Could q-Laguerre equation explain the claimed fractionation of the principal quantum number for hydrogen atom?

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1 Introduction

In [G2] a semiclassical model based on dark matter and hierarchy of Planck constants is developed for the fractionized principal quantum number n claimed by Mills [1] to have at least the values $n = 1/k$, $k = 2, 3, 4, 5, 6, 7, 10$.

This model could explain the claimed fractionization of the principal quantum number n for hydrogen atom [1] in terms of single electron transitions for all cases except $n = 1/2$: the basis reason is that Jones inclusions are characterized by quantum phases $q = exp(i\pi/n)$, $n > 2$. Since quantum deformation of the standard quantum mechanism is involved, this motivates an attempt to understand the claimed fractionization in terms of q-analog of hydrogen atom.

The Laguerre polynomials appearing in the solution of Schrödinger equation for hydrogen atom possess quantum variant, so called q-Laguerre polynomials [2], and one might hope that they would allow to realize this semiclassical picture at the level of solutions of appropriately modified Schrödinger equation and perhaps also resolve the difficulty associated with $n = 1/2$. Unfortunately, the polynomials discussed in [2] correspond to $0 < q \leq 1$ rather than complex values of $q = exp(i\pi/m)$ on circle and the extrapolation of the formulas for energy eigenvalues gives complex energies. It is however easy to modify the definition of q-derivative and it turns out that it is possible to reproduce $n = 1/2$ state exactly and $n = 1/m$, $m > 2$ states in a reasonable approximation as solutions of q-Laquerre equation for s-wave states. Also the generalization to associated q-Laquerre equation is straightforward.

2 Could q-Laguerre equation explain the claimed fractionation of the principal quantum number for hydrogen atom?

In [G2] a semiclassical model based on dark matter and hierarchy of Planck constants is developed for the fractionized principal quantum number n claimed by Mills [1] to have at least the values $n = 1/k$, $k = 2, 3, 4, 5, 6, 7, 10$. This model can explain the claimed fractionization of the principal quantum number n for hydrogen atom [1] in terms of single electron transitions for all cases. The original model could not cope with $n = 1/2$: the basic reason is that Jones inclusions are characterized by quantum phases $q = exp(i\pi/n)$, $n > 2$. Since quantum deformation of the standard quantum mechanism is involved, this motivated an attempt to understand the claimed fractionization in terms of q-analog of hydrogen atom. The safest interpretation for them would be as states which can exist in ordinary imbedding space (and also in other branches). The natural guess would be that they can occur as intermediate states in the phase transition changing $n_b/n_a = 1$ to $k = 2, 3, ...$

The Laguerre polynomials appearing in the solution of Schrödinger equation for hydrogen atom possess quantum variant, so called q-Laguerre polynomials [2], and one might hope that they would allow to realize this semiclassical picture at the level of solutions of appropriately modified Schrödinger equation and perhaps also resolve the difficulty associated with $n = 1/2$. Unfortunately, the polynomials discussed in [2] correspond to $0 < q \leq 1$ rather than complex values of $q = exp(i\pi/m)$ on circle and the extrapolation of the formulas for energy eigenvalues gives complex energies.

2.1 q-Laquerre equation for $q = exp(i\pi/m)$

The most obvious modification of the Laguerre equation for S-wave sates (which are the most interesting by semiclassical argument) in the complex case is based on the replacement

$$\begin{aligned} \partial_x &\rightarrow \frac{1}{2}(\partial_x^{q)} + \partial_x^{\overline{q})}) \\ \partial_x^{q)} f &= \frac{f(qx) - f(x)}{(q-1)x} \quad , \\ q &= exp(i\pi/m) \end{aligned} \tag{1}$$

to guarantee hermiticity. When applied to the Laguerre equation

$$x\frac{d^2L_n}{dx^2} + (1-x)\frac{dL_n}{dx} = nL_n \quad , \tag{2}$$

and expanding L_n into Taylor series

$$L_n(x) = \sum_{n \geq 0} l_n x^n \quad , \tag{3}$$

one obtains difference equation

$$\begin{aligned} a_{n+1}l_{n+1} &+ b_n l_n = 0 \quad , \\ a_{n+1} &= \frac{1}{4R_1^2}\left[R_{2n+1} - R_{2n} + 2R_{n+1}R_1 + 3R_1)\right] + \frac{1}{2R_1}\left[R_{n+1} + R_1\right] \\ b_n &= \frac{R_n}{2R_1} - n^{q)} + \frac{1}{2} \quad , \\ R_n &= 2cos\left[(n-1)\pi/m\right] - 2cos\left[n\pi/m\right] \quad . \end{aligned} \tag{4}$$

Here $n^{q)}$ is the fractionized principal quantum number determining the energy of the q-hydrogen atom. One cannot pose the difference equation on

l_0 since this together with the absence of negative powers of x would imply the vanishing of the entire solution. This is natural since for first order difference equations lowest term in the series should be chosen freely.

2.2 Polynomial solutions of q-Laquerre equation

The condition that the solution reduces to a polynomial reads as

$$b_n = 0 \tag{5}$$

and gives

$$n^{q)} = \frac{1}{2} + \frac{R_n}{2R_1} \ , \tag{6}$$

For $n = 1$ one has $n^{q)} = 1$ so that the ground state energy is not affected. At the limit $N \rightarrow \infty$ one obtains $n^{q)} \rightarrow n$ so that spectrum reduces to that for hydrogen atom. The periodicity $R_{n+2Nk} = R_n$ reflects the corresponding periodicity of the difference equation which suggests that only the values $n \leq 2m-1$ belong to the spectrum. Spectrum is actually symmetric with respect to the middle point $[N/2]$ which suggests that only $n < [m/2]$ corresponds to the physical spectrum. An analogous phenomenon occurs for representations of quantum groups. When m increases the spectrum approaches integer valued spectrum and one has $n > 1$ so that no fractionization in the desired sense occurs for polynomial solutions.

2.3 Non-polynomial solutions of q-Laquerre equation

One might hope that non-polynomial solutions associated with some fractional values of $n^{q)}$ near to those claimed by Mills might be possible. Since the coefficients a_n and b_n are periodic, one can express the solution ansatz as

$$\begin{aligned} L_n(x) &= P_a^{2m)}(x) \sum_k a^k x^{2mk} = P_a^{2m)}(x) \frac{1}{1 - ax^{2m}} \ , \\ P_a^{2m)}(x) &= \sum_{k=0}^{2m-1} l_k x^k \ , \\ a &= \frac{l_{2m}}{l_0} \ , \end{aligned} \tag{7}$$

This solution behaves as $1/x$ asymptotically but has pole at $x_\infty = (1/a)^{1/2m}$ for $a > 0$.

The expression for $l_{2m}/l_0 = a$ is

$$a = \prod_{k=1}^{2m} \frac{b_{2m-k}}{a_{2m-k+1}} \ . \tag{8}$$

This can be written more explicitly as

$$\begin{aligned} a &= (2R_1)^{2m} \prod_{k=1}^{2m} X_k \ , \\ X_k &= \frac{R_{2m-k} + (-2n^{q)} + 1)R_1}{R_{4m-2k+1} - R_{4m-2k} + 4R_{2m-k+1}R_1 + 2R_1^2 + 3R_1} \ , \\ R_n &= 2cos\left[(n-1)\pi/m\right] - 2cos\left[n\pi/m\right] \ . \end{aligned} \tag{9}$$

This formula is a specialization of a more general formula for $n = 2m$ and resulting ratios l_n/l_0 can be used to construct $P_a^{2m)}$ with normalization $P_a^{2m)}(0) = 1$.

2.4 Results of numerical calculations

Numerical calculations demonstrate following.

a) For odd values of m one has $a < 0$ so that a a continuous spectrum of energies seems to result without any further conditions.

b) For even values of m a has a positive sign so that a pole results.

For even value of m it could happen that the polynomial $P_a^{2m)}(x)$ has a compensating zero at x_∞ so that the solution would become square integrable. The condition for reads explicitly

$$P_a^{2m)}\left((\frac{1}{a})^{\frac{1}{2m}} \right) = 0 \ . \tag{10}$$

If $P_a^{2m)}(x)$ has zeros there are hopes of finding energy eigen values satisfying the required conditions. Laguerre polynomials and also q-Laguerre polynomials must posses maximal number of real zeros by their orthogonality implied by the hermiticity of the difference equation defining them. This suggests that also $P_a^{2m)}(x)$ possesses them if a does not deviate too much from zero. Numerical calculations demonstrate that this is the case for $n^{q)} < 1$.

For ordinary Laguerre polynomials the naive estimate for the position of the most distant zero in the units used is larger than n but not too much so. The naive expectation is that L_{2m} has largest zero somewhat above $x = 2m$ and that same holds true a small deformation of L_{2m} considered now since the value of the parameter a is indeed very small for $n^{q)} < 1$. The ratio $x_\infty/2m$ is below .2 for $m \leq 10$ so that this argument gives good hopes about zeros of desired kind.

One can check directly whether x_∞ is near to zero for the experimentally suggested candidates for $n^{q)}$. The table below summarizes the results of numerical calculations.

a) The table gives the exact eigenvalues $1/n_{q)}$ with a 4-decimal accuracy and corresponding approximations $1/n^{q)}_{\simeq} = k$ for $k = 3, ..., 10$. For a given value of m only single eigenvalue $n^{q)} < 1$ exists. If the observed anomalous spectral lines correspond to single electron transitions, the values of m for them must be different. The value of m for which $n^{q)} \simeq 1/k$ approximation is optimal is given with boldface. The value of k increases as m increases. The lowest value of m allowing the desired kind of zero of $P^{2m)}$ is $m = 18$ and for $k \in \{3, 10\}$ the allowed values are in range $18, .., 38$.

b) $n^{q)} = 1/2$ does not appear as an approximate eigenvalue so that for even values of m quantum calculation produces same disappointing result as the classical argument. Below it will be however found that $n^{q)} = 1/2$ is a universal eigenvalue for odd values of m.

m	$1/n^{q)}_{\simeq}$	$1/n^{q)}$	m	$1/n^{q)}_{\simeq}$	$1/n^{q)}$
18	**3**	2.7568	30	8	7.5762
20	**4**	3.6748	**32**	**8**	8.3086
22	5	4.5103	**34**	**9**	9.0342
24	**5**	5.3062	**36**	**10**	9.7529
26	**6**	6.0781	38	10	10.4668
28	**7**	6.8330			

Table 1. The table gives the approximations $1/n^{q})_{\simeq} = 1/k$ and corresponding exact values $1/n^{q})$ in the range $k = 3, ..., 10$ for which $P_a^{2m)}(x_\infty)$ is nearest to zero. The corresponding values of $m = 2k$ vary in the range, $k = 18, ..., 38$. For odd values of m the value of the parameter a is negative so that there is no pole. Boldface marks for the best approximation by $1/n^{q)}_{\simeq} = k$.

2.5 How to obtain $n^{q)} = 1/2$ state?

For odd values of m the quantization recipe fails and physical intuition tells that there must be some manner to carry out quantization also now. The following observations give a hunch about be the desired condition.

a) For the representations of quantum groups only the first m spins are realized. This suggests that there should exist a symmetry relating the coefficients l_n and l_{n+m} and implying $n^{q)} = 1/2$ for odd values of m. This symmetry would remove also the double degeneracy associated with the almost integer eigenvalues of $n^{q)}$. Also other fractional states are expected on basis of physical intuition.

b) For $n^{q)} = 1/2$ the recursion formula for the coefficients l_n involves only the coefficients R_m.

c) The coefficients R_k have symmetries $R_k = R_{k+2m}$ and $R_{k+m} = -R_m$.

There is indeed this kind of symmetry. From the formula

$$\begin{aligned} \frac{l_n}{l_0} &= (2R_1)^n \prod_{k=1}^{n} X_k \ , \\ X_k &= \frac{R_{n-k} + (-2n^{q)} + 1)R_1}{[R_{2n-2k+1} - R_{n-2k} + 4R_{n-k+1}R_1 + 2R_1^2 + 3R_1} \end{aligned} \tag{11}$$

one finds that for $n^{q)} = 1/2$ the formula giving l_{n+m} in terms of l_n changes sign when n increases by one unit

$$\begin{aligned} A_{n+1} &= (-1)^m A_n \ , \\ A_n &= \prod_{k=1}^{m} \frac{b_{n+m-k}}{a_{n+m-k+1}} = \prod_{k=1}^{m} (2R_1)^m \prod_{k=1}^{m} X_{k+n} \ . \end{aligned} \tag{12}$$

The change of sign is essentially due to the symmetries $a_{n+m} = -a_n$ and $b_{n+m} = b_n$. This means that the action of translations on A_n in the space of indices n are represented by group Z_2.

This symmetry implies $a = l_{2m}/l_0 = -(l_m)(l_0)^2$ so that for $n^{q)} = 1/2$ the polynomial in question has a special form

$$\begin{aligned} P_a^{2m)} &= P_a^{m)}(1 - Ax^m) \ , \\ A &= A_0 \ . \end{aligned} \tag{13}$$

The relationship $a = -A^2$ implies that the solution reduces to a form containing the product of m^{th} (rather than $(2m)^{th}$) order polynomial with a geometric series in x^m (rather than x^{2m}):

$$L_{1/2}(x) = \frac{P_a^{m)}(x)}{1 + Ax^m} \ . \tag{14}$$

Hence the n first terms indeed determine the solution completely. For even values of m one obtains similar result for $n^{q)} = 1/2$ but now A is negative so that the solution is excluded. This result also motivates the hypothesis that for the counterparts of ordinary solutions of Laguerre equation sum (even m) or difference (odd m) of solutions corresponding to n and $2m - n$ must be formed to remove the non-physical degeneracy.

This argument does not exclude the possibility that there are also other fractional values of n allowing this kind of symmetry. The condition for symmetry would read as

$$\begin{aligned} \prod_{k=1}^{m} (R_k + \epsilon R_1) &= \prod_{k=1}^{m} (R_k - \epsilon R_1) \ , \\ \epsilon &= (2n^{q)} - 1 \ . \end{aligned} \tag{15}$$

The condition states that the odd part of the polynomial in question vanishes. Both ϵ and $-\epsilon$ solutions so that $n^{q)}$ and $1 - n^{q)}$ are solutions. If one requires that the condition holds true for all values of m then the comparison of constant terms in these polynomials allows to conclude that $\epsilon = 0$ is the only universal solution. Since ϵ is free parameter, it is clear that the m:th order polynomial in question has at most m solutions which could correspond to other fractionized eigenvalues expected to be present on basis of physical intuition.

This picture generalizes also to the case of even n so that also now solutions of the form of Eq. 14 are possible. In this case the condition is

$$\prod_{k=1}^{m} (R_k + \epsilon R_1) = -\prod_{k=1}^{m} (R_k - \epsilon R_1) \ . \tag{16}$$

Obviously $\epsilon = 0$ and thus $n = 1/2$ fails to be a solution to the eigenvalue equation in this case. Also now one has the spectral symmetry $n_{\pm} = 1/2 \pm \epsilon$.

The symmetry $R_n = (-1)^m R_{n+m-1} = (-1)^m R_{n-m-1} = (-1)^m R_{m-n+1}$ can be applied to show that the polynomials associated with ϵ and $-\epsilon$ contain both the terms $R_n - \epsilon$ and $R_n + \epsilon$ as factors except for odd m for $n = (m+1)/2$. Hence the values of n can be written for even values of m as

$$n^{q)}(n) = \frac{1}{2} \pm \frac{R_n}{2R_1} \ , \ n = 1, ..., \frac{m}{2} \ , \tag{17}$$

and for odd values of m as

$$\begin{aligned} n_{\pm}^{q)}(n) &= \frac{1}{2} \pm \frac{R_n}{2R_1} \ , \ n = 1, ..., \frac{m+1}{2} - 1 \ , \\ n^{q)} &= 1/2 \ . \end{aligned} \tag{18}$$

Plus sign obviously corresponds to the solutions which reduce to polynomials and to $n^{q)} \simeq n$ for large m. The explicit expression for $n^{q)}$ reads as

$$n_{\pm}^{q)}(n) = \frac{1}{2} \pm \frac{(sin^2(\pi(n-1)/2m) - sin^2(\pi n/2m))}{2sin^2(\pi/2m)} \ . \tag{19}$$

At the limit of large m one has

$$n_{+}^{q)}(n) \simeq n \ , \ n_{-}^{q)}(n) \simeq 1 - n \ . \tag{20}$$

so that the fractionization $n \simeq 1/k$ claimed by Mills is not obtained at this limit. The minimum for $|n^{q)}|$ satisfies $|n^{q)}| < 1$ and its smallest value $|n^{q)}| = .7071$ corresponds to $m = 4$. Thus these zeros cannot correspond to $n^{q)} \simeq 1/k$ yielded by the numerical computation for even values of m based on the requirement that the zero of $P^{2m)}$ cancels the pole of the geometric series.

2.6 Some comments

Some closing comments are in order.

a) An open question is whether there are also zeros $|n^{q)}| > 1$ satisfying $P_a^{2m)}((1/a)^{1/2m}) = 0$ for even values of m.

b) The treatment above is not completely general since only s-waves are discussed. The generalization is however a rather trivial replacement $(1-x)d/dx \rightarrow (l+1-x)d/dx$ in the Laguerre equation to get associated Laguerre equation. This modifies only the formula for a_{n+1} in the recursion for l_n so that expression for $n^{q)}$, which depends on b_n:s only, is not affected.

Also the product of numerators in the formula for the parameter $a = l_{2m}/l_0$ remains invariant so that the general spectrum has the spectral symmetry $n^{q)} \rightarrow 1 - n^{q)}$. The only change to the spectrum occurs for even values of m and is due to the dependence of $x_\infty = (1/a)^{1/2m}$ on l and can be understood in the semiclassical picture. It might happen that the value of l is modified to its q counterpart corresponding to q-Legendre functions.

c) The model could partially explain the findings of Mills and $n^{q)} \simeq 1/k$ for $k > 2$ also fixes the value of corresponding m to a very high degree so that one would have direct experimental contact with generalized imbedding space, spectrum of Planck constants, and dark matter. The fact that the fractionization is only approximately correct suggests that the states in question could be possible for all sectors of imbedding space appear as intermediate states into sectors in which the spectrum of hydrogen atom is scaled by $n_b/n_a = k = 2, 3, ...$

d) The obvious question is whether q-counterparts of angular momentum eigenstates ($idf_m/d\phi = mf_m$) are needed and whether they make sense. The basic idea of construction is that the phase transition changing $\hbar$ does not involve any other modifications except fractionization of angular momentum eigenvalues and momentum eigenvalues having purely geometric origin. One can however ask whether it is possible to identify q-plane waves as ordinary plane waves. Using the definition $L_z = 1/2(\partial_u^q + \partial_u^{\overline{q}})$, $u = exp(i\phi)$, one obtains $f_n = exp(in\phi)$ and eigenvalues as $n^{q)} = R_n/R_1 \rightarrow n$ for $m \rightarrow \infty$. Similar construction applies in the case of momentum components.

3 An explanation in terms of quantized Planck constant

The recent view about quantization of Planck constants allows to understand the findings of Mills elegantly. The solutions of q-Laguerre equation could represent in this framework intermediate states facilitating the phase transition changing Planck constants without large change in the binding energy.

3.1 Quantization of Planck constants and the generalization of the notion of imbedding space

The recent geometric interpretation for the quantization of Planck constants is based on Jones inclusions of hyper-finite factors of type II_1 [C8].

a) Different values of Planck constant correspond to imbedding space metrics involving scalings of M^4 *resp.* CP_2 parts of the metric deduced from the requirement that distances scale as $\hbar(M^4)$ *resp.* $\hbar(M^4)$. Denoting the Planck constants by $\hbar(M^4) = n_a\hbar_0$ and $\hbar(CP_2) = n_b\hbar_0$, one has that covariant metric of M^4 is proportional to n_b^2 and covariant metric of CP_2 to n_a^2. In Kähler action only the effective Planck constant $\hbar_{eff}/\hbar_0 = \hbar(M^4)/\hbar(CP_2)$ appears and by quantum classical correspondence same is true for Schödinger equation. Elementary particle mass spectrum is also invariant. Same applies to gravitational constant. The alternative assumption that M^4 Planck constant is proportional to n_b would imply invariance of Schrödinger equation but would not allow to explain Bohr quantization of planetary orbits and would to certain degree trivialize the theory (to be honest I believed to this option for some time and it produced a lot of confusion).

b) M^4 and CP_2 Planck constants do not fully characterize a given sector $M^4_\pm \times CP_2$. Rather, the scaling factors of Planck constant given by the integer n characterizing the quantum phase $q = exp(i\pi/n)$ corresponds to the order of the maximal cyclic subgroup for the group $G \subset SU(2)$ characterizing the Jones inclusion $\mathcal{N} \subset \mathcal{M}$ of hyper-finite factors realized as subalgebras of the Clifford algebra of the "world of the classical worlds". This means that subfactor $\mathcal{N}$ gives rise to G-invariant configuration space spinors having interpretation as G-invariant fermionic states.

c) $G_b \subset SU(2) \subset SU(3)$ defines a covering of M^4_+ by CP_2 points and $G_a \subset SU(2) \subset SL(2,C)$ covering of CP_2 by M^4_+ points with fixed points defining orbifold singularities. Different sectors are glued together along CP_2 if G_b is same for them and along M^4_+ if G_a is same for them. The degrees of freedom lost by G-invariance in fermionic degrees of freedom are gained back since the discrete degrees of freedom provided by covering allow many-particle states formed from single particle states realized in G group algebra. Among other things these many-particle states make possible the notion of N-atom.

d) Phases with different values of scalings of M^4 and CP_2 Planck constants behave like dark matter with respect to each other in the sense that they do not have direct interactions except at criticality corresponding to a leakage between different sectors of imbedding space glued together along M^4 or CP_2 factors. In large $\hbar(M^4)$ phases various quantum time and length scales are scaled up which means macroscopic and macro-temporal quantum coherence. In particular, quantum energies associated with classical frequencies are scaled up by a factor n_a/n_b which is of special relevance for cyclotron energies and phonon energies (superconductivity). For large $\hbar(CP_2)$ the value of $\hbar_{eff}$ is small: this leads to interesting physics: in particular the binding energy scale of hydrogen atom increases by the factor n_b/n_a^2.

3.2 Quantization of Planck constant as an explanation for the findings of Mills

Also the small values of $\hbar_{eff}) = n_a/n_b$ are interesting since in this case hydrogen atom binding energy scale increases by factor $(n_b/n_a)^2$ as Planck constant decreases (this conforms with the interpretation about approach to chaos in systems like plasmas). The assumption $n_b/n_a = k = 2, 3, ...$ predicts exactly the binding energies reported of Mills. Also the fact that for $n_b/n_a > 137$ the binding energy becomes larger than electron rest mass remaining invariant in the phase transition implies trivially the upper bound $k \leq 137$.

More generally, this picture leads to the notion of N-atom. The space-time sheets can be regarded as $N(G_b)$-fold coverings of M^4 by CP_2 points related by subgroup $G_b \subset SU(2) \subset SU(3)$ (color group) and this meas that one can put one hydrogen atom to each sheet of the covering (analogous to multi-sheeted Riemann surface. The signature for N-atom would be scaled up binding energy spectrum whereas vibrational energies would be scaled downwards.

Another kind of N-atom results for $n_a/n_b > 1$. This N-atom would be like N-molecule having discrete spatial symmetry characterized by $G_a \subset SO(3)$: for large values of n_a the symmetry would consist of planar rotations and reflections with number-theoretically preferred values of n_a corresponding to Fermat polygons constructible using only ruler and compass. The only genuinely 3-D symmetry groups would correspond to tedrahedral and icosahedral symmetries which are encountered in the structure of water. Icosahedral and dual dodecahedral structures are very abundant in living matter.

In this case energies hf associated with classical frequencies are scaled up by factor $n_a/n_b > 1$ so that the vibrational modes need not be masked by the thermal noise. Note that also the quantum energies associated with cyclotron and plasma frequency are scaled up. For $n_a/n_b = n$ integer, one can ask whether the vibrational dark photons emitted by dark atoms could decay to n ordinary photons having ordinary vibrational energy. The signature would be the appearance of a compound such as water in places where it is not thermally stable.

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