{T'(\Sigma')} = \frac{2\pi}{T'(\Sigma)\cosh\psi} = \frac{\omega'(\Sigma)}{\cosh\psi},$$

or, taking into account Eq.(45),

$$\omega'(\Sigma) = \omega(\Sigma)\cosh\psi : \tag{53}$$

cyclic frequency of Σ' second rotation "seen" from Σ is $\cosh\psi$ times bigger than that of first rotation of Σ measured in itself and needed to chase Σ'.

Now compute change of the top's spin direction seen from Σ while in Σ'' being constantly pointed at a distant star, say, along $\mathbf{q}_{2''}$, space unit vector of Σ''. From Eq.(52) vector $\mathbf{q}_{2''}$ in projections onto unit vectors of Σ is

$$\begin{aligned} \mathbf{q}_{2''} &= -i\sinh\psi\sin\alpha\mathbf{q}_1 + (\cos\alpha\cos\alpha' + \cosh\psi\sin\alpha\sin\alpha')\mathbf{q}_2 + \\ &+ (\sin\alpha\cos\alpha' - \cosh\psi\cos\alpha\sin\alpha')\mathbf{q}_3 . \end{aligned} \tag{54}$$

Projections of $\mathbf{q}_{2''}$ onto spatial directions of Σ allow us determining apparent precession, e.g. projection onto $\mathbf{q}_3$

$$\left\langle \mathbf{q}_{2''} \right\rangle_3 = \sin(\alpha - \alpha') - \frac{1}{2}(\cosh\psi - 1)\sin\alpha\sin\alpha' , \tag{55}$$

or after some algebra

$$\left\langle \mathbf{q}_{2''} \right\rangle_3 = \sin\{[\omega(\Sigma) - \omega'(\Sigma)]t\} - \sinh^2\frac{\psi}{2}\sin\omega(\Sigma)t\sin\omega'(\Sigma)t . \tag{56}$$

Angular velocity of the first rhs term in Eq.(56) $\omega_T \equiv \omega - \omega'$ corresponds to "mostly noticeable" Thomas precession. Due to Eq. (53) it can be presented as

$$\omega_T = \omega(1 - \cosh\psi) ; \tag{59}$$

for small relative Σ - Σ'' velocities it takes the known form

$$\omega_T = -\frac{\omega}{2} v^2 \tag{60}$$

with $v = \omega R$. The second right-hand side term of Eq.(56) describes much "*less noticeable*" precession since its amplitude is v^2/c^2 less than that of the first term. For small relative velocities the second term is $-\frac{v^2}{4}\sin 2\omega t$.

It is worth to note that results given by Eqs.(55, 56) precisely coincide with those found in [3]. The only difference is that due to SO(1,2)-symmetry preserving choice of R- and H-rotation axes no time-like components of spin ever appear.

Section 5. Jupiter's satellite.

Presented relativistic treatment of circular motion suggests the following observational experiment aimed to control consistence of the theory. Consider a part of Solar system (Fig.2) where $\tilde{\Sigma}$ is constant Q-frame with Sun as reference body, Σ' is attached to Earth, and Σ belongs to Jupiter. In non-relativistic limit relative Jupiter-Earth velocity is

$$V^2 = v_E^2 + v_J^2 - 2v_E v_J \cos(\alpha - \beta) \tag{61}$$

with constant orbit velocities of the planets

$$v_E = \omega_E R_E \,,\; v_J = \omega_J R_J$$

and respective radius angles measured from $\mathbf{q}_{\tilde{1}}$ linearly depending on $\tilde{\Sigma}$ time

$$\alpha = \omega_E \tilde{t} \;, \qquad \beta = \omega_J \tilde{t} \;.$$

Picture of a satellite revolving about Jupiter and observed from Earth is similar of that regarded in Section 3. The only difference is that both object Σ and observer Σ' are non-inertial, their relative speed variable, and this is the crucial point. If the Earth's observer in a short period of time measures the satellite angular velocity $\omega(\Sigma')$ and neglects influence of relative Earth-Jupiter motion then in time t' he computes the rotation angle as

$$\varphi_{theor} = \omega(\Sigma')t' \,.$$

In fact, according to Eq.(53), the *apparent angular velocity* is

$$\omega(\Sigma') = \omega \cosh\psi \,,$$

where $\omega(\Sigma) \equiv \omega$ is genuine constant quantity measured on Jupiter. Since velocity of relative motion is variable, $\omega(\Sigma')$ is variable too. Hence the rotation angle really observed from Σ' is found as

$$\varphi_{real} = \int \omega(\Sigma')dt' = \omega \int \cosh\psi(t')dt' \,.$$

Difference between computed and observed values of the angle is

$$\Delta\varphi \equiv \varphi_{real} - \varphi_{theor} = \omega[\cosh\psi(t_0') - \int \cosh\psi(t')dt'] \,. \tag{63}$$

Ratio V/c is small, so non-relativistic value of relative velocity (61) is sufficient

$$\cosh\psi \approx 1 + \frac{1}{2}\left[\frac{V(t')}{c}\right]^2 \,.$$

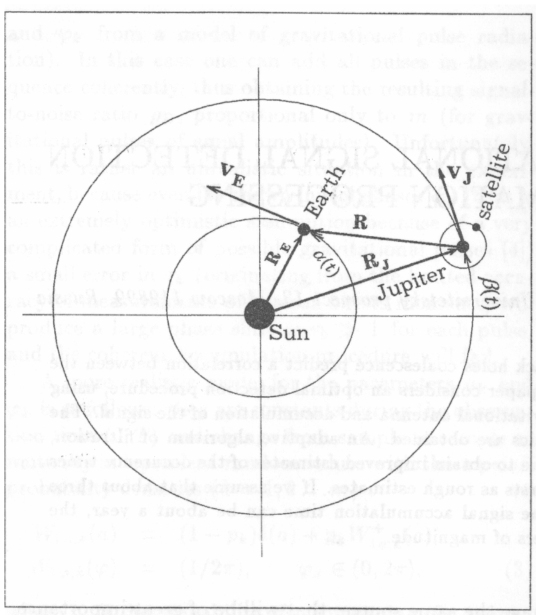

Figure 2: Jupiter's satellite

If at the moment of initial measurement t_0' velocities of Earth and Jupiter are parallel $\vec{v}_E(t_0') \uparrow\uparrow \vec{v}_E(t_0')$, then computed value of apparent angular velocity is minimal and Eq.(63) takes the form

$$\Delta\varphi \approx \frac{\omega v_E v_J}{c^2}\left[1-\frac{\sin(\alpha-\beta)}{\alpha-\beta}\right]t'. \tag{64}$$

The second term in brackets tends to zero with time so that final formula is

$$\Delta\varphi \approx \frac{\omega v_E v_J}{c^2}t'. \tag{65}$$

For the closest Jupiter satellite ("Metes") $\omega = 2.5\cdot 10^{-4}\sec$, $v_E = 30.4 km/\sec$, $v_J = 31.1 km/\sec$ in one Jupiter year $t' = 12 Earth\, years = 3.7\cdot 10^{8}\sec$ the angle difference $\Delta\varphi = 4.14\cdot 10^{-4}\, rad = 1.4'$ might be observable.

Section 6. Discussion.

Examples given in paper [1] and above show that quaternionic approach to relativistic kinematics seems to provide consistent description of inertial and non-inertial frames of reference. In the framework of the theory all cinematic effects of Einstein's Special Relativity are found as well as hyperbolic motion and Thomas precession, effects whose standard descriptions demand additional assumptions. The suggested relativistic scheme due to vectorial character of its space-time "interval" is obviously convenient to consider frames with curvilinear trajectories. In particular it permits to obtain plausible results of relativistic circular motion of general type and predict integral effects for frames with variable relative speed.

Nevertheless several examples whatever successful they were represent only separate pieces of unique theory. Kinematics of quaternionic relativity is not completed until the most general types of frames trajectories are taken into account. This is the task of next paper.

References

[1] A.P.Yefremov, *Gravitation & Cosmology*, Vol. 2 (1996), No.1(5), p.p. 77-83.
[2] L.D.Landau, E.M.Lifshts, "Theory of Fields", M., Nauka, 1973, p. 39.
[3] Ch.W.Misner, K.S.Thorn, J.A.Wheeler, "Gravitation", San Francisco, W.N.Freeman & Co., 1973, p.p. 166, 175.
[4] A.P.Yefremov, Izvestiya Vuzov, *Fizika*, (1985), **12**, p.p. 14-18.
[5] E.F.Tailor, J.A.Wheeler, "Spacetime Physics", San Francisco, W.N.Freeman & Co., 1966, p.p. 222-228.